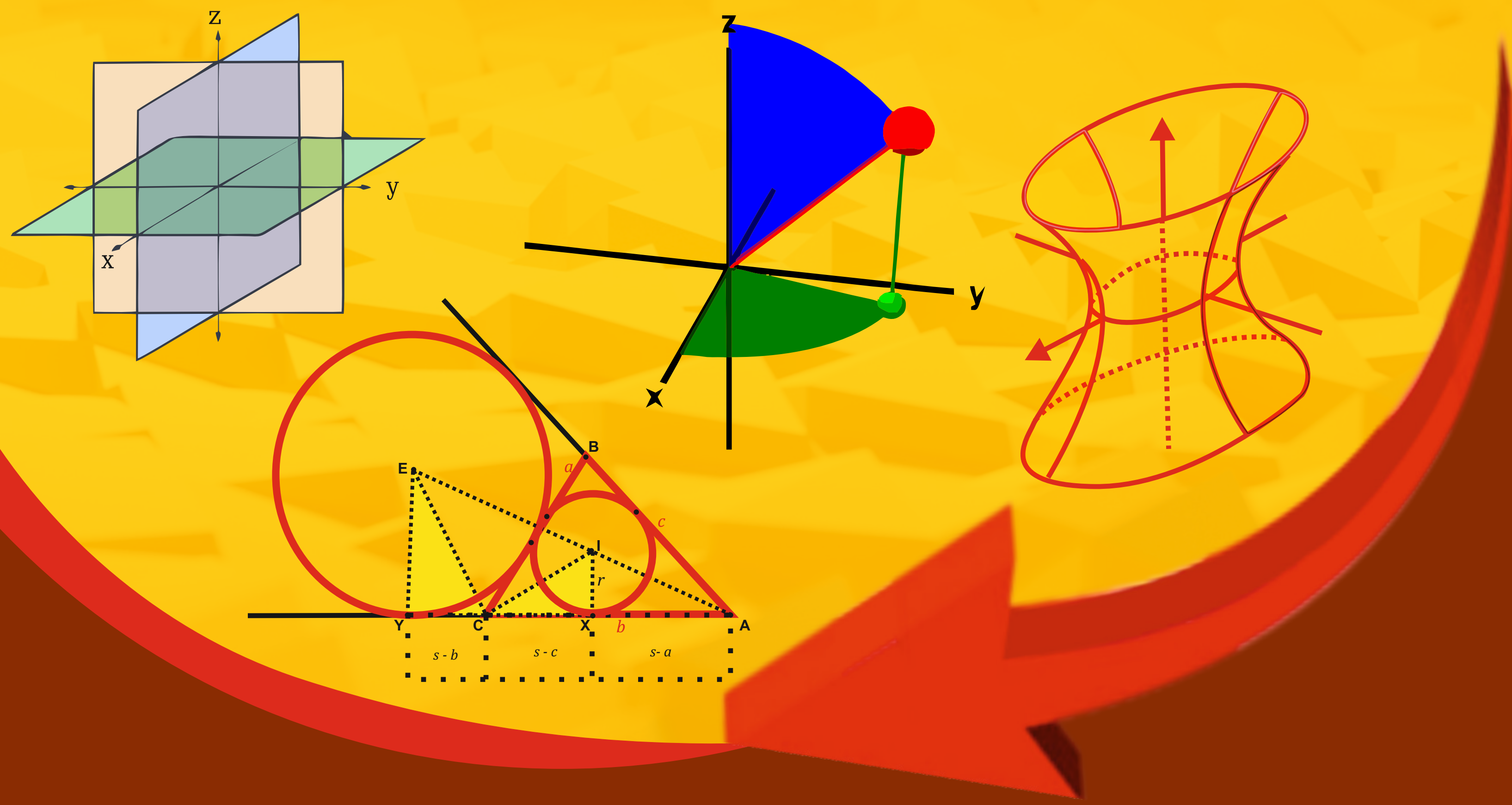

Coordinate Geometry Booster

with Problems & Solutions

for

JEE

Main and Advanced

Rejaul Makshud

McGraw Hill Education

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with Problems & Solutions

for

JEE

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About the Author

REJAUL MAKSHUD (RM)
Post Graduated from Calcutta University in PURE MATHEMATICS.
Presently, he trains IIT Aspirants at **RACE IIT Academy**, Jamshedpur.

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Dedicated to

My Parents

Preface

COORDINATE GEOMETRY BOOSTER with Problems & Solutions for JEE Main and Advanced is meant for aspirants preparing for the entrance examination of different technical institutions, especially NIT/IIT/BITSAT/IISc. In writing this book, I have drawn heavily from my long teaching experience at National Level Institutes. After many years of teaching, I have realised the need of designing a book that will help the readers to build their base, improve their level of mathematical concepts and enjoy the subject.

This book is designed keeping in view the new pattern of questions asked in JEE Main and Advanced Exams. It has six chapters. Each chapter has the concept booster followed by a large number of exercises with the exact solutions to the problems as given below:

Level - I	:	Problems based on Fundamentals
Level - II	:	Mixed Problems (Objective Type Questions)
Level - III	:	Problems for JEE Advanced Exam
Level - IV	:	Tougher Problems for JEE-Advanced Exam
(0.......9)	:	Integer Type Questions
Passages	:	Comprehensive Link Passages
Matching	:	Matrix Match
Previous years' papers	:	Questions asked in previous years' JEE-Advanced Exams

Remember friends, no problem in mathematics is difficult. Once you understand the concept, they will become easy. So please don't jump to exercise problems before you go through the Concept Booster and the objectives. Once you are confident in the theory part, attempt the exercises. The exercise problems are arranged in a manner that they gradually require advanced thinking.

I hope this book will help you to build your base, enjoy the subject and improve your confidence to tackle any type of problem easily and skilfully.

My special thanks goes to Mr. M.P. Singh (IISc. Bangalore), Mr. Manoj Kumar (IIT, Delhi), Mr. Nazre Hussain (B. Tech.), Dr. Syed Kashan Ali (MBBS) and Mr. Shahid Iqbal, who have helped, inspired and motivated me to accomplish this task. As a matter of fact, teaching being the best learning process, I must thank all my students who inspired me most for writing this book.

I would like to convey my affectionate thanks to my wife, who helped me immensely and my children who bore with patience my neglect during the period I remained devoted to this book.

I also convey my sincere thanks to Mr Biswajit Das of McGraw Hill Education for publishing this book in such a beautiful format.

I owe a special debt of gratitude to my father and elder brother, who taught me the first lessons of Mathematics and to all my learned teachers— Mr. Swapan Halder, Mr. Jadunandan Mishra, Mr. Mahadev Roy and Mr. Dilip Bhattacharya, who instilled the value of quality teaching in me.

I have tried my best to keep this book error-free. I shall be grateful to the readers for their constructive suggestions toward the improvement of the book.

Rejaul Makshud
M. Sc. (Calcutta University, Kolkata)

Contents

CHAPTER

1 Straight Lines

Rectangular Cartesian Co-ordinates

Concept Booster

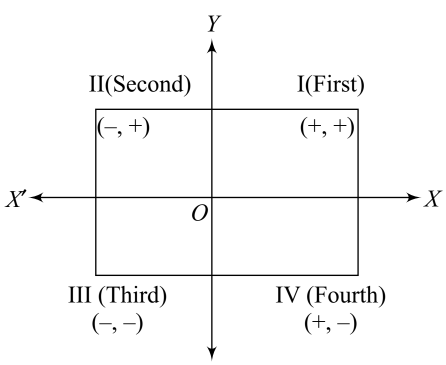
1. Introduction

A french mathematician and great philosopher Rane Descartes (1596–1665) introduced an idea to study geometry with the help of algebra.

It is called co-ordinate Geometry or analytical geometry. Here we use points, lines and curves by the different forms of algebraic equations.

2. Co-ordinate Axes and Rectangular Axes

The position of a point in a plane with reference to two intersecting lines called the co-ordinate axes and their point of intersection is called the origin. If these two axes cut at right angles, they are called rectangular axes, else they are called *oblique axes*.



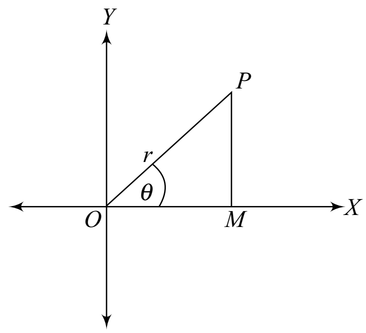
Let P be any point in the plane. Draw perpendiculars from P parallel to reference lines $X'OX$ and YOY', respectively. The lengths PN and PM are called the co-ordinates of the point P.

3. Cartesian Co-ordinates

This system of representing a point in 2-dimensions is called cartesian system. We normally denote PN by x and PM by y. Thus an ordered pair of two real numbers describes the cartesian co-ordinates of P.

The reference lines XOX' and YOY' are respectively, called x- and y-axis. These lines divide the plane into four equal parts and each part is called quadrant.

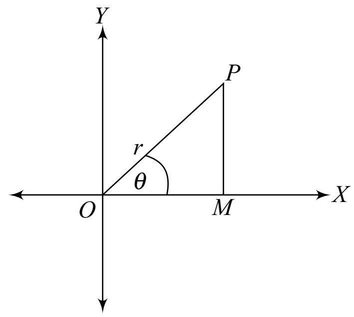
4. Polar Co-ordinates



The position of a point in a plane can also be described by other co-ordinate system, called polar co-ordinate system. In this case, we consider OX as initial line, O as origin. If P be any point on the plane such that $OP = r$ and $\angle POX = \theta$, then (r, θ) is called the polar co-ordinate of the point P, where $r > 0$ and $0 \le \theta < 2\pi$.

5. Relation between the Cartesian and Polar Co-ordinates

Let $P(x, y)$ be the cartesian co-ordinates with respect to OX and OY and $P(r, \theta)$ be the polars co-ordinates with respect to the pole O and the initial line OX.



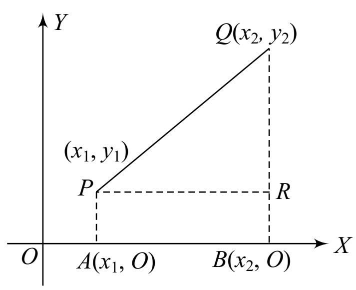
Now from the figure, $x = r\cos\theta$ and $y = r\sin\theta$

Thus, $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\dfrac{y}{x}\right)$.

Therefore, $(x, y) \Rightarrow (r\cos\theta, r\sin\theta)$

and $(r, \theta) \Rightarrow \left(\sqrt{x^2 + y^2}, \tan^{-1}\left(\dfrac{y}{x}\right)\right)$.

6. Distance between two Points (Cartesian form)



The distance between any two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

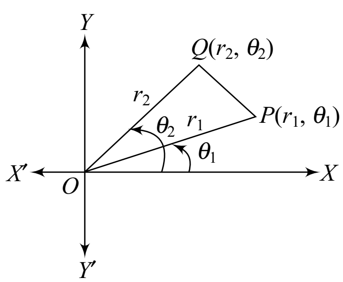
$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

> ***Note:*** If the three points P, Q, R are collinear, then $|PQ| \pm |QR| = |PR|$.

Polar Form

Let $P(r_1, \theta_1)$ and $Q(r_2, \theta_2)$ be any two points in polar form and θ be the angle between then.

Then



$$\cos\theta = \frac{r_1^2 + r_2^2 - PQ^2}{2r_1r_2}$$

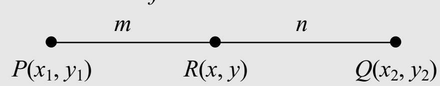
$\Rightarrow PQ^2 = r_1^2 + r_2^2 - 2r_1r_2\cos\theta$

$\Rightarrow PQ = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos\theta}$

> ***Notes:***
> **1. When 3 points are given and in order to prove**
> (i) an isosceles triangle, show that any two sides are equal.
> (ii) an equilateral triangle, show that all sides are equal
> (iii) a scalene triangle, show that all sides are unequal.
> (iv) a right-angled triangle, say ΔABC, show that $AB^2 + BC^2 = AC^2$
> **2. When 4 points are given and in order to prove**
> (i) a square, show that all sides and diagonals are equal.
> (ii) a rhombus, show that all sides are equal but diagonals are not equal.
> (iii) a rectangle, show that opposite sides and diagonals are equal.
> (iv) a parallelogram, show that opposite sides and diagonals are equal.

7. Section Formulae

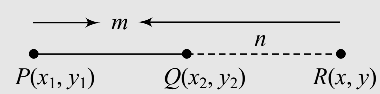
(i) *Internal Section formula*



If a point $R(x, y)$ divides a line segment joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ internally in the ratio $m : n$, then

$$x = \frac{mx_2 + nx_1}{m + n},\ y = \frac{my_2 + ny_1}{m + n}$$

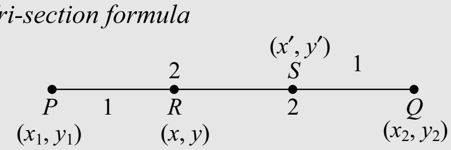
(ii) *External section formula*



If a point $R(x, y)$ divides a line segment joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ externally in the ratio $m : n$, then

$$x = \frac{mx_2 - nx_1}{m - n},\ y = \frac{my_2 - ny_1}{m - n}$$

(iii) *Tri-section formula*



Let the points P and Q be (x_1, y_1) and (x_2, y_2) respectively. Then the co-ordinates of the point R is $\left(\dfrac{x_2 + 2x_1}{3}, \dfrac{y_2 + 2y_1}{3}\right)$ and the co-ordinates of the point S is $\left(\dfrac{2x_2 + x_1}{3}, \dfrac{2y_2 + y_1}{3}\right)$.

(iv) *Mid-point formula*

If a point $R(x, y)$ divides a line segment joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ internally in the same ratio, then

$$x = \frac{x_1 + x_2}{2},\ y = \frac{y_1 + y_2}{2}$$

(v) If the line joining the points (x_1, y_1) and (x_2, y_2) is divided by the

(a) x-axis,then its ratio is $-\frac{y_1}{y_2}$

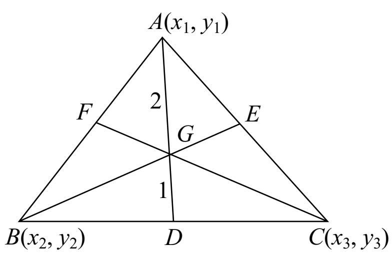
(b) y-axis , then its ratio is $-\frac{x_1}{x_2}$.

(vi) If a line $ax + by + c = 0$ divides the line joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ at R, then the ratio

$$\frac{PR}{RQ} = -\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}.$$

Important Points of a Triangle

(i) **Centroid:** The point of intersection of the medians of a triangle is called centroid.

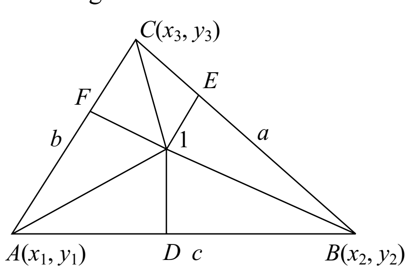


If the vertices of ΔABC be $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, the co-ordinates of its centroid is

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right).$$

(ii) If (a_1, b_1), (a_2, b_2) and (a_3, b_3) be the mid-points of the sides of a triangle, its centrod is also given as $\left(\frac{a_1 + a_2 + a_3}{3}, \frac{b_1 + b_2 + b_3}{3}\right)$.

(iii) **Incentre:** The point of intersection of an angle bisectors of a triangle is called its incentre.

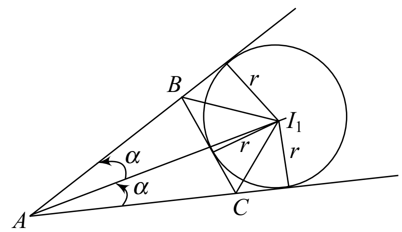


If the vertices of ΔABC be $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, the co-ordinates of its incentre is $\left(\frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c}\right)$,

where $a = BC$, $b = CA$, $c = AB$.

(iv) If ΔABC is an equilateral triangle, in-centre = centroid.

(v) **Ex-centre:** The point of intersection of the external bisectors of the angles of a triangle is called its ex-centre.



The circle opposite to the vertex A is called the escribed circle opposite A or the circle excribed to the side BC. If I_1 is the point of intersection of internal bisector of $\angle BAC$ and external bisector of $\angle ABC$ and $\angle ACB$, then

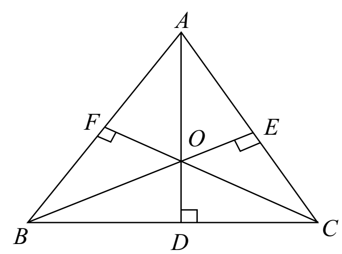
$$I_1 = \left(\frac{-ax_1 + bx_2 + cx_3}{-a + b + c}, \frac{-ay_1 + by_2 + cy_3}{-a + b + c}\right)$$

$$I_2 = \left(\frac{ax_1 - bx_2 + cx_3}{a - b + c}, \frac{ay_1 - by_2 + cy_3}{a - b + c}\right)$$

and $$I_3 = \left(\frac{ax_1 + bx_2 - cx_3}{a + b - c}, \frac{ay_1 + by_2 - cy_3}{a + b - c}\right)$$

respectively.

(vi) **Circumcentre**: The point of intersection of the perpendicular bisectors of a triangle is called its circumcentre.



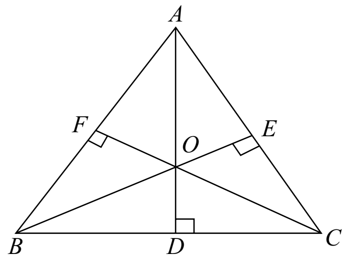
If the vertices of ΔABC be $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, then the co-ordinates of its circumcentre is

$$\left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}\right)$$

Notes:

1. If $O(x, y)$ be the circumcentre of ΔABC, its co-ordinates is determined by the relation $OA = OB = OC$.
2. In case of a right-angled triangle, mid-point of the hypotenuse is the circumcentre.

(vii) **Orthocentre:** The point of intersection of the altitudes of a triangle is called its orthocentre.

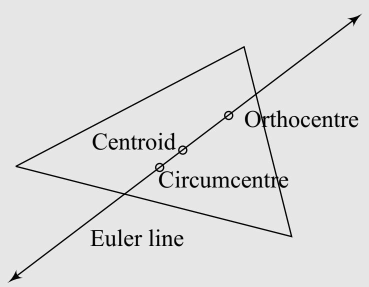


If the vertices of ΔABC be $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, the co-ordinates of its orthocentre is

$$\left(\frac{x_1\tan A + x_2\tan B + x_3\tan C}{\tan A + \tan B + \tan C},\right.$$
$$\left.\frac{y_1\tan A + y_2\tan B + y_3\tan C}{\tan A + \tan B + \tan C}\right)$$

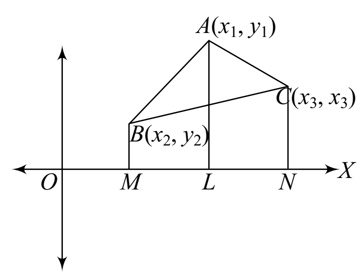
Notes:

1. In case of a right-angled triangle, orthocentre is the right-angled vertex.
2. In an isosceles triangle, G, O, I and C lie on the same line and in an equilateral triangle, all these four points coincide.
3. The orthocentre of ΔABC are (a, b), (b, a) and (a, a) is (a, a).
4. The orthocentre, the centroid and the circumcentre are always collinear and centroid divides the line joining ortho-centre and the circumcentre in the ratio 2 : 1.



5. In case of an obtuse-angled triangle, the circumcentre and orthocentre both lie outside of the triangle.

8. Area of a Triangle



If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of ΔABC, its area is given by

$$\frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)].$$

or
$$\frac{1}{2}\left\|\begin{matrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{matrix}\right\|$$

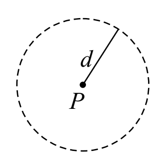
or
$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}$$

Notes:

1. The area of a polygon, with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3), …, (x_n, y_n) is
$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \cdots + \frac{1}{2}\begin{vmatrix} x_n & y_n \\ x_1 & y_1 \end{vmatrix}.$$
2. If the vertices of a triangle be $A(r_1, \theta_1)$, $B(r_2\theta_2)$ and (r_3, θ_3), its area is
$$\frac{1}{2}[r_1r_2\sin(a_1 - \theta_2) + r_2r_3\sin(\theta_2 - \theta_3) + r_3r_1\sin(\theta_3 - \theta_1)]$$
3. If $a_ix + b_iy + C_i = 0$, $i = 1, 2, 3$ be the sides of a triangle, its area is
$$\frac{1}{2C_1C_2C_3}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2,$$
where C_1, C_2 and C_3 are the cofactors of c_1, c_2 and c_3 in the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.
4. If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a triangle, its area is given by
$$\frac{1}{2}\begin{vmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{vmatrix}$$
5. If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be three collinear, points then
$$\begin{vmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{vmatrix} = 0$$
or
$$x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$$

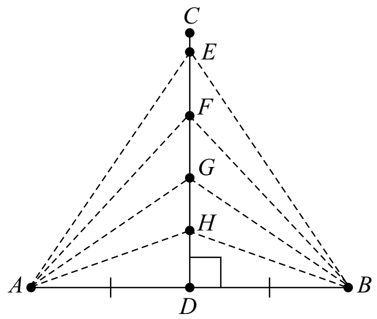
9. Locus and its Equation

The locus of a moving point is the path of a point which satisfies some geometrical conditions.



If a point moves in a plane in such a way that its distance from a fixed point is always the same. Then the locus of the movable point is called a circle.

If a points moves in a plane in such a way that its distances from two fixed points are always the same, the locus of the point is a perpendicular bisector.



Equation of a Locus

If the co-ordinates of every point on the locus satisfy the equation as well as if the co-ordinates of any point satisfy the equation, that point must lie on the locus.

Rule to Find Out the Locus of a Moving Point

(i) If x and y co-ordinates of the moving point are given in terms of a third variable, say t, eliminating t between x and y and then get the required locus.

(ii) Sometimes the co-ordinates of the moving point is taken as (x, y). In this case, the relation in x and y can be directly obtained by eliminating the variable.

(iii) If some geometrical conditions are given and we have to find the locus, we take a variable point (α, β) and write the given conditions in terms of α and β. Eliminating the variables and the relation between α and β is obtained. Finally replacing α by x and β by y the required locus is obtained.

10. Geometrical Transformations

When talking about geometric transformations, we have to be very careful about the object being transformed. We have two alternatives, either the geometric objects are transformed or the co-ordinate system is transformed. These two are very closely related, but the formulae that carry out the job are different. We only discuss transforming geometric objects here.

We shall start with the traditional Euclidean transformations that do not change length and angle measures.

Euclidean Transformations

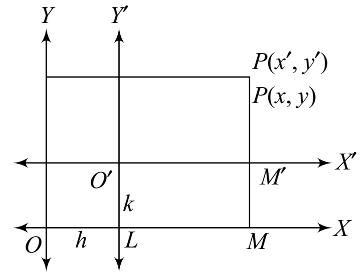
The Euclidean transformations are the most commonly used transformations. An Euclidean transformation is either a translation, a rotation, or a reflection. We shall discuss translations and rotations only.

(i) If the origin be shifted to a new point, say (h, k) and the new axes remain parallel to the original axes, the transformation is called *translation of axes*.

(ii) If the axes are rotated through an angle θ and the origin remain fixed, the transformation is called *rotation of axes*.

(iii) If the origin be shifted to a new point, say (h, k) and the axes also be rotated through an angle θ, the transformation is termed as *translation and rotation of axes*.

The coordinates of a point in a plane, when the origin is shifted from origin to a new point (h, k), the new axes remain parallel to the original axes.



Let O be the origin and, OX and OY the original axes.

Let the origin O be shifted to a new point O', whose co-ordinates are (h, k). Through O' draw $O'X'$ and $O'Y'$ parallel to OX and OY, respectively. O' is the new origin and, $O'X'$, and $O'Y'$ are the new axes.

Let P be any point in this plane, whose co-ordinates according to the original and new axes be (x, y) and (x', y') respectively.

Now we have to establish a relation between the new co-ordinates and the original co-ordinates.

From P, draw a perpendicular PM to OX which intersects $O'X'$ at M'.

Again from O', draw a perpendicular $O'L$ to OX.

Then $OL = h$, $O'L = k$;

$$OM = x,\ PM = y;$$

$$O'M' = x',\ PM' = y'$$

Thus, $x = OM = OL + LM$

$= OL + O'M$

$= h + x' = x' + h$

and $y = PM = PM' + M'M$

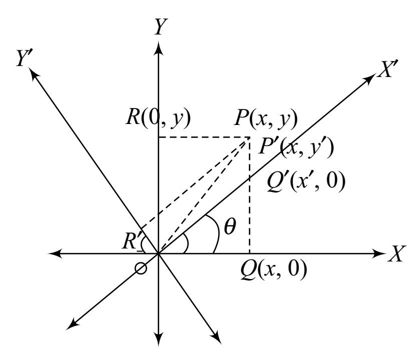
$= PM' + M'M$

$= y' + k$

Hence, we obtained the relation between the new and the original co-ordinates as

$$\begin{cases} x = x' + h \\ y = y' + k \end{cases} \text{and} \begin{cases} x' = x - h \\ y' = y - k \end{cases}$$

11. Rotation of Co-ordinate Axes



Let OX and OY be the original axes and OX' and OY' be the new axes obtained after rotating OX and OY through an angle θ in the anti-clockwise direction. Let P be any point in the plane having co-ordinates (x, y) with respect to axes OX and OY and (x', y') with respect to axes OX' and OY'.

Then $x = x'\cos\theta - y'\sin\theta$

$y = x'\sin\theta - y'\cos\theta$ …(i)

and $x' = x\cos\theta - y\sin\theta$

$y' = x\sin\theta - y\cos\theta$ …(ii)

STRAIGHT LINE

1. INTRODUCTION

The notion of a line or a straight line was introduced by ancient mathematicians to represent straight objects with negligible width and depth. Lines are an idealisation of such objects.

Euclid described a line as 'breadthless length', and introduced several postulates as basic unprovable properties from which he constructed the geometry, which is now called Euclidean geometry to avoid confusion with other geometries which have been introduced since the end of nineteenth century (such as non-Euclidean geometry, projective geometry, and affine geometry, etc.).

A line segment is a part of a line that is bounded by two distinct end-points and contains every point on the line between its end-points. Depending on how the line segment is defined, either of the two end-points may or may not be a part of the line segment. Two or more line segments may have some of the same relation as lines, such as being parallel, intersecting, or skew.

2. DEFINITIONS

Definition 1

It is the locus of a point which moves in a plane in a constant direction.

Definition 2

Every first-degree equation in x and y represents a straight line.

Definition 3

A straight line is also defined as the curve such that the line segment joining any two points on it lies wholly on it.

Definition 4

In 3D geometry, the point of intersection of two planes be a line.

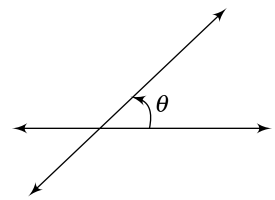
Notes:

1. The equation of a straight line is the relation between x and y, which is satisfied by the co-ordinates of each and every point on the line.
2. A straight line consists of only two arbitrary constants.

3. ANGLE OF INCLINATION OF A LINE

The angle of inclination of a line is the measure of the angle between the x-axis and the line measured in the anti-clockwise direction.

The angle of inclination of the line lies in between 0° and 180°.



Slope or Gradient of a Lines

If the inclination of a line be θ, then $\tan\theta$ where $0 < \theta < 180°$ and $\theta \neq \frac{\pi}{2}$ is called the slope or gradient of the line. It is usually denoted by m.

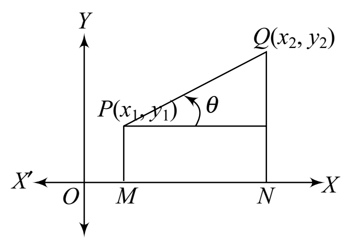
(i) If a line is parallel to x-axis, its slope is zero.

(ii) If a line is perpendicular to x-axis, the slope is not defined.

(iii) If a line is equally inclined with the axes, the slope is ± 1.

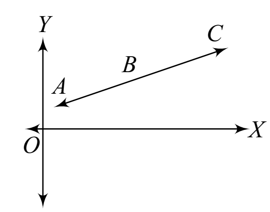
(iv) If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points on a line L, the slope of the line L is equal to

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$



(v) If three points A, B, C are collinear, then

$$m(AB) = m(BC) = m(CA).$$



(vi) If two lines, having slopes m_1 and m_2 and the angle between them be θ, then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|.$$

(vii) If two lines, having slopes m_1 and m_2, are parallel, then $m_1 = m_2$

(viii) If two lines, having slopes m_1 and m_2, are perpendicular, then $m_1 m_2 = -1$

(ix) If m is a slope of a line, the slope of a line perpendicular to it is $\frac{-1}{m}$.

(x) The equation of x-axis is $y = 0$.

(xi) The equation of y-axis is $x = 0$.

(xii) The equation of a line parallel to x-axis is y = constant = b (say)

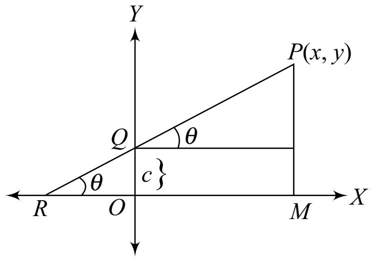
(xiii) The equation of a line parallel to y-axis is x = constant $= a$ (say).

4. Forms of a Straight Line

(i) Slope-intercept Form

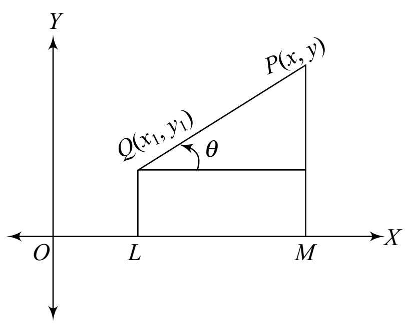
The equation of a straight line whose slope is m and cuts an intercept c on the y-axis is

$$y = mx + c.$$



> ***Notes:***
> 1. The general equation of a straight line is
> $$y = mx + c.$$
> 2. The equation of non-vertical lines in a plane is
> $$y = mx + c$$
> 3. The general equation of any line passing through the origin is
> $$y = mx.$$

(ii) Point-Slope Form

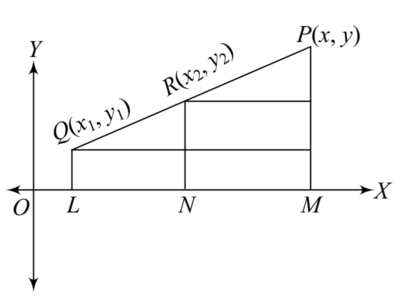


The equation of a line passing through the point (x_1, y_1) and having slope m is

$$y - y_1 = m(x - x_1).$$

> ***Note:*** The equation $y - y_1 = m(x - x_1)$ is also known as *one-point form of a line*.

(iii) Two Point Form



The equation of a line passing through (x_1, y_1) and (x_2, y_2) is

$$y - y_1 = \left(\frac{y_2 - y_2}{x_2 - x_1}\right) \times (x - x_1)$$

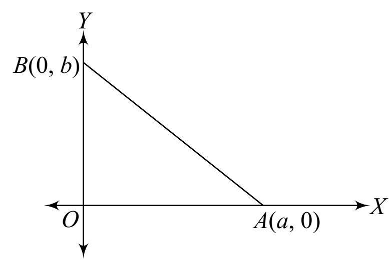
or

$$y - y_2 = \left(\frac{y_2 - y_2}{x_2 - x_1}\right) \times (x - x_2)$$

or $$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

or $$\begin{vmatrix} x - x_1 & y - y_1 \\ x_1 - x_2 & y_1 - y_2 \end{vmatrix} = 0$$

(iv) Intercept Form



The equation of a straight line which cuts off intercepts a and b on x- and y-axes, respectively is

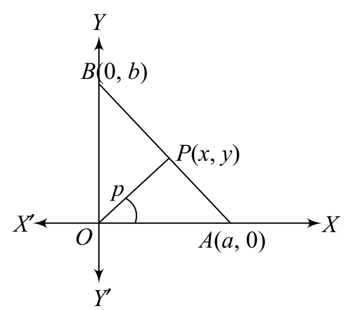
$$\frac{x}{a} + \frac{y}{b} = 1.$$

> ***Notes:***
> 1. The intercepts a and b may be positive or negative.
> 2. The intercept cut on negative side of x and y axes are taken as negative.

(v) Normal Form

The equation of a straight line upon which the length of a perpendicular from the origin is p and this normal makes an angle α with the positive direction of x-axis is

$$x \cos \alpha + y \sin \alpha = p.$$



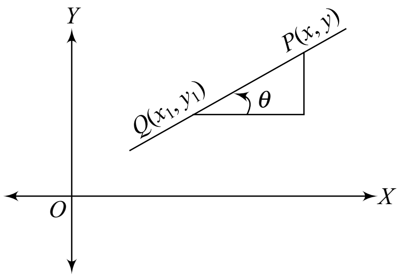
> ***Note:*** Here, p is always taken is positive and α is measured from the positive direction of x-axis in anti-clockwise direction such that $0 \le \alpha < 2\pi$.

(vi) Distance Form or Parametric Form or Symmetric Form

The equation of a line passing through the point (x_1, y_1) and making angle θ with the positive direction of x-axis is

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r,$$

where r is the distance of the point (x, y) from the point (x_1, y_1).



> ***Notes:***
> 1. If $Q(x_1, y_1)$ be a point on a line AB which makes an angle θ with the positive direction of x-axis, there will be two points on the line AB at a distance r from $Q(x_1, y_1)$ and their co-ordinates will be $(x_1 \pm r\cos\theta, y_1 \pm r\sin\theta)$.
> 2. If a point, say P, lies above $Q(x_1, y_1)$ on the line AB, we consider r is positive.
> Thus, the co-ordinates of P will be
> $(x_1 + r\cos\theta, y_1 + r\sin\theta)$.
> 3. If a point, say R, lies below $Q(x_1, y_1)$ on the line AB, we consider r is negative.
> Thus, the co-ordinates of R will be
> $(x_1 - r\cos\theta, y_1 - r\sin\theta)$.

5. Reduction of General Equation into Standard Form

Let $Ax + By + C = 0$ be the general equation of a straight line, where $A^2 + B^2 \neq 0$.

(i) **Reduction into slope-intercept form**

The given equation is $Ax + By + C = 0$.

$\Rightarrow \quad By = -Ax - C$

$$\Rightarrow \quad y=-\left(\frac{A}{B}\right)x-\left(\frac{C}{B}\right)$$

which is in the form of $y = mx + c$.

(ii) **Reduction into intercept form**

The given equation is $Ax + By + C = 0$.

$\Rightarrow \quad Ax + By = -C$

$$\Rightarrow \quad \frac{Ax}{-C}+\frac{By}{-C}=1$$

$$\Rightarrow \quad \frac{x}{\left(-\frac{C}{A}\right)}+\frac{y}{\left(-\frac{C}{B}\right)}=1$$

which is in the form of $\frac{x}{a}+\frac{y}{b}=1$.

(iii) **Reduction into normal form**

The given equation is $Ax + By + C = 0$.

$\Rightarrow \quad -Ax - By = C$

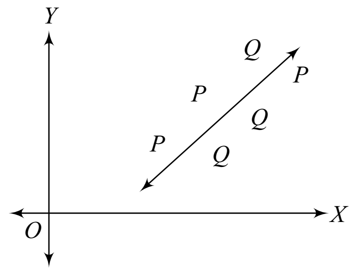
$$\Rightarrow \quad \left(\frac{-A}{\sqrt{A^2+B^2}}\right)x-\left(\frac{-B}{\sqrt{A^2+B^2}}\right)y=\frac{C}{(\sqrt{A^2+\mathrm{B}^2})}$$

which is the normal form of the line $Ax + By + C = 0$.

6. Position of Two Points with Respect to a Given Line

The points $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on the same or on the opposite sides of the line $ax + by + c = 0$ according as

$$\frac{ax_1+by_1+c}{ax_2+by_2+\mathrm{c}}>0 \text{ or } <0.$$



> ***Notes:***
> 1. The side of the line, where origin lies, is known as origin side.
> 2. A point (α, β) will lie on the origin side of the line $ax + by + c = 0$, if $a\beta + b\beta + c$ and c have the same sign.
> 3. A point (α, β) will lie on the non-origin side of the line $ax + by + c = 0$, if $a\alpha + b\beta + c$ and c have the opposite signs.

7. Equation of a Line Parallel to a Given Line

The equation of any line parallel to a given line

$ax + by + c = 0$ is $ax + by + k = 0$

> ***Note:*** The equation of a line parallel to $ax + by + c = 0$ and passing through (x_1, y_1) is
> $a(x - x_1) + b(y - y_1) = 0$

8. Equation of a Line Perpendicular to a Given Line

The equation of a line perpendicular to a given line

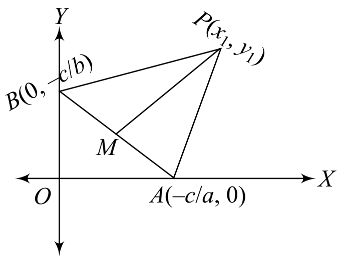
$ax + by + c = 0$ is $bx - ay + k = 0$.

> ***Note:*** The equation of a line perpendicular to $ax + by + c = 0$ and passing through (x_1, y_1) is
> $b(x - x_1) - a(y - y_1) = 0$.

9. Distance from a Point to a Line

The length of the perpendicular from a point (x_1, y_1) to the line $ax + by + c = 0$ is given by

$$\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|.$$



Area of ΔPAB

$$=\frac{1}{2}\times AB\times PM$$

We know that

$$\Delta=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1\\ -\frac{c}{a} & 0 & 1\\ 0 & -\frac{c}{b} & 1\end{vmatrix}$$

$$=\frac{1}{2}\left(\frac{cx_1}{b}+\frac{c}{a}\left(y_1+\frac{c}{b}\right)\right)$$

$$=\frac{1}{2}\left(\frac{cx_1}{b}+\frac{cy_1}{a}+\frac{c^2}{ab}\right)$$

$$=\frac{c}{2ab}(ax_1+by_1+c)$$

Also, $PM=\sqrt{a^2+b^2}$

Thus, $\frac{1}{2}\times PM\times AB=\frac{c}{2ab}(ax_1+by_1+c)$

$$\Rightarrow\quad \frac{1}{2}\times PM\times\sqrt{\frac{c^2}{a^2}+\frac{c^2}{b^2}}=\frac{c}{2ab}(ax_1+by_1+c)$$

$$\Rightarrow\quad PM\times\frac{c}{2ab}\sqrt{a^2+b^2}=\frac{c}{2ab}(ax_1+by_1+c)$$

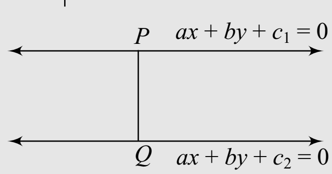
$$\Rightarrow\quad PM=\left|\frac{(ax_1+by_1+c)}{\sqrt{a^2+b^2}}\right|..$$

Rule 1. The length of the perpendicular from the origin to the line $ax + by + c = 0$ is

$$\left|\frac{c}{\sqrt{a^2+b^2}}\right|.$$

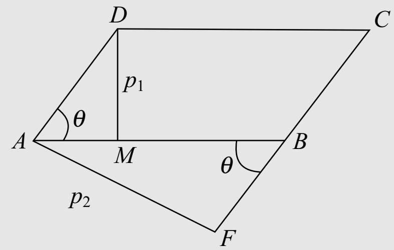
Rule 2. The distance between two parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is given by

$$\left|\frac{c_1-c_2}{\sqrt{a^2+b^2}}\right|.$$



Rule 3. The area of a parallelogram whose sides are $a_1x + b_1y + c_1 = 0$, $a_1x + b_1y + d_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_2x + b_2y + d_2 = 0$ is given by

$$\frac{p_1\times p_2}{\sin\theta}=\left|\frac{(c_1-d_1)(c_2-d_2)}{(a_1b_2-a_2b_1)}\right|.$$



The distance between two parallel sides $a_1x + b_1y + c_1 = 0$, $a_1x + b_1y + d_1 = 0$ is given by

$$p_1=\left|\frac{(c_1-d_1)}{\sqrt{a_1^2+b_1^2}}\right|$$

The distance between two parallel sides $a_2x + b_2y + c_2 = 0$, $a_2x + b_2y + d_2 = 0$ is given by

$$p_2=\left|\frac{(c_2-d_2)}{\sqrt{a_2^2+b_2^2}}\right|.$$

Now, $$\tan\theta=\left|\frac{\frac{b_1}{a_1}-\frac{b_2}{a_2}}{1+\frac{b_1}{a_1}\cdot\frac{b_2}{a_2}}\right|=\left|\frac{a_2b_1-a_1b_2}{a_1a_2+b_1b_2}\right|$$

$$\Rightarrow\quad \sin\theta=\frac{a_1b_2-a_2b_1}{\sqrt{a_1^2+b_1^2}\sqrt{a_2^2+b_2^2}}$$

Thus, the area of a parallelogram $=\frac{p_1\times p_2}{\sin\theta}$

$$=\left|\frac{(c_1-d_1)(c_2-d_2)}{(a_1b_2-a_2b_1)}\right|$$

Rule 4. If $p_1 = p_2$, the parallelogram becomes a rhombus and its area is given by

$$\frac{(c_1-d_1)^2}{|(a_1b_2-a_2b_1)|\sqrt{\left(\frac{a_1^2+b_1^2}{a_2^2+a_2^2}\right)}}.$$

Rule 5. The area of a parallelogram whose sides are $y = mx + a$, $y = mx + b$, $y = nx + c$ and $y = nx + d$ is given by

$$\left|\frac{(a-b)(c-d)}{(m-n)}\right|.$$

10. Point of Intersection of two Lines

Let $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ be two lines.

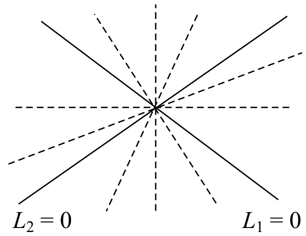
(i) Simply solve the given equations and get the point of intersection P.

(ii) **Concurrent lines:** If three or more lines meet at a point, we say that these lines are concurrent lines and the meeting point is known as *point of concurrency*.

The three lines $a_1x + b_1y + c_1 = 0$, $i = 1, 2, 3$ are said to be concurrent if

$$\begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}=0$$

(iii) **Family of lines:** Any line passing through the point of intersection of the lines L_1: $a_1x + b_1y + c_1 = 0$ and L_2: $a_2x + b_2y + c_2 = 0$ can be defined as
$(a_1x + b_1y + c_1) + \lambda(a_2x + b_2y + c_2) = 0$, where $\lambda \in R$.

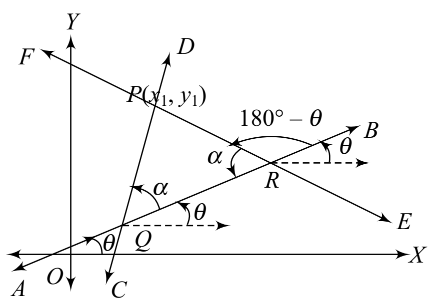


11. Equation of Straight Lines Passing through a Given Point and Making a Given Angle with a Given Line

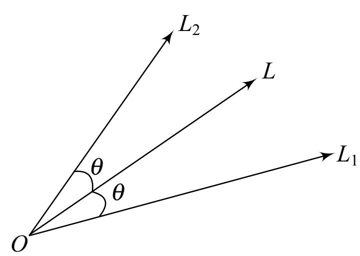
The equation of the straight lines which pass through a given point (x_1, y_1) and make an angle α with the given straight line $y = mx + c$ are

$$y - y_1 = \tan(\theta \pm \alpha)(x - x_1),$$

where $m = \tan\theta$.



12. A Line is Equally Inclined with Two Lines



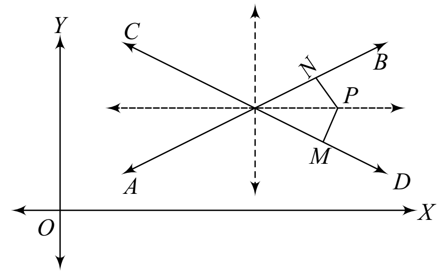
If two lines $y = m_1x + c_1$ and $y = m_2x + c_2$ be equally inclined to a line $y = mx + c$, then

$$\left(\frac{m_1 - m}{1 + mm_1}\right) = -\left(\frac{m_2 - m}{1 + mm_2}\right).$$

13. Equation of Bisectors

The equation of the bisectors of the angles between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is given by

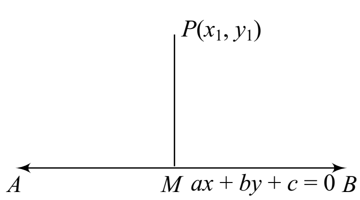
$$\left(\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}}\right) = \pm\left(\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}\right).$$



Notes:

1. When we shall find the equation of bisectors, first we make c_1 and c_2 positive.
2. Two bisectors are perpendicular to each other.
3. The positive bisector, the equation of the bisector contains the origin.
4. The negative bisector, the equation of the bisector does not contain the origin.
5. If $a_1a_2 + b_1b_2 > 0$, then the negative bisector is the acute-angle bisector and the positive bisector is the obtuse-angle bisector.
6. If $a_1a_2 + b_1b_2 < 0$, then the positive bisector is the acute-angle bisector and the negative bisector is the obtuse-angle bisector.
7. If $a_1a_2 + b_1b_2 > 0$, the origin lies in obtuse angle and if $a_1a_2 + b_1b_2 < 0$, the origin lies in acute angle.
8. The equation of the bisector of the angle between the two lines containing the point (α, β) will be positive bisector according as $(a_1\alpha + b_1\beta + c_1)$ and $(a_2\alpha + b_2\beta + c_2)$ are of the same sign and will be negative bisector if $(a_1\alpha + b_1\beta + c_1)$ and $(a_2\alpha + b_2\beta + c_2)$ are of the opposite signs.

14. Foot of the Perpendicular Drawn From the Point (x_1, y_1) to the Line $ax + by + c = 0$



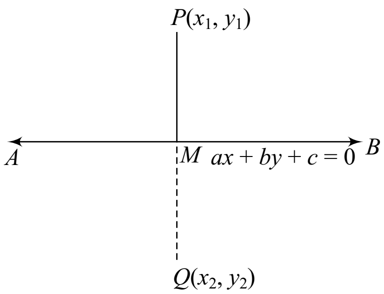
The foot of the perpendicular from the point (x_1, y_1) to the line $ax + by + c = 0$ is given by

$$\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = -\frac{(ax_1 + by_1 + c)}{(a^2 + b^2)}.$$

15. Image of a Point (x_1, y_1) with Respect to a Line Mirror $ax + by + c = 0$

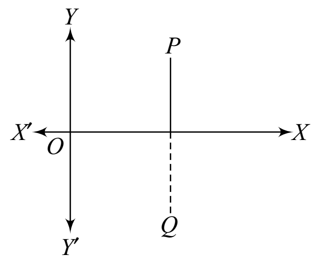
The image of a point (x_1, y_1) with respect to a line mirror $ax + by + c = 0$ is given by

$$\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = -2\left(\frac{ax_1 + by_1 + c}{a^2 + b^2}\right)$$



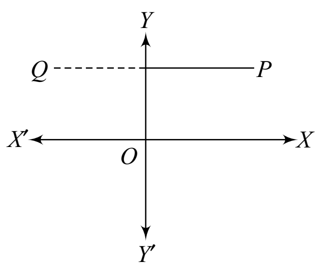
Rule 1 The image of a point $P(\alpha, \beta)$ with respect to x-axis is given by $Q(\alpha, -\beta)$.

> ***Note:*** The image of the line $ax + by + c = 0$ about x-axis is given by $ax - by + c = 0$.



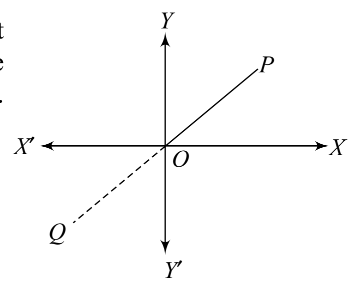
Rule 2 The image of a point $P(\alpha, \beta)$ with respect to y-axis is given by $Q(-\alpha, \beta)$.

> ***Note:*** The image of the line $ax + by + c = 0$ about y-axis is given by $-ax + by + c = 0$.



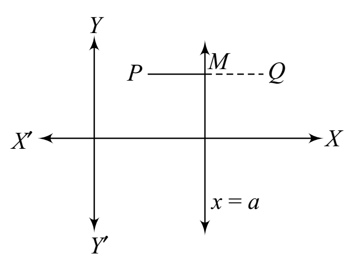
Rule 3 The image of a point $P(\alpha, \beta)$ with respect to the origin is given by $Q(-\alpha, -\beta)$.

> ***Note:*** The image of the line $ax + by + c = 0$ with respect to origin is given by $-ax - by + c = 0$.



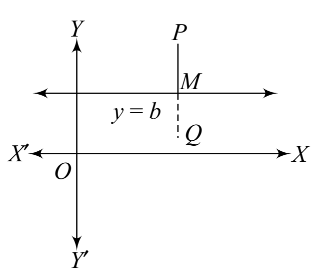
Rule 4 The image of a point $P(\alpha, \beta)$ with respect to the line $x = a$ is given by $Q(2a - \alpha, \beta)$.

> ***Note:*** The image of the line $ax + by + c = 0$ with respect to the line $x = \lambda$ is given by $a(2\lambda - x) + by + c = 0$.



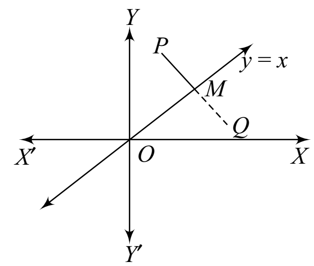
Rule 5 The image of a point $P(\alpha, \beta)$ with respect to the line $y = b$ is given by $Q(\alpha, 2b - \beta)$.

> ***Note:*** The image of the line $ax + by + c = 0$ about the line $y = \mu$ is $ax + b(2\mu - y) + c = 0$.



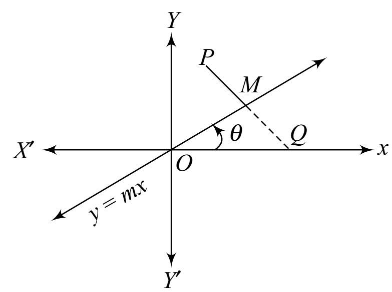
Rule 6 The image of a point $P(\alpha, \beta)$ with respect to the line $y = x$ is $Q(\beta, \alpha)$.

> ***Note:*** The image of a line $ax + by + c = 0$ about the line $y = x$ is $ay + bx + c = 0$.



Rule 7 The image of a point $P(\alpha, \beta)$ with respect to the line $y = mx$, where $m = \tan\theta$, is given by

$$Q(\alpha\cos 2\theta + \beta\sin 2\theta,\ \alpha\sin 2\theta - \beta\cos 2\theta).$$



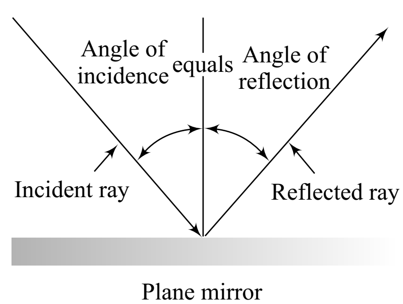
> ***Note:*** The image of a line $ax + by + c = 0$ with respect to the line $y = mx$, where $m = \tan\theta$, is given by
>
> $$a(\alpha\cos 2\theta + \beta\sin 2\theta) + b(\alpha\sin 2\theta - \beta\cos 2\theta) + c = 0.$$

16. Reflection of Light

When you play billiards, you will notice that when a ball bounces from a surface, the angle of rebound is equal to the angle of incidence. This observation is also true with light. When an incident light ray strikes a smooth surface (like a plane mirror) at an angle, the angle formed by the incident ray measured from the normal is equal to the angle formed by the reflected ray.

Law of Reflection

The angle of incidence is equal to the angle of reflection. The reflected and incident rays lie in a plane that is normal to the reflecting surface.



EXERCISES

LEVEL I

(Problems based on Fundamentals)

ABC OF COORDINATES

1. Find the polar co-ordinates of the points whose cartesian co-ordinates are (3, –4) and (–3, 4).
2. Transform the equation $r^2 = a^2$ into cartesian form.
3. Transform the equation $r = 2a\cos\theta$ into cartesian form.
4. Transform the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ into polar form.
5. Transform the equation $2x^2 + 3xy + 2y^2 = 1$ into polar form.

DISTANCE BETWEEN TWO POINTS

6. Find the distance between the points $(a\cos\alpha, a\sin\alpha)$ and $(a\cos\beta, a\sin\beta)$, where $a > 0$.
7. Find the distance between the points $\left(3, \frac{\pi}{4}\right)$ and $\left(7, \frac{5\pi}{4}\right)$.
8. If the point (x, y) be equidistant from the points (6, –1) and (2, 3) such that $Ax + By + C = 0$, find the value of $A + B + C + 10$.
9. The vertices of a triangle ABC are $A(-2, 3)$, $B(2, -1)$ and $C(4, 0)$. Find $\cos A$.
10. Prove that the points (–4, –1), (–2, –4), (4, 0) and (2, 3) are the vertices of a rectangle.
11. Two vertices of an equilateral triangle are (3, 4) and (–2, 3). Find the co-ordinates of the third vertex.

SECTION FORMULAE

12. Find the point, which divides the line joining the points (2, 3) and (5, –3) in the ratio 1 : 2.
13. In what ratio does y-axis divide the line segment joining $A(-3, 5)$ and $B(7, 2)$.
14. Find the ratio in which the join of the points (1, 2) and (–2, 3) is divided by the line $3x + 4y = 7$.
15. Find the co-ordinates of the points which trisect the line segment joining (1, –2) and (–3, 4).
16. The co-ordinates of the mid-points of the sides of a triangle are (1, 1), (3, 2) and (4, 1). Find the co-ordinates of its vertices.
17. The co-ordinates of three consecutive vertices of a parallelogram are (1, 3), (–1, 2) and (2, 5). Find its fourth vertex.
18. Find the centroid of ΔABC, whose vertices are $A(2, 4)$, $B(6, 4)$, $C(2, 0)$.
19. Two vertices of a triangle are (–1, 4) and (5, 2). If its centroid is (0, –3), find its third vertex.
20. Find the in-centre of ΔABC, whose vertices are $A(1, 2)$, $B(2, 3)$ and $C(3, 4)$.
21. If a triangle has its orthocentre at (1, 1) and circumcentre at (3/2, 3/4), find its centroid.
22. The vertices of a ΔABC are $A(0, 0)$, $B(0, 2)$ and $C(2, 0)$. Find the distance between the circum-centre and orthocentre.

AREA OF A TRIANGLE

23. Find the area of a triangle whose vertices are (3, –4), (7, 5) (–1, 10).
24. Find the area of a triangle whose vertices are $(t, t + 2)$, $(t + 3, t)$ and $(t + 2, t + 2)$.
25. If $A(x, y)$, $B(1, 2)$ and $C(2, 1)$ be the vertices of a triangle of area 6 sq.u., prove that $x + y + 9 = 0$.
26. Find the area of a quadrilateral whose vertices are (1, 1), (7, –3), (12, 2) and (7, 21).
27. Find the area of a pentagon whose vertices are (4, 3), (–5, 6) (0, 07) (3, –6) and (–7, –2).
28. Prove that the points $(a, b + c)$, $(b, c + a)$ and $(c, a + b)$ are collinear.
29. Let the co-ordinates of A, B, C and D are (6, 3), (–3, 5), (4, –2) and $(x, 3x)$, respectively and $\frac{\Delta DBC}{\Delta ABC} = \frac{1}{2}$, find x.
30. Find the area of a triangle formed by the lines $7x - 2y + 10 = 0$, $7x + 2y - 10 = 0$ and $9x + y + 2 = 0$.

LOCUS OF A POINT

31. If the co-ordinates of a variable point P be $(a\cos\theta, a\sin\theta)$, where θ is a parameter, find the locus of P.
32. If the co-ordinates of a variable point P be $(at^2, 2at)$, find the locus of P.
33. Find the locus of a movable point P, where its distance from the origin is 3 time its distance from y-axis.
34. If a point moves in a plane in such a way that its distance from the point $(a, 0)$ to its distance is equal to its distance from y-axis.
35. If the co-ordinates of a variable point P be $\left(t + \frac{1}{t}, t - \frac{1}{t}\right)$, where t is a parameter, find the locus of P.
36. Find the locus of a movable point P, for which the sum of its distance from (0, 3) and (0, –3) is 8.
37. A stick of length l slides with its ends on two mutually perpendicular lines. Find the locus of the mid-point of the stick.
38. If O be the origin and A be a point on the line $y^2 = 8x$. Find the locus of the mid-point of OA.
39. If P be the mid-point of the straight line joining the points $A(1, 2)$ and Q where Q is a variable point on the curve $x^2 + y^2 + x + y = 0$. Find the locus of P.
40. A circle has the centre (2, 2) and always touches x and y axes. If it always touches a line AB (where A and B lie on positive x and y axes), find the locus of the circumcentre of the ΔOAB, where O is the origin.
41. Find the locus of a movable point P, for which the difference of its distances from (2, 0) and (–2, 0) is 6.

42. If $A(1, 1)$ and $B(-2, 3)$ are two fixed points, find the locus of a point P so that the area of ΔPAB is 9 sq.u.
43. Find the locus of a point whose co-ordinates are given by $x = t^2 + t$, $y = 2t + 1$, where t is parameter.
44. If a variable line $\frac{x}{a} + \frac{y}{b} = 1$ intersects the axes at A and B, respectively such that $a^2 + b^2 = 4$ and O be the origin, find the locus of the circumcentre of ΔOAB.
45. A variable line cuts the x and y axes at A and B respectively where $OA = a$, $OB = b$ (O the origin) such that $a^2 + b^2 = 27$, find the locus of the centroid of ΔOAB.

TRANSFORMATION OF COORDINATES

46. Find the equation of the curve $2x^2 + y^2 - 3x + 5y - 8 = 0$, when the origin is shifted to the point $(-1, 2)$ without changing the direction of the axes.
47. The equation of a curve referred to the new axes retaining their directions and origin is $(4, 5)$ is $x^2 + y^2 = 36$. Find the equation referred to the original axes.
48. At what point, the origin be shifted if the co-ordinates of a point $(-1, 8)$ become $(-7, 3)$?
49. If the axes are turned through 45°, find the transformed form of the equation $3x^2 + 3y^2 + 2xy = 2$.
50. Transform to parallel axes through the point $(1, -2)$, the equation $y^2 - 4x + 4y + 8 = 0$.
51. If a point $P(1, 2)$ is translated itself 2 units along the positive direction of x-axis and then it is rotated about the origin in anti-clockwise sense through an angle of 90°, find the new position of P.
52. If the axes are rotated through $\frac{\pi}{4}$, the equation $x^2 + y^2 = a^2$ is transformed to $\lambda xy + a^2 = 0$, find the value of $\lambda = 10$.
53. Transform to axes inclined at $\frac{\pi}{3}$ to the original axes, the equation $x^2 + 2\sqrt{3}y - y^2 = 2a^2$.
54. What does the equation $2x^2 + 2xy + 3y^2 - 18x - 22y + 50 = 0$ become when referred to new rectangular axes through the point $(2, 3)$, the new set making $\frac{\pi}{4}$ with the old?
55. If a point $P(2, 3)$ is rotated through an angle of 90° about the origin in anti-clockwise sense, say at Q, find the co-ordinates of Q.
56 If a point $P(3, 4)$ is rotated through an angle of 60° about the point $Q(2, 0)$ in anti-clockwise sense, say at R, find the co-ordinates of R.

Straight Line

Level I

(Problems Based on Fundamentals)

ABC OF STRAIGHT LINE

1. Find the slope of the line PQ, where $P(2, 4)$ and $Q(3, 10)$.
2. Find the value of λ, if 2 is slope of the line through $(2, 5)$ and $(\lambda, 7)$.
3. Prove that the line joining the points $(2,-3)$ and $(-5, 1)$ is parallel to the line joining $(7,-1)$ and $(0,3)$.
4. Prove that the points $(a, b + c)$, $(b, c + a)$ and $(c, a + b)$ are collinear.
5. Find the angle between the lines joining the points $(0, 0)$, $(2, 2)$ and $(2, -2)$ $(3, 5)$.
6. If the angle between two lines is 45° and the slope of one of them is 1/2, find the slope of the other line.
7. Find the equation of a line passing through the point $(2, -3)$ and is parallel to x-axis.
8. Find the equation of a line passing through the point $(3, 4)$ and is perpendicular to y-axis.
9. Find the equation of a line which is equidistant from the lines $x = 6$ and $x = 10$.

VARIOUS FORMS OF A STRAIGHT LINE

10. Find the equation of a straight line which cuts off an intercept of 7 units on y-axis and has the slope 3.
11. Find the equation of a straight line which makes an angle of 135° with the positive direction of x-axis and cuts an intercept of 5 units on the positive direction of y-axis.
12. Find the equation of a straight line which makes an angle of $\tan^{-1}\left(\frac{3}{5}\right)$ with the positive direction of x-axis and cuts an intercept of 6 units in the negative direction of y-axis.
13. Find the equation of a straight line which cuts off an intercept of 4 units from y-axis and are equally inclined with the axes.
14. Find the equations of bisectors of the angle between the co-ordinate axes.
15. Find the equation of a line passing though the point $(2, 3)$ and making an angle of 120° with the positive direction of x-axis.
16. Find the equation of the right bisectors of the line joining the points $(1, 2)$ and $(5, 7)$.
17. Find the equation of a line which passes through the point $(1, 2)$ and makes an angle θ with the positive direction of x-axis, where $\cos\theta = -\frac{3}{5}$.
18. A line passes through the point $A(2, 0)$ which makes an angle of 30° with the positive direction of x-axis and is rotated about A in clockwise direction through an angle of 15°. Find the equation of the straight line in the new position.
19. Find the equation of a line passing through the points $(1, 2)$ and $(3, 4)$.
20. The vertices of a triangle are $A(10, 4)$, $B(-4, 9)$ and $C(-2, -1)$. Find the equation of its altitude through A.
21. The vertices of a triangle are $A(1, 2)$, $B(2, 3)$ and $C(5, 4)$. Find the equation of its median through A.
22. Find the equation of the internal bisector of $\angle BAC$ of the ΔABC, whose vertices are $A(5, 2)$, $B(2, 3)$ and $C(6, 5)$.

23. A square is inscribed in a ΔABC, whose co-ordinates are $A(0, 0)$, $B(2, 1)$ and $C(3, 0)$. If two of its vertices lie on the side AC, find the vertices of the square.
24. The line joining the points $A(2, 0)$ and $B(3, 1)$ is rotated about A in the anti-clockwise direction through an angle of 15°. Find the equation of a line in the new position.
25. Find the equation of a line which passes through the point (3, 4) and makes equal intercepts on the axes.
26. Find the equation of a line which passes through the point (2, 3) and whose x-intercept is twice of y-intercept.
27. Find the equation of a line passes through the point (2, 3) so that the segment of the line intercepted between the axes is bisected at this point.
28. Find the equation of a line passing though the point (3, –4) and cutting off intercepts equal but of opposite signs from the two axes.
29. Find the equation of a line passing through the point (3, 2) and cuts off intercepts a and b on x- and y-axes such that $a - b = 2$.
30. Find the area of a triangle formed by lines $x\ y = 0$ and $2x + 3y = 6$.
31. Find the equation of a straight line passing through the point (3, 4) so that the segment of the line intercepted between the axes is divided by the point in the ratio 2 : 3.
32. Find the equation of the straight line which passes through the origin and trisect the intercept of the line $3x + 4y = 12$.
33. Find the area of a triangle formed by the lines $xy - x - y + 1 = 0$ and $3x + 4y = 12$.
34. A straight line cuts off intercepts from the axes of co-ordinates, the sum of the reciprocals of which is a constant. Show that it always passes through a fixed point.
35. The length of the perpendicular from the origin to a line is 5 and the line makes angle of 60° with the positive direction of y-axis. Find the equation of the line.
36. Find the equation of the straight line upon which the length of the perpendicular from the origin is 2 and the slope of the perpendicular is 5/12.

DISTANCE FORM OF A STRAIGHT LINE

37. A line passing through the point (3, 2) and making an angle θ with the positive direction of x-axis such that $\tan\theta = 3/4$. Find the co-ordinates of the point on the line that are 5 units away from the given point.
38. Find the co-ordinates of the points at a distance $4\sqrt{2}$ units from the point (–2, 3) in the direction making an angle of 45° with the positive direction of x-axis.
39. A point $P(3, 4)$ moves in a plane in the direction of $\hat{i} + \hat{j}$. Find the new position of P.
40. A line joining two points $A(2, 0)$ and $B(3, 1)$ is rotated about A in anti-clockwise direction through an angle of 15°. Find the new position of B.
41. Find the direction in which a straight line must be drawn through the point (1, 2) so that its point of intersection with the line $x + y = 4$ may be at a distance $\sqrt{\frac{2}{3}}$ from the point (1, 2).
42. The centre of a square is at the origin and one vertex is $P(2, 1)$. Find the co-ordinates of other vertices of the square.
43. The extremities of a diagonal of a square are (1, 1) and (–2, –1). Find the other two vertices.
44. A line through (2, 3) makes an angle $\frac{3\pi}{4}$ with the negative direction of x-axis. Find the length of the line segment cut off between (2, 3) and the line $x + y = 7$.
45. If the straight line drawn through the point $P(\sqrt{3}, 2)$ and making an angle of $\frac{\pi}{6}$ with the x-axis meets the line $\sqrt{3}x - 4y + 8 = 0$ at Q. Find the length of PQ.
46. Find the distance of the point (2, 3) from the line $2x - 3y + 9 = 0$ measured along the line $2x - 2y + 5 = 0$.
47. Find the distance of the point (3, 5) from the line $2x + 3y = 14$ measured parallel to the line $x - 2y = 1$
48. Find the distance of the point (2, 5) from the line $3x + y + 4 = 0$ measured parallel to the line $3x - 4y + 8 = 0$.
49. A line is drawn through $A(4, -1)$ parallel to the line $3x - 4y + 1 = 0$. Find the co-ordinates of the two points on this line which are at a distance of 5 units from A.

REDUCTION OF A STRAIGHT LINE

50. Reduce $x + \sqrt{3}y + 4 = 0$ into the
    (i) slope intercept form and also find its slope and y-intercept.
    (ii) intercept form and also find the lengths of x and y intercepts.
    (iii) normal form and also find the values of p and α.

PARALLEL/PERPENDICULAR FORM OF A STRAIGHT LINE

51. Find the location of the points (2, 2) and (3, 5) with respect to the line
    (i) $2x + 3y + 4 = 0$ (ii) $3x - 2y + 2 = 0$
    (iii) $x + y - 7 = 0$
52. Determine whether the point (2, –7) lies on the origin side of the line $2x + y + 2 = 0$ or not.
53. Write the parallel line to each of the following lines.
    (i) $3x - 4y + 10 = 0$ (ii) $2x + 5y + 10 = 0$
    (iii) $-x + y - 2012 = 0$ (iv) $y = x$
    (v) $x = 5$ (vi) $y = 2013$
54. Find the equation of a line parallel to $3x + 4y + 5 = 0$ and passes through (2, 3).
55. Find the equation of a line parallel to $3x - 4y + 6 = 0$ and passing through the mid-point of of the line joining the points (2, 3) and (4, –1).
56. Find the equation of a line passing through (2, 1) and parallel to the line joining the points (2, 3) and (3, –1).
57. Write the perpendicular line to each of the following lines.
    (i) $5x - 3y + 2010 = 0$ (ii) $3x + 5y + 2011 = 0$

(iii) $-x + y - 2012 = 0$ (iv) $y = 2011x$
(v) $x = 2014$ (vi) $y = 2013$

58. Find the equation of a line perpendicular to $2x + 3y - 2012 = 0$ and passing through (3, 4).
59. Find the equation of a line perpendicular to $2x - 3y - 5 = 0$ and cutting an intercept 1 on the x-axis.
60. Find the equation of the right bisectors of the line joining the points (1, 2) and (3, 5).
61. Find the equation of the altitude through A of ΔABC, whose vertices are $A(1, 2)$, $B(4, 5)$ and $C(3, 8)$.
62. Find the equation of a line passing through the point of intersection of the lines $2x + y = 8$ and $x - y = 10$ and is perpendicular to $3x + 4y + 2012 = 0$.
63. Find the distance of the point (4, 5) from the straight line $3x - 5y + 7 = 0$.
64. The equation of the base of an equilateral triangle be $x + y = 2$ and one vertex is (2, –1). Find the length of the sides of the triangle.
65. If a and b be the intercepts of a straight line on the x and y axes, respectively and p be the length of the perpendicular from the origin, prove that $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2}$.
66. Find the distance between the lines $3x - 4y = 5$ and $6x - 8y + 11 = 0$.
67. Let L has intercepts a and b on the co-ordinate axes. When the axes are rotated through an angle, keeping the origin fixed, the same line L has intercepts p and q, prove that $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2} + \frac{1}{q^2}$.

ARIA OF A PARALLELOGRAM

68. If the area of a parallelogram formed by the lines $x + 3y = a$, $3x - 2y + 3a = 0$, $x + 3y + 4a = 0$ and $3x - 2y + 7a = 0$ is ma^2, find the value of $11m + 30$.
69. Prove that the four lines $ax \pm by \pm c = 0$ enclose a rhombus, whose area is $\frac{2c^2}{|ab|}$.

FAMILY OF STRAIGHT LINES

70. Find the point of intersection of the lines $x - y + 4 = 0$ and $2x + y = 10$.
71. Prove that the three lines $2x - 3y + 5 = 0$, $3x + 4y = 7$ and $9x - 5y + 8 = 0$ are concurrent.
72. Find the equation of a line which is passing through the point of intersection of the lines $x + 3y - 8 = 0$, $2x + 3y + 5 = 0$ and (1, 2).
73. Find the value of m so that the lines $y = x + 1$, $2x + y = 16$ and $y = mx - 4$ may be concurrent.
74. If the lines $ax + y + 1 = 0$, $x + by + 1 = 0$ and $x + y + c = 0$ (where a, b, c are distinct and different from 1) are concurrent, find the value of $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c}$.
75 If $2a + 3b + 6c = 0$, the family of straight lines $ax + by + c = 0$ passes through a fixed point. Find the co-ordinates of the fixed point.
78. If $4a^2 + 9b^2 - c^2 + 12ab = 0$, the family of the straight lines $ax + by + c = 0$ is either concurrent at (m, n) or (p, q). Find the value of $m + n + p + q + 10$.
79. The family of lines $x(a + 2b) + y(a - 3b) = a - b$ passes through a fixed point for all values of a and b. Find the co-ordinates of the fixed point.
80. Find the equation of a line passing through the point of intersection of $2x + 3y + 1 = 0$, $3x - 5y - 5 = 0$ and equally inclined to the axes.
81. Find the slope of the lines which make an angle of 45° with the line $3x - y + 5 = 0$.

EQUATION OF STRAIGHT LINES PASSING THROUGH A GIVEN POINT AND MAKING A GIVEN ANGLE WITH A GIVEN LINE

82. Find the equations of the lines through the
The line makes an angle 45° with the line $x - 2y = 3$.
83. A vertex of an equilateral triangle is (2, 3) and the equation of the opposite side $x + y = 2$. Find the equation of the other sides of the triangle.
84. A line $4x + y = 1$ through the point $A(2, -7)$ meets the line BC, whose equation is $3x - 4y + 1 = 0$ at the point B. Find the equation of the line AC so that $AB = AC$.
85. Find the equations of straight lines passing through (–2, –7) and having an intercept of length 3 between the straight lines $4x + 3y = 12$ and $4x + 3y = 3$.
86. Find the equations of the lines passing through the point (2, 3) and equally inclined to the lines $3x - 4y = 7$ and $12x - 5y + 6 = 0$.
87. Two straight lines $3x + 4y = 5$ and $4x - 3y = 15$ intersect at A. Points B and C are chosen on these lines such that $AB = AC$. Determine the possible equations of the line BC passing through the point (1, 2).
88. Two equal sides of an isosceles triangle have the equations $7x - y + 3 = 0$ and $x + y = 3$ and its third side passes through the point (1, –10). Find the equation of the third side.

EQUATIONS OF BISECTORS

89. Find the equations of the bisectors of the angle between the straight lines $3x - 4y + 7 = 0$ and $12x + 5y - 2 = 0$.
90. Find the equation of the bisectors bisecting the angle containing the origin of the straight lines $4x + 3y = 6$ and $5x + 12y + 9 = 0$.
91. Find the bisector of the angle between the lines $2x + y = 6$ and $2x - 4y + 7 = 0$, which contains the point (1, 2).
92. Find the equation of the bisector of the obtuse angle between the lines $3x - 4y + 7 = 0$ and $12x + 5y = 2$.
93. Find the bisector of the acute angle between the lines $x + y = 3$ and $7x - y + 5 = 0$.
94. Prove that the length of the perpendiculars drawn from any point of the line $7x - 9y + 10 = 0$ to the lines $3x + 4y = 5$ and $12x + 5y = 7$ are the same.
95. Find the co-ordinates of the incentre of the triangle whose sides are $x + 1 = 0$, $3x - 4y = 5$, $5x + 12y = 27$.
96. The bisectors of the angle between the lines $y = \sqrt{3}x + 3$ and $\sqrt{3}y = x + 3\sqrt{3}$ meet the x-axis respectively, at P and Q. Find the length of PQ.

97. Two opposite sides of a rhombus are $x + y = 1$ and $x + y = 5$. If one vertex is $(2, -1)$ and the angle at that vertex be 45°. Find the vertex opposite to the given vertex.
98. Find the foot of perpendicular drawn from the point (2, 3) to the line $y = 3x + 4$.

IMAGE OF A POINT WITH RESPECT TO A STRAIGHT LINE

99. Find the image of the point (–8, 12) with respect to the line mirror $4x + 7y + 13 = 0$.
100. Find the image of the point (3, 4) with respect to the line $y = x$.
101. If (–2, 6) be the image of the point (4, 2) with respect to the line $L = 0$, find the equation of the line L.
102. The image of the point (4, 1) with respect to the line $y = x$ is P. If the point P is translated about the line $x = 2$, the new position of P is Q. Find the co-ordinates of Q.
103. The image of the point (3, 2) with respect to the line $x = 4$ is P. If P is rotated through an angle $\frac{\pi}{4}$ about the origin in anti-clockwise direction. Find the new position of P.
104. The equations of the perpendicular bisectors of the sides AB and AC of ΔABC are $x - y + 5 = 0$ and $x + 2y = 0$, respectively. If the point A is $(1, -2)$, find the equation of the line BC.

REFLECTION OF A STRAIGHT LINE

105. A ray of light is sent along the line $x - 2y = 3$. Upon reaching the line $3x - 2y = 5$, the ray is reflected from it. Find the equation of the line containing the reflected ray.
106. A ray of light passing through the point (1, 2) is reflected on the x-axis at a point P and passes through the point (5, 3). Find the abscissa of the point P.
107. A ray of light is travelling along the line $x = 1$ and gets reflected from the line $x + y = 1$, find the equation of the line which the reflected ray travel.
108. A ray of light is sent along the line $x - 6y = 8$. After refracting across the line $x + y = 1$, it enters the opposite side after turning by 15° away from the line $x + y = 1$. Find the equation of the line along with the reflected ray travels.

Level II

(Mixed Problems)

1. If $P(1, 2)$, $Q(4, 6)$, $R(5, 7)$ and $S(a, b)$ are the vertices of a parallelogram $PQRS$, then
   (a) $a = 2, b = 4$ (b) $a = 3, b = 4$
   (c) $a = 2, b = 3$ (d) $a = 3, b = 5$
2. The extremities of the diagonal of a parallelogram are the points (3, –4) and (–6, 5). Third vertex is the point (–2, 1), the fourth vertex is
   (a) (1, 1) (b) (1, 0)
   (c) (0, 1) (d) (–1, 0)
3. The centroid of a triangle is (2, 3) and two of its vertices are (5, 6) and (–1, 4). Then the third vertex of the triangle is
   (a) (2, 1) (b) (2, –1)
   (c) (1, 2) (d) (1, –2)
4. If a and b are real numbers between 0 and 1 such that the points $(a, 1)$, $(1, b)$ and $(0, 0)$ form an equilateral triangle, then a, b are
   (a) $2-\sqrt{3}, 2-\sqrt{3}$ (b) $\sqrt{3}-1, \sqrt{3}-1$
   (c) $\sqrt{2}-1, \sqrt{2}-1$ (d) None
5. If O be the origin and $Q_1(x_1, y_1)$ and $Q_2(x_2, y_2)$ be two points, then $OQ_1 OQ_2 \cos(\angle Q_1OQ_2)$ is
   (a) $x_1y_2 + x_2y_1$ (b) $(x_1^2 + y_1^2)(x_2^2 + y_2^2)$
   (c) $(x_1 - x_2) + (y_1 - y_2)$ (d) $x_1x_2 + y_1y_2$
6. If the sides of a triangle are $3x + 4y$, $4x + 3$ and $5x + 5y$ units, where $x > 0, y > 0$, the triangle is
   (a) right angled (b) acute angled
   (c) obtuse angled (d) isosceles
7. A triangle is formed by the co-ordinates (0, 0), (0, 21) and (21, 0). Find the number of integral co-ordinates strictly inside the triangle (integral co-ordinates of both x and y)
   (a) 190 (b) 105 (c) 231 (d) 205
8. The set of all real numbers x, such that $x^2 + 2x$, $2x + 3$ and $x^2 + 3x + 8$ are the sides of a triangle, is
   (a) $x \geq 4$ (b) $x \geq 5$ (c) $x \leq 5$ (d) $x \leq 4$
9. The area of a triangle with vertices at the points $(a, b + c)$, $(b, c + a)$ and $(c, a + b)$ is
   (a) 0 (b) $a + b + c$
   (c) $ab + bc + ca$ (d) none
10. If the vertices of a triangle ABC are $(\lambda, 2 - 2\lambda)$, $(-\lambda + 1, 2\lambda)$ and $(-4 - \lambda, 6 - 2\lambda)$. If its area be 70 sq. units, the number of integral values of λ is
    (a) 1 (b) 2 (c) 3 (d) 4
11. If the co-ordinates of points A, B, C and D are (6, 3), (–3, 5), (4, –2) and $(x, 3x)$, respectively and if $\frac{\Delta ABC}{\Delta DBC} = \frac{1}{2}$, then x is
    (a) 8/11 (b) 11/8 (c) 7/9 (d) 0
12. The area of a triangle is 5 and two of its vertices are $A(2, 1)$, $B(3, -2)$. The third vertex which lies on the line $y = x + 3$ is
    (a) $\left(\frac{7}{2}, \frac{13}{2}\right)$ (b) $\left(\frac{5}{2}, \frac{5}{2}\right)$
    (c) $\left(-\frac{3}{2}, \frac{3}{2}\right)$ (d) (0, 0)
13. If the points $(2k, k)$, $(k, 2k)$ and (k, k) with $k > 0$ encloses a triangle of area 18 sq. units, the centroid of the triangle is equal to
    (a) (8, 8) (b) (4, 4)
    (c) (–4, –4) (d) $(4\sqrt{2}, 4\sqrt{3})$
14. If r be the geometric mean of p and q, the line $px + qy + r = 0$
    (a) has a fixed direction
    (b) passes through a fixed point
    (c) forms with the axes of a triangle of
    (d) sum of its intercepts on the axes Constant area is constant.
15. A line passing through the point (2, 2) cuts the axes of co-ordinates at A and B such that area $OAB = k$ $(k > 0)$. The intercepts on the axes are the roots of the equation

(a) $x^2 - kx + 2k = 0$ (b) $x^2 - 2kx + k = 0$
(c) $x^2 + kx + 2k = 0$ (d) $x^2 + kx + k = 0$

16. If A and B be two points on the line $3x + 4y + 15 = 0$ such that $OA = OB = 9$ units, the area of the triangle OAB is
(a) $9\sqrt{2}$ (b) $18\sqrt{2}$ (c) $12\sqrt{2}$ (d) None.

17. The line segment joining the points (1, 2) and (–2, 1) is divided by the line $3x + 4y = 7$ in the ratio
(a) 3 : 4 (b) 4 : 3 (c) 9 : 4 (d) 4 : 9

18. If a straight line passes through (x_1, y_1) and its segment between the axes is bisected at this point, its equation is given by
(a) $\frac{x}{x_1} + \frac{y}{y_1} = 2$ (b) $2(xy_1 + x_1y) = x_1y_1$
(c) $xy_1 + x_1y = x_1y_1$ (d) None.

19. A straight line through the point $P(3, 4)$ is such that its intercept between the axes is bisected at P. Its equation is
(a) $3x - 4y + 7 = 0$ (b) $4x + 3y = 24$
(c) $3x + 4y = 25$ (d) $x + y = 7$

20. If the lines $4x + 3y = 1$, $y = x + 5$ and $bx + 5y = 3$ are concurrent, then b is
(a) 1 (b) 3 (c) 6 (d) 0

21. The lines $ax + by + c = 0$, $bx + cy + a = 0$ and $cx + ay + b = 0$ are concurrent if
(a) $\Sigma a^3 = 3abc$ (b) $\Sigma a^3 = 0$
(c) $\Sigma a^2 = \Sigma ab$ (d) None

22. The lines $ax + 2y + 1 = 0$, $bx + 3y + 1 = 0$ and $cx + 4y + 1 = 0$ are concurrent if a, b, c are in
(a) AP (b) GP (c) HP (d) none

23. If the lines $ax + y + 1 = 0$, $x + by + 1 = 0$ and $x + y + c = 0$ (a, b, c are distinct and not equal to 1) are concurrent, the value of $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c}$ is
(a) 0 (b) 1 (c) 2 (d) none

24. The points $(-a, -b)$, (0, 0), (a, b) and (a^2, ab) are
(a) collinear (b) vertices of a rectangle
(c) vertices of a parallelogram (d) none

25. If $25p^2 + 9q^2 - r^2 - 30pq = 0$, a point on the line $px + qy + r = 0$ is
(a) (–5, 3) (b) (1, 2) (c) (0, 0) (d) (5, 3)

26. The set of lines $ax + by + c = 0$ where $3a + 2b + 4c = 0$ are concurrent at the point
(a) (3, 2) (b) (2, 4) (c) (3/4, ½) (d) None.

27. If a, b, c are in AP, the straight line $a + by + c = 0$ will always pass through the point
(a) (1, 1) (b) (2, 2) (c) (–2, 1) (d) (1, –2)

28. The equation of the line which passes through the point (–3, 8) and cuts off positive intercepts on the axes whose sum is 7, is
(a) $3x - 4y = 12$ (b) $4x + 3y = 12$
(c) $3x + 4y = 12$ (d) $4x - 3y = 12$

29. If a pair of opposite vertices of parallelogram are (1, 3) and (–2, 4) and the sides are parallel to $5x - y = 0$ and $7x + y = 0$, the equation of a side through (1, 3) is
(a) $5x - y = 2$ (b) $7x + y = 10$
(c) $5x - y + 14 = 0$ (d) $7x + y + 10 = 0$

30. The point $P(a, b)$ and $Q(b, a)$ lie on the lines $3x + 2y - 13 = 0$ and $4x - y - 5 = 0$. The equation of the line PQ is
(a) $x - y = 5$ (b) $x + y = 5$
(c) $x - y = -5$ (d) $x + y = -5$

31. The sides AB, BC, CD and DA of a quadrilateral are $x + 2y = 3$, $x = 1$, $x - 3y = 4$, $5x + y + 12 = 0$ respectively. The angle between diagonals AC and BD is
(a) 45° (b) 60° (c) 90° (d) 30°

32. Let PS be the median of the triangle with vertices $P(2, 2)$, $Q(6, -1)$ and R (7, 3). The equation of the line passing through (1, –1) and parallel to PS is
(a) $2x - 9y - 7 = 0$ (b) $2x - 9y - 11 = 0$
(c) $2x + 9y - 11 = 0$ (d) $2x + 9y + 7 = 0$

33. The equation of the base of an equilateral triangle is $x + y = 2$ and the vertex is (2, –1). The length of its side is
(a) $\sqrt{\frac{1}{2}}$ (b) $\sqrt{\frac{3}{2}}$ (c) $\sqrt{\frac{2}{3}}$ (d) $\sqrt{2}$

34. The distance between the lines $4x + 3y = 11$ and $8x + 6y = 15$ is
(a) 7/2 (b) 7/3 (c) 7/5 (d) 7/10

35. A variable point $\left(1 + \frac{\lambda}{\sqrt{2}}, 2 + \frac{\lambda}{\sqrt{2}}\right)$ lies in between two parallel lines $x + 2y = 1$ and $2x + 4y = 15$, the range of λ is given by
(a) $0 < \lambda < \frac{5\sqrt{2}}{6}$ (b) $-\frac{4\sqrt{2}}{5} < \lambda < \frac{5\sqrt{2}}{6}$
(c) $-\frac{4\sqrt{2}}{5} < \lambda < 0$ (d) none

36. The sum of the abcissae of all the points on the line $x + y = 4$ that lie at a unit distance from the line $4x + 3y - 10 = 0$ is
(a) – 4 (b) –3 (c) 3 (d) 4

37. If the algebraic sum of the perpendicular distances from the points (2, 0), (0, 2), (1,1) to a variable straight line be zero, the line passes through a fixed point whose co-ordinates are
(a) (1, 1) (b) (2, 2) (c) (0, 0) (d) None

38. If a, b and c are related by $4a^2 + 9b^2 - 9c^2 + 12ab = 0$, the greatest distance between any two lines of the family of lines $ax + by + c = 0$ is
(a) 4/3 (b) $\frac{2}{3} \times \sqrt{13}$
(c) $3\sqrt{13}$ (d) 0

39. If the axes are turned through an angle $\tan^{-1} 2$, the equation $4xy - 3x^2 = a^2$ becomes
(a) $x^2 - 4y^2 = 2a^2$ (b) $x^2 - 4y^2 = a^2$
(c) $x^2 + 4y^2 = a^2$ (d) $x^2 - 2xy^2 = a^2$

40. The number of integral values of m for which the x co-ordinate of the point of intersection of the lines $3x + 4y = 9$ and $y = mx + 1$ is also an integer, is
(a) 2 (b) 0 (c) 4 (d) 1

41. Given the family of lines
$a(2x + y + 4) + b(x - 2y - 3) = 0$.
The number of lines belonging to the family at a distance $\sqrt{10}$ from any point (2, –3) is
(a) 0 (b) 1 (c) 2 (d) 4

42. If $\frac{x}{c}+\frac{y}{d}=1$ be any line through the intersection of $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{b}+\frac{y}{a}=1$, then
(a) $\frac{1}{c}+\frac{1}{d}=\frac{1}{a}+\frac{1}{b}$ (b) $\frac{1}{a}+\frac{1}{d}=\frac{1}{c}+\frac{1}{b}$
(c) $\frac{1}{b}+\frac{1}{d}=\frac{1}{a}+\frac{1}{c}$ (d) None.

43. The point of intersection of the lines $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{b}+\frac{y}{a}=1$ lies on the line
(a) $x-y=0$
(b) $(x+y)(a+b)=2ab$
(c) $(px+qy)(a+b)=(p+q)ab$
(d) $(px-qy)(a-b)=(p-q)ab$

44. The equation of the right bisector of the line segment joining the points (7, 4) and (–1, –2) is
(a) $4x+3y=10$ (b) $3x-4y+7=0$
(c) $4x+3y=15$ (d) none

45. The points (1, 3) and (5, 1) are two opposite vertices of a rectangle. The other two vertices lie on the line $y=2x+c$, the other vertices and c are
(a) (1, 1), (2, 3), $c=4$ (b) (4, 4), (2, 0), $c=-4$
(c) (0, 0), (5, 4), $c=3$ (d) none

46. The four lines $ax\pm by\pm c=0$ enclose a
(a) square
(b) parallelogram
(c) rectangle
(d) rhombus of area $\frac{2c^2}{ab}$

47. The area bounded by the curves $y=|x|-1$ and $y=-|x|+1$ is
(a) 1 (b) 2 (c) $2\sqrt{2}$ (d) 4

48. The area of the parallelogram formed by the lines $y=mx$, $y=mx+1$, $y=nx$ and $y=nx+1$ is
(a) $\frac{|m+n|}{(m-n)^2}$ (b) $\frac{2}{|m+n|}$
(c) $\frac{1}{|m+n|}$ (d) $\frac{1}{|m-n|}$

49. The line which is parallel to x-axis and crosses the curve $y=\sqrt{x}$ at an angle of 45° is
(a) $x=¼$ (b) $y=¼$ (c) $y=½$ (d) $y=1$

50. The reflection of the point (4, –13) in the line $5x+y+6=0$ is
(a) (–1, –14) (b) (3, 4)
(c) (1, 2) (d) (–4, 13)

51. The area enclosed within the curve $|x|+|y|=1$ is
(a) 4 (b 2 (c) 1 (d) 3

52. The incentre of the triangle formed by the lines $x=0$, $y=0$ and $3x+4y=12$ is
(a) $\left(\frac{1}{2},\frac{1}{2}\right)$ (b) (1, 1) (c) $\left(1,\frac{1}{2}\right)$ (d) $\left(\frac{1}{2},1\right)$

53. The incentre of the triangle formed by the axes and the line $\frac{x}{a}+\frac{y}{b}=1$ is
(a) $\left(\frac{a}{2},\frac{b}{2}\right)$ (b) $\left(\frac{a}{3},\frac{b}{3}\right)$
(c) $\left(\frac{ab}{a+b+\sqrt{a^2+b^2}},\frac{ab}{a+b+\sqrt{a^2+b^2}}\right)$
(d) $\left(\frac{ab}{a+b+\sqrt{ab}},\frac{ab}{a+b+\sqrt{ab}}\right)$

54. The orthocentre of a triangle whose vertices are (0, 0), (3, 4), (4, 0) is
(a) $\left(3,\frac{7}{3}\right)$ (b) $\left(3,\frac{5}{4}\right)$
(c) (5, –2) (d) $\left(3,\frac{3}{4}\right)$

55. The orthocentre of the triangle formed by the lines $xy=0$ and $x+y=1$ is
(a) $\left(\frac{1}{2},\frac{1}{2}\right)$ (b) $\left(\frac{1}{3},\frac{1}{3}\right)$
(c) (0, 0) (d) $\left(\frac{1}{4},\frac{1}{4}\right)$

56. The mid-points of the sides of a triangle are (5, 0), (5, 12) and (0, 12). The orthocentre of the triangle is
(a) (0, 0) (b) (10, 0)
(c) (0, 24) (d) $\left(\frac{13}{3},8\right)$

57. One side of an equilateral triangle is the line $3x+4y+8=0$ and its centroid is at $O(1, 1)$. The length of its side is
(a) 2 (b) $\sqrt{5}$ (c) $6\sqrt{3}$ (d) $\sqrt{7}$

58. The incentre of the triangle with vertices $(1,\sqrt{3})$, (0, 0) and (2, 0) is
(a) $\left(1,\frac{\sqrt{3}}{2}\right)$ (b) $\left(\frac{2}{3},\frac{1}{\sqrt{3}}\right)$
(c) $\left(\frac{2}{3},\frac{\sqrt{3}}{2}\right)$ (d) $\left(1,\frac{1}{\sqrt{3}}\right)$

59. Let $P(-1, 0)$, $Q(0, 0)$ and $R(3, 3\sqrt{3})$ be three points. Then the equation of the bisector of the $\angle PQR$ is
(a) $\frac{\sqrt{3}}{2}x+y=0$ (b) $x+\sqrt{3}y=0$
(c) $y+x\sqrt{3}=0$ (d) $x+\frac{\sqrt{3}}{2}y=0$

60. The vertices of a triangle ABC are (1, 1), (4, –2) and (5, 5), respectively. The equation of the perpendicular dropped from C to the internal bisector of $\angle A$ is
(a) $y=5$ (b) $x=5$
(c) $2x+3y=7$ (d) none

61. The vertices of a triangle are $A(-1, -7)$, $B(5, 1)$ and $C(1, 4)$. The equation of the internal bisector of the $\angle ABC$ is

(a) $3x - 7y = 8$ (b) $x - 7y + 2 = 0$
(c) $3x - 3y - 7 = 0$ (d) none

62. The bisector of the acute angle formed between the lines $4x - 3y + 7 = 0$ and $4x - 4y + 14 = 0$ has the equation
(a) $x + y = 7$ (b) $x - y + 3 = 0$
(c) $2x + y = 11$ (d) $x + 2y = 12$

63. The opposite angular points of a square are (3, 4) and (1, –1), the other two vertices are
(a) $\left(-\frac{1}{2}, \frac{5}{2}\right), \left(\frac{9}{2}, \frac{1}{2}\right)$ (b) $\left(\frac{9}{2}, \frac{1}{2}\right), \left(\frac{7}{2}, \frac{1}{2}\right)$
(c) $\left(\frac{7}{2}, \frac{1}{2}\right), \left(-\frac{7}{2}, -\frac{1}{2}\right)$ (d) $\left(-\frac{7}{2}, -\frac{1}{2}\right), \left(-\frac{7}{2}, -\frac{9}{2}\right)$

64. A line through $A(-5, -4)$ meets the lines $x + 3y + 2 = 0$, $2x + y + 4 = 0$ and $x - y - 5 = 0$ at B, C and D, respectively. If $\left(\frac{15}{AB}\right)^2 + \left(\frac{10}{AC}\right)^2 = \left(\frac{6}{AD}\right)^2$, the equation of the line is
(a) $2x + 3y + 22 = 0$ (b) $5x - 4y + 7 = 0$
(c) $3x - 2y + 3 = 0$ (d) none

65. The equation of the lines through the point (2, 3) and making an intercept of length 2 units between the lines $2x + y = 3$ and $2x + y = 5$ are
(a) $x + 3 = 0$, $3x + 4y = 12$
(b) $y - 2 = 0$, $4x - 3y = 6$
(c) $x - 2 = 0$, $3x + 4y = 18$
(d) none

66. A line is such that its segment between the straight lines $5x - y = 4$ and $3x + 4y = 4$ is bisected at the point (1, 5). Its equation is
(a) $23x - 7y + 6 = 0$ (b) $7x + 4y + 3 = 0$
(c) $83x - 35y + 92 = 0$ (d) None

67. A straight line through the origin O meets the parallel lines $4x + 2y = 9$ and $2x + y + 6 = 0$ at points P and Q, respectively. Then the point O divides the segment PQ in the ratio
(a) 1 : 2 (b) 3 : 4 (c) 2 : 1 (d) 4 : 3

Level III

(Problems for JEE Advanced)

1. Derive the conditions to be imposed on β so that $(0, \beta)$ should lie on or inside the triangle having sides $y + 3x + 2 = 0$, $3y - 2x - 5 = 0$ and $4y + x - 14 = 0$
2. Find the number of integral values of m, for which the x-co-ordinate of the point of intersection of the lines $3x + 4y = 9$ and $y = mx + 1$ is also an integer.
3. Let $P = (-1, 0)$, $Q = (0, 0)$ and $R = (3, 3\sqrt{3})$ be three points. Find the equation of the bisector of the $\angle PQR$.
4. A straight line through the point (2, 2) intersects the lines $\sqrt{3}x + y = 0$ and $\sqrt{3}x - y = 0$ at the points A and B. Find the equation to the line AB so that the ΔOAB is equilateral.
5. The area of the triangle formed by the intersection of a line parallel to x-axis and passing through $P(h, k)$ with the lines $y = x$ and $x + y = 2$ is $4h^2$. Find the locus of the point P.
6. Two rays in the first quadrant $x + y = |a|$ and $ax - y = 1$ intersects each other in the interval $a \in (a_0, \infty)$. Find the value of a_0.
7. A variable line is at constant distance p from the origin and meets the co-ordinate axes in A and B. Show that the locus of the centroid of the ΔOAB is $\frac{1}{x^2} + \frac{1}{y^2} = \frac{9}{p^2}$.
8. The line segment joining $A(3, 0)$ and $B(5, 2)$ is rotated about A in the anti-clockwise direction through an angle of 45° so that B goes to C. If $D(x, y)$ is the image of C with respect to y-axis, find the value of $x + y + 7$.
9. Find the equation of the line passing through the point (4, 5) and equally inclined to the lines $3x - 4y = 7$ and $5y - 12x = 6$.
10. If $A(3, 0)$ and $C(-2, 5)$ be the opposite vertices of a square, find the co-ordinates of remaining two vertices.
11. For what values of the parameter m does the point $P(m, m + 1)$ lie within the ΔABC, the vertices of which are $A(0, 3)$, $B(-2, 0)$ and $C(6,1)$?
12. A straight line passes through the point (h, k) and this point bisects the part of the intercept between the axes. Show that the equation of the straight line is
$$\frac{x}{2h} + \frac{y}{2k} = 1$$
13. Find the values of the parameter m for which the points (0, 0) and $(m, 3)$ lie on the opposite lines $3x + 2y - 6 = 0$ and $x - 4y + 16 = 0$.
14. If $A(0, 3)$ and $B(-2, 5)$ be the adjacent vertices of a square. Find the possible co-ordinates of remaining two vertices.
15. Find the co-ordinates of two points on the line $x + y = 3$ which are situated at a distance $\sqrt{8}$ from the point (2, 1) on the line.
16. If a and b be variables, show that the lines $(a + b)x + (2a - b)y = 0$ pass through a fixed point.
17. Determine all values of α for which the point (α, α^2) lies inside the triangle formed by the lines $2x + 3y - 1 = 0$, $x + 2y - 3 = 0$ and $5x - 6y - 1 = 0$.
18. The points (1, 3) and (5, 1) are two opposite vertices of a rectangle. The other two vertices lie on the line $y = 2x + c$. Find c and the remaining two vertices.
19. A vertex of an equilateral triangle is (2, 3) and the equation of the opposite side is $x + y = 2$. Find the equation of the other two sides.
20. Two consecutive sides of a parallelogram are $4x + 5y = 0$ and $7x + 2y = 0$. If the equation to one diagonal is $11x + 7y = 9$, find the equation of the other diagonal.
21. A ray of light coming from the point (1, 2) is reflected at a point A on the x-axis and then passes through the point (5, 3). Find the point A.

22. A man starts from the point $P(-3, 4)$ and reaches the point $Q(0, 1)$ after touching the line $2x + y = 7$ at R. Find R on the line so that he travels along the shortest path.
23. A ray of light is sent along the straight line $y = \frac{2}{3}x - 4$. On reaching the x-axis, it is reflected. Find the point of incidence and the equation of the reflected ray.
24. If the point (a, a) is placed in between the lines $|x + y| = 4$, find a.
25. The equations of two sides of a triangle are $3x - 2y + 6 = 0$ and $4x + 5y = 20$ and the orthocentre is (1, 1). Find the equation of the third side.
26. Two vertices of a triangle are (4, –3) and (–2, 5) . If the orthocentre of the triangle is (1, 2), prove that the third vertex is (33, 26).
27. The equations of the perpendicular bisectors of the sides AB and AC of a ΔABC are $x - y + 2 = 0$ and $x + 2y = 0$, respectively. If the point A is $(1, -2)$, find the equation of the line BC.
28. A line $4x + y = 1$ through the point $A(2, -7)$ meets the line BC whose equation is $3x - 4y + 1 = 0$ at the point B. Find the equation of the line AC, so that $AB = AC$.
29. Find the equations of the straight lines passing through (–2, –7) and having an intercept of length 3 between the straight lines $4x + 3y = 12$ and $8x + 6y = 6$.
30. If $A(1, p^2)$, $B(0, 1)$, $C(p, 0)$ are the co-ordinates of three points, find the value of p for which the area of the ΔABC is minimum.
31. The straight lines $3x + 4y = 5$ and $4x - 3y = 15$ intersect at A. Points B and C are chosen on these lines, such that $AB = AC$. Determine the possible equations of the line BC passing through the point (1, 2).
32. The centre of a square is at the origin and one vertex is $A(2, 1)$. Find the co-ordinates of other vertices of the square.
33. Two equal sides of an isosceles triangle are $7x - y + 3 = 0$ and $x + y - 3 = 0$ and its third side passes through the point (1, –10). Find the equation of the third side.
34. Two opposite sides of a rhombus are $x + y = 1$ and $x + y = 5$. If one vertex is (2, –1) and the angle at the vertex is 45°. Find the vertex opposite to the given vertex.
35. Two sides of a rhombus $ABCD$ are parallel to the lines $y = x + 2$ and $y = 7x + 3$. If the diagonals of the rhombus intersect at the point (1, 2) and the vertex A is on y-axis. Find the possible co-ordinates of A.

Level IV

(Tougher Problems for JEE Advanced)

1. The equation of two sides of a parallelogram are $3x - 2y + 12 = 0$ and $x - 3y + 11 = 0$ and the point of intersection of its diagonals is (2, 2). Find the equations of other two sides and its diagonal.
2 Let the point B is symmetric to A (4, –1) with respect to the bisector of the first quadrant. Find AB.
3 A line segment AB through the point $A(2, 0)$ which makes an angle of 30° with the positive direction of x-axis is rotated about A in anti-clockwise direction through an angle of 15°. If C be the new position of the point $B(2 + \sqrt{3}, 1)$, find the co-ordinates of C.
4. The point (1, –2) is reflected in the x-axis and then translated parallel to the positive direction of x-axis through a distance of 3 units, find the co-ordinates of the point in the new position.
5. A line through the point $A(2, 0)$ which makes an angle of 30° with the positive direction of x-axis is rotated about A in anti-clockwise direction through an angle of 15°. Find the equation of the straight line in the new position.
6. A line $x - y + 1 = 0$ cuts the y-axis at A. This line is rotated about A in the clockwise direction through 75°. Find the equation of the line in the new position.
7. The point (1, 1) is translated parallel to the line $y = 2x$ in the first quadrant through a unit distance. Find the new position of the point.
8. Two particles start from the same point (2, –1), one moving 2 units along the line $x + y = 1$ and the other 5 units along the line $x - 2y = 4$. If the particles move towards increasing y, find their new positions and the distance between them.
9. If a line AB of length $2l$ moves with the end A always on the x-axis and the end B always on the line $y = 6x$. Find the equation of the locus of the mid-point of AB.
10 The opposite angular points of a square are (3, 4) and (1, –1). Find the co-ordinates of the other two vertices. **[Roorkee, 1985]**
11. Two vertices of a triangle are (4, –3) and (–2, 5). If the orthocentre of the triangle is at (1, 2). Find the co-ordinates of the third vertex. **[Roorkee, 1987]**
12. A line is such that its segment between the straight lines $5x - y - 4 = 0$ and $3x + 4y - 4 = 0$ is bisected at the point (1, 5). Obtain its equation. **[Roorkee, 1988]**
13. The extremities of the diagonal of a square are (1, 1) and (–2, –1). Obtain the other two vertices and the equation of the other diagonal. **[Roorkee, 1989]**
14 A variable straight line passes through the point of intersection of the lines $x + 2y = 1$ and $2x - y = 1$ and meets the co-ordinate axes in A and B. Find the locus of the mid-point of AB. **[Roorkee, 1989]**
15. A variable line, drawn through the point of intersection of the straight lines $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{b} + \frac{y}{a} = 1$, meets the co-ordinates axes in A and B. Find the locus of the mid-point of AB. **[Roorkee, 1989]**
16 A line joining $A(2, 0)$ and $B(3, 1)$ is rotated about A in anti-clockwise direction through. If B goes to C in the new position, what will be the co-ordinates of C? **[Roorkee, 1989]**

17. Which pair of points lie on the same side of $3x - 8y - 7 = 0$?
    (a) (0, –1) and (0, 0) (b) (4, –3) and (0, 1)
    (c) (–3, –4) and (1, 2) (d) (–1, –1) and (3, 7)
    [Roorkee, 1990]
18 Determine the conditions to be imposed on β so that $(0, \beta)$ should lie on or inside the triangle having sides $y + 3x + 2 = 0$, $3y - 2x - 5 = 0$ and $4y + x - 14 = 0$. **[Roorkee Main, 1990]**
19. A ray of light is sent along the line $x - 2y - 3 = 0$. Upon reaching the line $3x - 2y - 5 = 0$, the ray is reflected from it. Find the equation of the line containing the reflected ray. **[Roorkee Main, 1990]**
20. A line parallel to the straight line $3x - 4y - 2 = 0$ and at a distance 4 units from it is
    (a) $3x - 4y + 30 = 0$ (b) $4x - 3y + 12 = 0$
    (c) $3x - 4y + 18 = 0$ (d) $3x - 4y + 22 = 0$
    [Roorkee, 1991]
21. The equation of a straight line passing through (–5, 4) which cuts an intercept of $\sqrt{2}$ between the lines $x + y + 1 = 0$ and $x + y - 1 = 0$ is
    (a) $x - 2y + 13 = 0$ (b) $2x - y + 14 = 0$
    (c) $x - y + 9 = 0$ (d) $x - y + 10 = 0$
    [Roorkee, 1991]
22. $P(3, 1)$, $Q(6, 5)$ and $R(x, y)$ are three points such that the $\angle PRQ$ is a right angle and the area of $\Delta RPQ = 7$, the number of such points R is
    (a) 0 (b) 1 (c) 2 (d) infinite
    [Roorkee, 1992]
23. P is a point on either of the two lines $y - \sqrt{3}|x| = 2$ at a distance of 5 units from their point of intersection, the co-ordinates of the foot of the perpendicular from P on the bisector of the angle between them are
    (a) $\left(0, \frac{4+5\sqrt{3}}{2}\right)$ or $\left(0, \frac{4-5\sqrt{3}}{2}\right)$ depending on which line P is taken
    (b) $\left(0, \frac{4+5\sqrt{3}}{2}\right)$
    (c) $\left(0, \frac{4-5\sqrt{3}}{2}\right)$
    (d) $\left(\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)$ **[Roorkee, 1992]**
24. If one of the diagonals of the square is along the line $x = 2y$ and one of its vertices is $A(3, 0)$, its sides through the vertex A are given by
    (a) $y + 3x + 9 = 0$; $3y + x - 3 = 0$
    (b) $y - 3x + 9 = 0$; $3y + x - 3 = 0$
    (c) $y - 3x + 9 = 0$; $3y - x + 3 = 0$
    (d) $y - 3x + 3 = 0$; $3y + x + 9 = 0$ **[Roorkee, 1993]**
25 The sides AB, BC, CD and DA of a quadrilateral have the equations $x + 2y = 3$, $x = 1$, $x - 3y = 4$ and $5x + y + 12 = 0$ respectively. Find the angle between the diagonals AC and BD. **[Roorkee Main, 1993]**
26. Given vertices $A(1, 1)$, $B(4, -2)$ and $C(5, 5)$ of a triangle, find the equation of the perpendicular dropped from C to the interior bisector of the $\angle A$. **[Roorkee Main,1994]**
27. The co-ordinates of the foot of the perpendicular from the point (2, 4) on the line $x + y = 1$ are
    (a) (1/2, 3/2) (b) (–1/2, 3/2)
    (c) (4/3, ½) (d) (3/4, –1/2)
    [Roorkee, 1995]
28. All points lying outside the triangle formed by the points (1, 3), (5, 0) and (–1, 2) satisfy
    (a) $2x + y \geq 0$ (b) $2x + y - 13 \geq 0$
    (c) $2x + y - 12 \leq 0$ (d) $-2x + y \leq 0$.
    [Roorkee, 1995]
29. In a ΔABC, the equation of the perpendicular bisector of AC is $3x - 2y + 8 = 0$. If the co-ordinates of the points A and B are (1, –1) and (3, 1), respectively, find the equation of the line BC and the centre of the circumcircle of the ΔABC. **[Roorkee Main, 1995]**
30. Two sides of a rhombus, lying in the first quadrant are given by $3x - 4y = 0$ and $12x - 5y = 0$. If the length of the longer diagonal is 12, find the equations of the other two sides of the rhombus. **[Roorkee Main, 1996]**
31. What is the equation of a straight line equally inclined to the axes and equidistant from the points (1, –2) and (3, 4)? **[Roorkee, 1997]**
32. If the points $(2a, a)$, $(a, 2a)$ and (a, a) enclose a triangle of area 8, find the value of a.
    (a) $x - y - 2 = 0$ (b) $x + y - 2 = 0$
    (c) $x - y - 1 = 0$ (d) $x + y - 1 = 0$
    [Roorkee, 1997]
33. One diagonal of a square $7x + 5y = 35$ intercepted by the axes. Obtain the extremities of the other diagonal. **[Roorkee Main,1997]**
34. The equations of two equal sides AB and AC of an isosceles triangle ABC are $x + y = 5$ and $7x - y = 3$, respectively. Find the equations of the side BC if the area of the ΔABC is 5 sq. units. **[Roorkee, 1999]**
35. Find the position of the point (4, 1) after it undergoes the following transformations successively:
    (i) reflection about the line $y = x - 1$
    (ii) translation by one unit along x-axis in the positive direction.
    (iii) rotation through an angle $\frac{\pi}{4}$ about the origin in the anti-clockwise direction.
    [Roorkee Main, 2000]
36 Two vertices of a triangle are at (–1, 3) and (2, 5) and its orthocentre is at (1, 2). Find the co-ordinates of the third vertex.
    [Roorkee Main, 2001]

Integer Type Questions

1. Let the algebraic sum of the perpendicular distances from the points (3, 0), (0, 3) and (2, 2) to a variable straight line be zero, the line passing through a fixed point whose co-ordinates are (p, q), where $3(p+q)-2$ is....
2. If the distance of the point (2, 3) from the line $2x-3y+9=0$ measured along the line $2x-2y+5=0$ is $d\sqrt{2}$, find $(d+2)$.
3. Find the number of possible straight lines passing through (2, 3) and forming a triangle with co-ordinate axes, whose area is 12 sq. units.
4. Find the number of integral values of m for which the x-co-ordinate of the point of intersection of the lines $3x+4y=9$ and $y=mx+2$ is also an integer.
5. Find the area of the parallelogram formed by the lines $y=2x$, $y=2x+1$, $y=x$ and $y=x+1$.
6. If one side of a rhombus has end-points (4, 5) and (1, 1) such that the maximum area of the rhombus is $5m$ sq. units, find m.
7. Find the area of a rhombus enclosed by the lines $x \pm 2y \pm 2 = 0$.
8. $P(x, y)$ be a lattice point if $x, y \in N$. If the number of lattices points lies inside of a triangle form by the line $x+y=10$ and the co-ordinate axes is $m(m+5)$, find m.
9. $P(x, y)$ be an IIT point if $x, y \in I^+$. Find the number of IIT points lying inside the quadrilateral formed by the lines $2x+y=6$, $x+y=9$, $x=0$ and $y=0$.
10. If the point $P(a^2, a)$ lies in the region corresponding to the acute angle between the lines $2y=x$ and $4y=x$ such that the value of a lies in (p, q), where $p, q \in N$, find the value of $(p+q+1)$.

Comprehensive Link Passage

Passage I

Sometimes we do not take x-axis and y-axis at 90° and assume that they are inclined at an angle ω. Let OX and OY, the x-axis and the y-axis respectively, are inclined at an angle ω. Let P be a point on the plane. Draw parallel lines from P parallel to y- and x-axis. We will write $PN=x$ and $PM=y$ and will say that the co-ordinates of P are (x, y), where such axes are called oblique axes.

1. The distance of $P(x, y)$ from origin must be
   (a) $\sqrt{x^2+y^2}$ (b) $\sqrt{x^2+y^2-2xy\sin\omega}$
   (c) $\sqrt{x^2+y^2-2xy\cos\omega}$ (d) none
2. If $M(x_1, y_1)$ and $N(x_2, y_2)$ be two points in oblique system, the co-ordinates of the mid-points of A and B are
   (a) $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$
   (b) $\left(\frac{x_1+x_2}{2}\sin\omega, \frac{y_1+y_2}{2}\sin\omega\right)$
   (c) $\left(\frac{x_1+x_2}{2}\cos\omega, \frac{y_1+y_2}{2}\cos\omega\right)$
   (d) none
3. If a line makes intercepts a and b on x and y axes (obviously oblique axes), its equation be
   (a) $\frac{x}{a\cos\omega}+\frac{y}{b\cos\omega}=1$
   (b) $\frac{x}{a}+\frac{y}{b}=1$
   (c) $\frac{x}{b\cos\omega}+\frac{y}{a\cos\omega}=1$
   (d) none
4. If axes are inclined at 45°, the radius of the circle $x^2+xy+y^2-4x-5y-2=0$ is
   (a) 2 (b) 4 (c) 3 (d) $\sqrt{5}$

Passage II

Let us consider a rectangular co-ordinate system OX and OY be rotated through an angle θ in the anti-clockwise direction. Then we get a new co-ordinate system OX' and OY'. If we consider a point $P(x, y)$ in the old co-ordinate system and $P(X, Y)$ in the new co-ordinate system, we can write

$$x = X\cos\theta - Y\sin\theta \text{ and } y = X\sin\theta + Y\cos\theta.$$

Then

1. The value of X is
   (a) $X\cos\theta + Y\sin\theta$ (b) $X\sin\theta + Y\cos\theta$
   (c) $X\cos\theta - Y\sin\theta$ (d) $X\sin\theta - Y\cos\theta$.
2. The equation of a curve in a plane $17x^2-16xy+17y^2=225$.Through what angle must the axes be rotated, so that the equation becomes $9X^2+25Y^2=225$?
   (a) 30° (b) 45° (c) 60° (d) 75°
3. If the axes be rotated at 45°, the equation $17x^2-16xy+17y^2=225$ reduces to $AX^2+BY^2=C^2$, the value of $A+B+C$ is
   (a) 59 (b) 50 (c) 58 (d) 57.
4. If the axes are rotated through 45°, the equation $3x^2+2xy+3y^2=2$ reduces to
   (a) $X^2+2Y^2=1$ (b) $2X^2+Y^2=1$
   (c) $2X^2+3Y^2=1$ (d) $5X^2+3Y^2=1$.
5. The equation $4xy-3x^2=a^2$ become when the axes are turned through an angle $\tan^{-1}2$ is
   (a) $x^2+4y^2=a^2$ (b) $x^2-4y^2=a^2$
   (c) $4x^2+y^2=a^2$ (d) $4x^2-y^2=a^2$.

Passage III

The equations of adjacent sides of a parallelogram are $x+y+1=0$ and $2x-y+2=0$. If the equation of one of its diagonal is $13x-2y-32=0$. Then the

1. equation of the diagonal must be
   (a) $7x-8y+1=0$ (b) $2x-y=0$
   (c) $2x-y=7$ (d) $3x+4y=5$
2. area of the given parallelogram must be
   (a) 45 (b) 45/2 (c) $3\sqrt{5}$ (d) $4\sqrt{2}$

3. equation of the side of the parallelogram opposite to the given side $2x - y + 2 = 0$ must be
   (a) $2x - y + 5 = 0$ (b) $2x - y = 0$
   (c) $2x - y = 7$ (d) $x + 3y = 4$

Passage IV

The vertex C of a right-angled isosceles ΔABC is (2, 2) and the equation of the hypotenuse AB is $3x + 4y = 4$. Then

1. the equations of the sides AC and AB must be
   (a) $7y - x = 12$, $7x + y = 16$
   (b) $3x - 4y + 2 = 0$, $4x + 3y = 14$
   (c) $x + y = 4$, $2x - 3y = 10$
   (d) $x - y = 2$, $3x + 2y = 5$.
2. the area of the ΔABC must be
   (a) 1 sq. units (b) 2 sq. units
   (c) $2\sqrt{2}$ sq. units (d) 4 sq. units
3. The in-radius of the ΔABC must be
   (a) $\dfrac{2}{2+\sqrt{2}}$ (b) $\dfrac{4}{2+\sqrt{2}}$
   (c) $\dfrac{2-\sqrt{2}}{2+\sqrt{2}}$ (d) $\dfrac{1}{\sqrt{2}}$

Passage V

The vertex A of a ΔABC is (3, –1). The equations of median BE and angular bisector CF are, respectively $x - 4y + 10 = 0$ and $6x + 10y - 59 = 0$. Then the

1. equation of AB must be
   (a) $x + y = 2$ (b) $18x + 13y = 41$
   (c) $23x + y = 70$ (d) $x + 4y = 0$.
2. slope of the side BC must be
   (a) 1/7 (b) 1/9 (c) 2/9 (d) 3/4
3. length of the side AC must be
   (a) $\sqrt{83}$ (b) $\sqrt{85}$ (c) $\sqrt{89}$ (d) $\sqrt{88}$

Passage VI

In a ΔABC, the equation of altitudes AM and BN and the side AB are given by the equations $x + 5y = 3$ and $x + y = 1$. Then

1. the equation of the third altitude CL must be
   (a) $3x - y = 2$ (b) $3x - y = 1$
   (c) $3x + y + 1 = 0$ (d) $x + 3y + 1 = 0$.
2. the equation of BC must be
   (a) $5x - y = 5$ (b) $5x + y + 5 = 0$
   (c) $x - 2y = 3$ (d) $x + 2y + 3 = 0$
3. if R is the circum-radius of the triangle, $2R \cos B$ must be equal to
   (a) $\dfrac{1}{\sqrt{3}}$ (b) $\dfrac{1}{\sqrt{2}}$ (c) $\dfrac{1}{3}$ (d) $\dfrac{1}{2}$

Passage VII

Let the curve be S: $f(x, y) = 0$ and the line mirror L: $ax + by + c = 0$. We take a point P on the given curve in parametric form. Suppose Q be the image or reflection of point P about the line mirror $L = 0$, which again contains the same parameter. Let $Q = (\varphi(t), \psi(t))$, where t is a parameter. Now, let $x = \varphi(t)$ and $y = \psi(t)$. Eliminating t, we get the equation of the reflected curve S'. Then

1. the image of the line $3x - y = 2$ in the line $y = x - 1$ is
   (a) $x + 3y = 2$ (b) $3x + y = 2$
   (c) $x - 3y = 2$ (d) $x + y = 2$
2. the image of the circle $x^2 + y^2 = 4$ in the line $x + y = 2$ is
   (a) $x^2 + y^2 - 2x - 2y = 0$
   (b) $x^2 + y^2 - 4x - 4y + 6 = 0$
   (c) $x^2 + y^2 - 2x - 2y + 2 = 0$
   (d) $x^2 + y^2 - 4x - 4y + 4 = 0$
3. the image of the parabola $x^2 = 4y$ in the line $x + y = a$ is
   (a) $(x - a)^2 = -4(y - a)$ (b) $(y - a)^2 = -4(x - a)$
   (c) $(x - a)^2 = 4(y + a)$ (d) $(y - a)^2 = 4(x + a)$
4. the image of an ellipse $9x^2 + 16y^2 = 144$ in the line $y = x$ is
   (a) $16x^2 + 9y^2 = 144$ (b) $9x^2 + 16y^2 = 144$
   (c) $16x^2 + 25y^2 = 400$ (d) $25x^2 + 16y^2 = 400$
5. the image of the rectangular hyperbola $xy = 9$ in the line $y = 3$ is
   (a) $xy + 9 = 0$ (b) $xy - 6x + 9 = 0$
   (c) $xy + 6y - 9 = 0$ (d) $xy + 6x + 9 = 0$.

Matrix Match
(For JEE-Advanced Examination Only)

1. Match the following columns:
   In ΔABC, $AB : x + 2y = 3$,
   BC: $2x - y + 5 = 0$, $AC : x - 2 = 0$
   be the sides, then the

	Column I		Column II
(A)	circumcentre of ΔABC is (P)	(P)	$\left(\frac{-7}{5}, \frac{11}{5}\right)$
(B)	centroid of ΔABC is	(Q)	$\left(\frac{13}{15}, \frac{131}{30}\right)$
(C)	orthocentre of ΔABC is (R)	(R)	$\left(2, \frac{19}{4}\right)$

2. Match the following columns:
   A line cuts x-axis at A and y-axis at B such that $AB = l$, the loci of the

	Column I		Column II
(A)	circumcentre of ΔABC is	(P)	$x^2 + y^2 = \frac{l^2}{9}$
(B)	orthocentre of ΔABC is	(Q)	$x^2 + y^2 = \frac{l^2}{4}$
(C)	incentre of ΔABC is	(R)	$x^2 + y^2 = 0$
(D)	centroid of ΔABC is	(S)	$y = x$

3. Match the following columns:
   The vertex C of a ΔABC is (4, –1). The equation of altitude AD and median AE are $2x - 3y + 12 = 0$ and $2x + 3y = 0$, respectively then

Column I		Column II	
(A)	slope of side AB	(P)	–3/7
(B)	slope of side BC	(Q)	–3/2
(C)	slope of side AC	(R)	–9/11

4. Match the following columns:
The general equation of 2nd degree is
$$\lambda x^2 + 2y^2 + 4xy + 2x + 4y + \lambda = 0$$
It represents

Column I		Column II	
(A)	distinct lines if	(P)	$k = 1$
(B)	parallel lines if	(Q)	$k = 3$
(C)	imaginary lines if	(R)	$k = \phi$

5. Match the following columns:

Column I		Column II	
(A)	If $3a + 2b + 6c = 0$, the family of lines $ax + by + c = 0$ passes through a fixed point. Then the fixed point is	(P)	$(-2, -3)$
(B)	The family of lines $x(a + 2b) + y(a + 3b) = a + b$ passes through a fixed point. Then the point is	(Q)	$\left(\frac{1}{2}, \frac{1}{3}\right)$
(C)	If $4a^2 + 9b^2 - c^2 + 12ab = 0$, the family of straight lines $ax + by + c = 0$ passes through a fixed point. Then the fixed point is	(R)	$(2, -1)$

6. Match the following columns:

Column I		Column II	
The area of a parallelogram formed by the lines			
(A)	$3x - 4y + 1 = 0$, $3x - 4y + 3 = 0$, $4x - 3y - 1 = 0$ and $4x - 3y - 2 = 0$ is	(P)	20/11 units
(B)	$x + 3y = 1$, $3x - 2y + 3 = 0$, $x + 3y + 4 = 0$ and $3x - 2y + 7 = 0$ is	(Q)	2/7 units
(C)	$y = 2x + 3$, $y = 2x + 5$. $y = 7x + 4$ and $y = 7x + 5$ is	(R)	2/5 units

7. Match the following columns:

Column I		Column II	
(A)	The equation of the obtuse angle bisector of the lines $3x - 4y + 7 = 0$ and $12x + 5y - 2 = 0$ is	(P)	$6x + 2y - 5 = 0$
(B)	The equation of the acute-angle bisector of the lines $x + y = 3$ and $7x - y + 5 = 0$ is	(Q)	$21x + 77y - 101 = 0$
(C)	The bisector of the angle between the lines $2x + y = 6$ and $2x - 4y + 7 = 0$ which contains the point (1, 2) is	(R)	$6x - 2y = 5$
(D)	The bisector of the angle between the lines $4x + 3y = 6$ and $5x + 12y + 9 = 0$ which containing the origin is	(S)	$7x + 9y = 3$
(E)	The bisector of the angle between the lines $4x + 3y = 6$ and $5x + 12y + 9 = 0$ which does not containing the origin is	(T)	$9x - 7y = 41$

8. Match the following columns:

Column I		Column II	
(A)	The image of a line $2x + 3y + 4 = 0$ w.r.t. x-axis is	(P)	$2x - y - 11 = 0$
(B)	The image of a line $7x - 3y - 10 = 0$ w.r.t. y-axis is	(Q)	$5x + 3y + 7 = 0$
(C)	The image of a line $3x + 5y + 7 = 0$ w.r.t. the line $y = x$ is	(R)	$x - 4y + 29 = 0$
(D)	The image of a line $2x + y + 3 = 0$ w.r.t. the line $x = 2$ is	(S)	$7x + 3y + 10 = 0$
(E)	The image of a line $x + 4y + 5 = 0$ w.r.t. the line $y = 3$ is	(T)	$2x - 3y + 4 = 0$

9. Match the following Columns:

Column I		Column II	
(A)	The image of a point (2, 3) w.r.t. x-axis is	(P)	(4, 5)
(B)	The image of a point (3, 4) w.r.t. y-axis is	(Q)	(1, –1)
(C)	The image of a point (5, 4) w.r.t. the line $y = x$ is	(R)	(2, –3)
(D)	The image of a point (2, 5) w.r.t. the line $x = 3$ is	(S)	(–3, 4)
(E)	The image of a point (1, 5) w.r.t. the line $y = 2$ is	(T)	(4, 5)
(F)	The image of a point (2, 4) w.r.t. the line $y = \sqrt{3}x$ is	(U)	$(2\sqrt{3} - 1, 2 + \sqrt{3})$

10. Match the following columns:

Column I		Column II	
(A)	The orthocentre of the triangle formed by the lines $xy = 0$ and $x + y = 4$ is	(P)	(3, 1)
(B)	The orthocentre of the triangle formed by the lines $x + y = 4$, $x - y = 2$ and $2x + 3y = 6$ is	(Q)	$\left(-\frac{1}{4}, -\frac{1}{6}\right)$
(C)	The orthocentre of the triangle formed the lines $6x^2 + 5xy - 6y^2 + 3x - 2y = 0$ and $x + 4y = 5$ is	(R)	(0, 0)

11. Match the following columns:

Column I		Column II	
(A)	A light beam emanating from the point (3, 10) reflects from the straight line $2x + y = 6$ and the passes through the point $B(7, 2)$. Then the equation of the reflected ray is	(P)	$2x + 3y = 12$
(B)	A ray of light is sent along the line $y = \frac{2}{3}x - 4$. On reaching the x-axis it is reflected. Then the equation of the reflected ray is	(Q)	$x + 3y = 13$
(C)	A ray of light is sent along the line $x - 2y + 5 = 0$. Upon reaching the line $3x + 2y + 7 = 0$, the ray is reflected from it. Then the equation of the line containing the reflected ray is	(R)	$19x - 22y = -9$

Questions asked in Previous Years' JEE-Advanced Examinations

1. The area of a triangle is 5, two of its vertices are (2, 1) and (3, –2). The third vertex lies on $y = x + 3$. Find the third vertex. **[IIT-JEE, 1978]**
2. One side of a rectangle lies along the line $4x + 7y + 5 = 0$. Two of its vertices are (–3, 1) and (1, 1). Find the equation of the other three vertices. **[IIT-JEE, 1978]**
3. A straight line segment of length l moves with its ends on two mutually perpendicular lines. Find the locus of the point which divides the line in the ratio 1 : 2. **[IIT-JEE, 1978]**
4. Two vertices of a triangle are (5, –1) and (–2, 3). If the orthocentre of the triangle is the origin, find the co-ordinates of the third vertex. **[IIT-JEE, 1979]**
5. Find the equation of the line which bisects the obtuse angle between the lines $x - 2y + 4 = 0$ and $4x - 3y + 2 = 0$. **[IIT-JEE, 1979]**
6. The points $(-a, -b)$, (0, 0) and (a, b) are
   (a) collinear
   (b) vertices of a rectangle
   (c) vertices of a parallelogram
   (d) none **[IIT-JEE, 1979]**
7. The points $(-a, -b)$, (0, 0), (a, b) and (a^2, ab) are
   (a) collinear
   (b) vertices of a parallelogram
   (c) vertices of a rectangle
   (d) None **[IIT-JEE, 1979]**
8. A straight line L is perpendicular to the line $5x - y = 1$. The area of the triangle formed by the line L and the co-ordinate axes is 5. Find the equation of the line L. **[IIT-JEE, 1980]**
9. Given the four lines with the equations $x + 2y = 3$, $3x + 4y = 7$, $2x + 3y = 4$ and $4x + 5y = 6$. Then
   (a) they all are concurrent
   (b) they are the sides of a quadrilateral
   (c) only three lines are concurrent
   (d) None **[IIT-JEE, 1980]**
10. The point (4, 1) undergoes the following three transformations successively
    (i) reflection about the line $y = x$
    (ii) transformation through a distance 2 units along the positive direction of x-axis.
    (iii) rotation through an angle $\frac{\pi}{4}$ about the origin in the counterwise direction. Then the final position of the point is given by the co-ordinates
    (a) $\left(\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$ (b) $(-\sqrt{2}, 7\sqrt{2})$
    (c) $\left(-\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$ (d) $(\sqrt{2}, 7\sqrt{2})$ **[IIT-JEE, 1980]**
11. The points (1, 3) and (5, 1) are two opposite vertices of a rectangle. The other two vertices lie on the line $y = 2x + c$. Find c and the remaining two vertices. **[IIT-JEE, 1981]**
12. The set of lines $ax + by + c = 0$, where $2a + 3b + 4c = 0$ is concurrent at the point… **[IIT-JEE, 1982]**
13. The straight lines $x + y = 0$, $3x + y = 4$ and $x + 3y = 4$ form a triangle which is
    (a) iscosceles (b) equilateral
    (c) right angled (d) none **[IIT-JEE, 1983]**
14. The ends A and B of a straight line segment of length c slide upon the fixed rectangular axes OX and OY, respectively. If the rectangle $OABP$ be completed, show

that the locus of the perpendicular drawn from P to AB is $x^{2/3} + y^{2/3} = a^{2/3}$. **[IIT-JEE, 1983]**

15. The vertices of a triangle are $[at_1t_2, a(t_1 + t_2)]$, $[at_2t_3, a(t_2 + t_3)]$, $[at_3t_1, a(t_1 + t_3)]$. Find the orthocentre of the triangle. **[IIT-JEE, 1983]**
16. The straight line $5x + 4y = 0$ passes through the point of intersection of the straight lines $x + 2y = 10$ and $2x + y + 5 = 0$. Is it true/false? **[IIT-JEE, 1983]**
17. The co-ordinates of A, B and C are (6, 3), (–3, 5) and (4, –2) respectively and P is any point (x, y), show that the ratio of the area of the ΔPBC and ABC is $\left|\frac{x+y-2}{7}\right|$ **[IIT-JEE, 1983]**
18. Two equal sides of an isosceles triangle are given by the equations $7x - y + 3 = 0$ and $x + y - 3 = 0$ and its third side passes through the point (1, –10). Determine the equation of the third side. **[IIT-JEE, 1984]**
19 If a, b and c are in AP, the straight line $ax + by + c = 0$ will always pass through a fixed point, whose co-ordinates are... **[IIT-JEE, 1984]**
20 Three lines $px + qy + r = 0$, $qx + ry + p = 0$ and $rx + py + q = 0$ are concurrent if
(a) $p + q + r = 0$
(b) $p^2 + q^2 + r^2 = pq + qr + rp$
(c) $p^3 + q^3 + r^3 = 3pqr$
(d) none of these **[IIT-JEE, 1985]**
21. The orthocentre of the triangle formed by the lines $x + y = 1$, $2x + 3y = 6$ and $4x - y + 4 = 0$ lies in quadrant number... **[IIT-JEE, 1985]**
22. Two sides of a rhombus $ABCD$ are parallel to the lines $y = x + 2$ and $y = 7x + 3$. If the diagonals of the rhombus intersect at the point (1, 2) and the vertex A is on the y-axis, find the possible co-ordinates of A. **[IIT-JEE, 1985]**
23. If $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$, the two triangles with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) and (a_1, b_1), (a_2, b_2), (a_3, b_3), must be congruent. Is it true or false? **[IIT-JEE, 1985]**
24. One of the diameter of the circle circumscribing the rectangle $ABCD$ is $4y = x + 7$. If A and B are the points (–3, 4) and (5, 4), respectively, find the area of the rectangle. **[IIT-JEE, 1985]**
25. The set of all real numbers a such that $a^2 + 2a$, $2a + 3$ and $a^2 + 3a + 8$ are the sides of a triangle is... **[IIT-JEE, 1985]**
26. All points inside the triangle formed by the points (1, 3), (5, 0) and (–1, 2) satisfy
(a) $3x + 2y \geq 0$ (b) $2x + y - 13 \geq 0$
(c) $2x - 3y - 12 \leq 0$ (d) $-2x + y \leq 0$
(e) None **[IIT-JEE, 1986]**
27. The equation of the perpendicular bisectors of the sides AB and AC of a triangle ABC are $x - y + 5 = 0$ and $x + 2y = 0$, respectively. If the point A is (1, –2), find the equation of the line BC. **[IIT-JEE, 1986]**
28. The points $\left(0, \frac{8}{3}\right)$, (1, 3) and (82, 30) are the vertices of
(a) an obtuse-angled triangle
(b) an acute-angled triangle
(c) right-angled triangle
(d) an isosceles triangle
(e) None **[IIT-JEE, 1986]**

No questions asked in 1987.

29. The lines $2x + 3y + 19 = 0$ and $9x + 6y - 17 = 0$ cut the co-ordinate axes in concylic points. (T/F) **[IIT-JEE, 1988]**
30. Lines L_1: $ax + by + c = 0$ and L_2: $ax + my + n = 0$ intersect at the point P and make an angle θ with each other. Find the equation of a line L different from L_2 which passes through P and makes the same angle θ with L_1. **[IIT-JEE, 1988]**
31. Let ABC be a triangle with $AB = AC$. If D is the mid-point of BC, E the foot of the perpendicular drawn from D to AC and F the mid-point of DE. Prove that AF is perpendicular to BE. **[IIT-JEE, 1989]**
32. Striaght lines $3x + 4y = 5$ and $4x - 3y = 15$ intersect at the point A. Points B and C are chosen on these two lines such that $AB = AC$. Determine the possible equations of the line BC passing through the point (1, 2). **[IIT-JEE, 1990]**
33. A line L has intercepts a and b on the co-ordinate axes. When the axes are rotated through a given angle, keeping the origin fixed, the same line L has intercepts p and q, then
(a) $a^2 + b^2 = p^2 + q^2$ (b) $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2} + \frac{1}{q^2}$
(c) $a^2 + p^2 = b^2 + q^2$ (d) $\frac{1}{p^2} + \frac{1}{a^2} = \frac{1}{b^2} + \frac{1}{q^2}$
[IIT-JEE, 1990]
34. A line cuts the x-axis at $A(7, 0)$ and the y-axis at $B(0, -5)$. A variable line PQ is drawn perpendicular to AB cutting the x-axis in P and the y-axis in Q. If AQ and BP intersect in R, find the locus of R. **[IIT-JEE, 1990]**
35. Find the equation of the line passing through the point (2, 3) and making intercept of length 2 units between the lines $y + 2x = 3$ and $y + 2x = 5$. **[IIT-JEE, 1991]**
36. Let the algebraic sum of the perpendicular distance from the points (2, 0), (0, 2) and (1, 1) to a variable straight line be zero, the line passes through a fixed point whose co-ordinates are.... **[IIT-JEE, 1991]**
37. If the sum of the distances of a point from two mutually perpendicular lines in a plane is 1, its locus is
(a) square (b) circle
(c) straight line (d) two intersecting lines
[IIT-JEE, 1992]

38. Determine all values of α for which the point (α, α^2) lies inside the triangle formed by the lines $2x + 3y - 1 = 0$, $x + 2y - 3 = 0$ and $5x - 6y - 1 = 0$. **[IIT-JEE, 1992]**
39. The vertices of a triangle $A(-1, -7)$, $B(5, 1)$ and $C(1, 4)$. The equation of the bisector of the angle $\angle ABC$ is... **[IIT-JEE, 1993]**
40. A line through $A(-5, -4)$ meets the lines $x + 3y + 2 = 0$, $2x + y + 4 = 0$ and $x - y - 5 = 0$ at the points B, C and D, respectively. If $\left(\frac{15}{AB}\right)^2 + \left(\frac{10}{AC}\right)^2 = \left(\frac{6}{AD}\right)^2$, find the equation of the line. **[IIT-JEE, 1993]**
41. The orthocentre of the triangle formed by the lines $xy = 0$ and $x + y = 1$ is
(a) $\left(\frac{1}{2}, \frac{1}{2}\right)$ (b) $\left(\frac{1}{3}, \frac{1}{3}\right)$ (c) $(0, 0)$ (d) $\left(\frac{1}{4}, \frac{1}{4}\right)$
[IIT-JEE, 1995]
42. A rectangle $PQRS$ has its sides PQ parallel to the line $y = mx$ and vertices P, Q and S on the lines $y = a$, $x = b$ and $x = -b$, respectively. Find the locus of the vertex R. **[IIT-JEE, 1996]**

No questions asked in 1997.

43. The diagonals of a parallelogram $PQRS$ are along the lines $x + 3y = 4$ and $6x - 2y = 7$. Then $PQRS$ must be a:
(a) rectangle (b) square
(c) cyclic quad. (d) rhombus
[IIT-JEE, 1998]

No questions asked in 1999.

44. Let PS be the median of the triangle with vertices $P(2, 2)$, $Q(6, -1)$ and $R(7, 3)$. The equation of the line passing through $(1, -1)$ and parallel to PS is
(a) $2x - 9y - 7 = 0$ (b) $2x - 9y - 11 = 0$
(c) $2x + 9y - 11 = 0$ (d) $2x + 9y + 7 = 0$
[IIT-JEE, 2000]
45. A straight line through the origin O meets the parallel lines $4x + 2y = 9$ and $2x + y + 6 = 0$ at points P and Q, respectively. The point O divides the segment PQ in the ratio
(a) 1 : 2 (b) 3 : 4 (c) 2 : 1 (d) 4 : 3
[IIT-JEE, 2000]
46. For points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ of the co-ordinate plane, a new distance $d(P, Q)$ is defined as $d(P, Q) = |x_1 - x_2| + |y_1 - y_2|$.
Let $O = (0, 0)$ and $A = (3, 2)$. Prove that the set of points in the first quadrant which are equidistant (with respect to the new distance) from O and A consists of the union of a line segment of finite length and an infinite ray. Sketch this set in a labelled diagram. **[IIT-JEE, 2000]**
47. The area of the parallelogram formed by the lines $y = mx$, $y = mx + 1$, $y = nx$ and $y = nx + 1$ equals
(a) $\frac{|m+n|}{(m-n)^2}$ (b) $\frac{2}{|m+n|}$
(c) $\frac{1}{|m+n|}$ (d) $\frac{1}{|m-n|}$
[IIT-JEE, 2001]
48. The number of integral values of m, for which the x-co-ordinate of the point of intersection of the lines $3x + 4y = 9$ and $y = mx + 1$ is also an integer, is
(a) 2 (b) 0 (c) 4 (d) 1
[IIT-JEE, 2001]
49. Let $P = (-1, 0)$, $Q = (0, 0)$ and $R = (3, 3\sqrt{3})$ be three points. The equation of the bisector of the angle PQR is
(a) $\frac{\sqrt{3}}{2}x + y = 0$ (b) $x + \sqrt{3}y = 0$
(c) $\sqrt{3}x + y = 0$ (d) $x + \frac{\sqrt{3}}{2}y = 0$
[IIT-JEE, 2002]
50. A straight line L through the origin meets the lines $x + y = 1$ and $x + y = 3$ at P and Q, respectively.
Through P and Q two straight lines L_1 and L_2 are drawn parallel to $2x - y = 5$ and $3x + y = 5$, respectively. Lines L_1 and L_2 intersect at R. Show that the locus of R as L varies, is a straight line. **[IIT-JEE, 2002]**
51. A straight line L with negative slope passes through the point (8, 2) and meets the positive co-ordinate axes at points P and Q. Find the absolute minimum value of $OP + OQ$ as L varies, where O is the origin. **[IIT-JEE, 2002]**
52. A straight line through the point (2, 2) intersects the lines $\sqrt{3}x + y = 0$ and $\sqrt{3}x - y = 0$ at the points A and B. The equation to the line AB so that the triangle OAB is equilateral, is
(a) $x - y = 0$ (b) $y - 2 = 0$
(c) $x + y - 4 = 0$ (d) none of these
[IIT-JEE, 2002]
53. A straight line through the origin O meets the parallel lines $4x + 2y = 9$ and $2x + y + 6 = 0$ at points P and Q respectively. The point O divides the segment PQ in the ratio
(a) 1 : 2 (b) 3 : 4 (c) 2 : 1 (d) 4 : 3.
[IIT-JEE, 2002]
54. The orthocentre of the triangle with vertices (0, 0), (3, 4) and (4, 0) is
(a) $\left(3, \frac{5}{4}\right)$ (b) (3, 12)
(c) $\left(3, \frac{3}{4}\right)$ (d) (3, 9)
[IIT-JEE, 2003]
55. If the equation of the locus of a point equidistant from the points (a_1, b_1) and (a_2, b_2) is $(a_1 - a_2)x + (b_1 - b_2)y + c = 0$, then the values of '$c$' is
(a) $\frac{1}{2}(a_2^2 + b_2^2 - a_1^2 - b_1^2)$ (b) $(a_1^2 - a_2^2 + b_1^2 - b_2^2)$
(c) $\frac{1}{2}(a_1^2 + a_2^2 + b_1^2 + b_2^2)$ (d) $\sqrt{(a_1^2 - a_2^2 + b_1^2 - b_2^2)}$
[IIT-JEE, 2003]
56. The area of the triangle formed by the lines $x + y = 3$ and angle bisector of the pair of straight lines $x^2 - y^2 + 2y - 1 = 0$ is
(a) 2 (b) 4 (c) 6 (d) 8
[IIT-JEE, 2004]

57. The area of the triangle formed by the intersection of a line parallel to x-axis and passing through $P(h, k)$ with the lines $y = x$ and $x + y = 2$ is $4h^2$. Find the locus of the point P. **[IIT-JEE, 2005]**

58. Two rays in the first quadrant $x + y = |a|$ and $ax - y = 1$ intersects each other in the interval $a \in (a_0, \infty)$. Find the value of a_0. **[IIT-JEE, 2006]**

59. Lines $L_1: y - x = 0$, $L_2: 2x + y = 0$ intersect the line $L_3: y + 2 = 0$ at P and Q, respectively. The bisector of the acute angle between L_1 and L_2 intersect L_3 at R.
Statement I: The ratio $PR : RQ$ equals $2\sqrt{2} : \sqrt{5}$.
Statement II: In any triangle bisector of an angle, divides the triangle into two similar triangles.
[IIT-JEE, 2007]

60. Consider the lines given by
$L_1: x + 3y - 5 = 0$, $L_2: 3x - ky - 1 = 0$,
and $L_3: 5x + 2y - 12 = 0$

Column I	Column II
(a) L_1, L_2, L_3 are concurrent if	(p) $k = -9$
(b) One of L_1, L_2, L_3 is parallel to at least of the other two, if	(q) $k = -6/5$
(c) L_1, L_2, L_3 form *a* triangle, if	(s) $k = 5/6$
(d) L_1, L_2, L_3 do not form a triangle, if	(t) $k = 5$

[IIT-JEE, 2008]

61. The locus of the orthocentre of the triangle formed by the lines $(1 + p)x - py + p(1 + p) = 0$,
$(1 + q)x - qy + q(1 + q) = 0$ and $y = 0$,
where $p \neq q$, is
(a) a hyperbola (b) a parabola
(c) an ellipse (d) a straight line
[IIT-JEE, 2009]

No questions asked in 2010.

62. A straight line L through the point $(3, -2)$ is inclined at an angle 60° to the line $\sqrt{3}x + y = 1$. If L also intersects the x-axis, the equation of L is
(a) $y + \sqrt{3}x + (2 - 3\sqrt{3}) = 0$
(b) $y - \sqrt{3}x + (2 + 3\sqrt{3}) = 0$
(c) $-x + \sqrt{3}y + (3 + 2\sqrt{3}) = 0$
(d) $x + \sqrt{3}y + (-3 + 2\sqrt{3}) = 0$ **[IIT-JEE, 2011]**

No questions asked in 2012.

63. For $a > b > c > 0$, the distance between (1, 1) and the point of intersection of the lines $ax + by + c = 0$ and $bx + ay + c = 0$ is less than $2\sqrt{2}$, then
(a) $a + b - c > 0$ (b) $a - b + c < 0$
(c) $a - b + c > 0$ (d) $a + b - c < 0$
[IIT-JEE, 2013]

64. For a point P in the plane, let $d_1(P)$ and $d_2(P)$ be the distances of the point P from the lines $x - y = 0$ and $x + y = 0$, respectively.
The area of the region R consisting of all points P lying in the first quadrant of the plane and satisfying $2 \leq d_1(P) + d_2(P) \leq 4$, is ...
[IIT-JEE, 2014]

Answers

Level I

(CARTESIAN CO-ORDINATES)

1. $\left(5, -\tan^{-1}\left(\frac{4}{3}\right)\right), \left(5, \pi - \tan^{-1}\left(\frac{4}{3}\right)\right)$
2. $(x^2 + y^2)^2 = a^2(x^2 - y^2)$
3. $(x^2 + y^2) = 2ax$
4. $\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2} = \frac{1}{r^2}$
5. $r^2 = \frac{2}{(4 + 3\sin 2\theta)}$
6. $2a\left|\sin\left(\frac{\alpha - \beta}{2}\right)\right|$
7. 10
8. 34
9. $\frac{3}{\sqrt{10}}$
10.
11. $\left(\frac{1 \pm \sqrt{3}}{2}, \frac{7 \pm 5\sqrt{3}}{2}\right)$
12. (3, 1)
13. 3/7
14. 12:1
15. $\left(-\frac{2}{3}, 0\right)$ and $\left(-\frac{5}{3}, 2\right)$
16. (0, 1), (1, 0) and (3, 1)
17. (4, 6)
18. $\left(\frac{10}{3}, \frac{8}{3}\right)$
19. (–4, –15)
20. (2, 3)
21. $\left(\frac{4}{3}, \frac{5}{6}\right)$
22. $\sqrt{2}$
23. 46
24. 2
25. $x + y + 9 = 0$, $x + y - 15 = 0$
26. 132

27. 54
28. 0
29. $x = 7/4$
30. $\dfrac{2304}{3850}$
31. $x^2 + y^2 = a^2$
32. $y^2 = 4ax$
33. $y^2 = 8x^2$
34. $y^2 = 2ax - a^2$
35. $x^2 - y^2 = 4$
36. $\dfrac{x^2}{7} + \dfrac{y^2}{16} = 1$
37. $x^2 + y^2 = \dfrac{l^2}{4}$
38. $y^2 = 4x$
39. $2(x^2 + y^2) - x - y + 1 = 0$
40. $xy = x + y + \sqrt{x^2 + y^2}$
44. $x^2 + y^2 = 4$
45. $x^2 + y^2 = 3$
46. $2x^2 + y^2 - 7x + 9y + 1 = 0$
47. $x^2 + y^2 - 8x - 10y + 5 = 0$
48. $(-6, -5)$
49. $2x^2 + y^2 = 1$
50. $y^2 = 4x$
51. $(-2, 3)$
52. 12
53. $x^2 - y^2 = a^2$
54. $4x^2 + 2y^2 = 1$
55. $(-3, 2)$
56. $\left(\dfrac{1 - 4\sqrt{3}}{2}, \dfrac{4 + \sqrt{3}}{2}\right)$

Level I

(STRAIGHT LINES)

1. 6
2. 3
5. $\theta = \tan^{-1}\left(\dfrac{3}{4}\right)$
6. 3 or $-1/3$
7. $y + 3 = 0$
8. $y = 4$
9. $x = 8$
10. $y = 3x + 7$
11. $y = x + 5$
12. $y = mx + c = \dfrac{3}{5}x - 6$
13. $y = mx + c = \pm x + 4$
14. $y = mx + c = \pm x.$
15. $\sqrt{3}x + y = 2\sqrt{3} + 3$
16. $8x + 10y = 69$
17. $4x + 5y = 14$
18. $(2 - \sqrt{3})x - y - 2(2 - \sqrt{3}) = 0$
19. $x - y + 1 = 0$
20. $x - 5y + 10 = 0$
21. $3x - 5y + 7 = 0$
22. $y - 2 = \dfrac{4-2}{4-5}(x - 5) = -2(x - 5)$
23. (3/2, 0), (9/4, 0), (9/4, 3/4) and (3/2, 3/4)
24. $y - 0 = \sqrt{3}(x - 2),\ x\sqrt{3} - y - 2\sqrt{3} = 0$
25. $x + y = 7$
26. $2x + y = 7$
27. $3x + 2y = 12$
28. $x - y = 7$
29. $x - y = 1,\ 2x - 3y = 12$
30. 3
31. $2x + y = 10$
32. $3x - 8y = 0,\ 3x - 2y = 0$
35. $x + \sqrt{3}\,y = 10$
36. $12x + 5y = 26$
37. $(7, 5), (-1, -1)$
38. $(2, 7)$ and $(-6, -1)$
39. $(4, 5)$
40. $\left(2 + \sqrt{2}\cdot\dfrac{1}{2}, \sqrt{2}\cdot\dfrac{\sqrt{3}}{2}\right) = \left(2 + \dfrac{1}{\sqrt{2}}, \sqrt{\dfrac{3}{2}}\right)$
41. 75°, 15°
42. $R(-2, -1)$, $Q(-1, 2)$ and $S(1, -2)$
43. $\left(-\dfrac{3}{2}, \dfrac{3}{2}\right)$ and $\left(\dfrac{1}{2}, -\dfrac{3}{2}\right)$
44. $\sqrt{2}$
45. 6
46. $4\sqrt{2}$
47. $\sqrt{5}$
48. 5
49. $(8, 2)$ and $(0, -4)$
54. $3x + 4y - 18 = 0$
55. $3x - 4y - 5 = 0$
56. $4x + y = 9$
57.
58. $3x - 2y - 1 = 0$
59. $3x + 2y + 3 = 0$
60. $4x + 6y = 29$
61. $x - 3y + 5 = 0.$
62. $4x - 3y - 36 = 0$
63. $\dfrac{6}{\sqrt{34}}$
64. $\sqrt{\dfrac{2}{3}}$

65. $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{p^2}$
66. $2\frac{1}{10}$
67. $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{p^2}+\frac{1}{q^2}$
68. 50
69. $\frac{2c^2}{|ab|}$
70. (2, 6)
71. 0
72. $7x - 48y + 89 = 0$
73. $m = 2$
74. $\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}=1$
75. $x=\frac{1}{3}$ and $y=\frac{1}{2}$
78. 10
81. $m=-2$ and $m=\frac{1}{2}$
82. $3x - y - 7 = 0$ and $x + 3y - 9 = 0$
83. $(2+\sqrt{3})x - y - (2\sqrt{3}+1) = 0$ and $(2-\sqrt{3})x - y + (2\sqrt{3}-1) = 0$
84. $52x + 89y + 519 = 0$
85. $x + 2 = 0$ and $7x - 24y + 182 = 0$
86. $9x - 7y + 3 = 0$ and $7x + 9y = 41$
87. $x - 7y + 13 = 0$ and $7x + 7 = 7$
88. $3x + y + 7 = 0$ and $x - 3y = 31$
89. $21x + 77y = 101$ and $3x - y + 3 = 0$
90. $7x + 9y = 3$
91. $6x - 2y = 5$
92. $21x + 77y = 101$
93. $6x + 2y = 5$
94. $7x - 9y + 10 = 0$
95. $\left(\frac{1}{3},\frac{2}{3}\right)$
96. $PQ = 6$
97. $(2-2\sqrt{2}, 3+2\sqrt{2})$
98. (–1/10, 37/10)
99. (0, 28
100. (4, 3).
101. $3x - 2y + 5 = 0$
102. (6, 1)
103. $\left(\frac{3}{\sqrt{2}},\frac{7}{\sqrt{2}}\right)$
105. $29x - 2y = 31$
106. $\frac{13}{5}$
107. $y = 0$
108. $(70-37\sqrt{3})x - 13y - 153 + 74\sqrt{3} = 0$

Level II

1. (c)	2. (d)	3. (b)	4. (a)	5. (a)
6. (a)	7. (a)	8. (b)	9. (a)	10. (a)
11. (b)	12. (a,c)	13. (d)	14. (c)	15. (a)
16. (b)	17. (d)	18. (a)	19. (b)	20. (c)
21. (a)	22. (a)	23. (b)	24. (a)	25. (a, d)
26. (c)	27. (d)	28. (b)	29. (a, b)	30. (b)
31. (c)	32. (d)	33. (c)	34. (d)	35. (b)
36. (a)	37. (a)	38. (b)	39. (b)	40. (a)
41. (b)	42. (a)	43. (a,bc)	44. (c)	45. (b)
46. (d)	47. (b)	48. (d)	49. (c)	50. (a)
51. (b)	52. (b)	53. (c)	54. (d)	55. (c)
56. (a)	57. (c)	58. (d)	59. (c)	60. (b)
61. (d)	62. (b)	63. (a)	64. (a)	65. (c)
66. (c)	67. (b)	68. ()	69. ()	70. ()

Level III

1. $\frac{5}{3}\le\beta\le\frac{7}{2}$ 2. 2
3. $\sqrt{3}x + y = 0$ 4. $y = 2$
5. $2x - y + 1 = 0$, $2x + y - 1 = 0$
6. $a_0 = 1$
8. $2(2+\sqrt{2})$
9.
10. (5, 3) and (–3, 5)
11. $\frac{4}{5} < m < 4$
12. ...
13. R – (–4, 0)
14. (–2,1) and (–4, 3)
15. $-2\sqrt{2}$
16. (0, 3), (4, −1)
17. $\alpha\in\left(-\frac{3}{2},-1\right)\cup\left(\frac{1}{2},1\right)$
18. $c = -1$; (4, 4), (2, 0)
19. $y=(2\pm\sqrt{3})x-1\pm 2\sqrt{3}$
20. $y = x$.
21. (13/5, 0)
22. (42/25, 91/25)
23. (6, 0) ; $2x + 3y = 12$.
24. $-2 < a < 2$
25. $26x - 122y = 1675$.
26. 27.B(–7, 6); C(11/5, 2/5)
28. $52x + 89y + 519 = 0$
29. $x + 2 = 0$ and $7x + 24y + 182 = 0$.
30. $p=\frac{1}{\sqrt{3}}$
31. $x - 7y + 13 = 0$ and $7x + y = 9$.
32. R(42/25, 91/25)

33. $3x + y + 7 = 0$ and $x - 3y - 31 = 0$
34. D: $(6+2\sqrt{2}, -1-2\sqrt{2})$
or D: $(2-2\sqrt{2}, 3+2\sqrt{2})$
35. (0, 0) or (0, 5/2).

Level IV

1. Sides are: $x - 3y = 3$, $3x - 2y = 16$,
Diagonals are $x + 4y = 10$, $5x - 8y + 6 = 0$.
2. $5\sqrt{2}$
3. $C = \left(\frac{3\sqrt{2}-1}{\sqrt{2}}, \frac{(\sqrt{2}+1)}{\sqrt{2}}\right)$
4. (4, 2)
5. $x - y = 2$
6. $x + y\sqrt{3} - \sqrt{3} = 0$
7. $\left(1 \pm \frac{1}{\sqrt{5}}, 1 \pm \frac{2}{\sqrt{5}}\right)$
8. $(2-\sqrt{2}, \sqrt{2}-1), (2\sqrt{5}+2, \sqrt{5}-1), \sqrt{29+2\sqrt{10}}$
9. $9x^2 - 6xy + 10y^2 = 9l^2$

INTEGER TYPE QUESTIONS

1. 8	2. 6	3. 3	4. 1	5. 1
6. 5	7. 4	8. 4	9. 6	10. 7

COMPREHENSIVE LINK PASSAGES

Passage I:	1. (c)	2. (a)	3. (b)	4. (c)	
Passage II:	1. (a)	2. (b)	3. (a)	4. (b)	5. (b)
Passage II:	1. (b)	2. (a)	3. (c)		
Passage IV:	1. (a)	2. (b)	3. (b)		
Passage V:	1. (b)	2. (c)	3. (b)		
Passage VI:	1. (b)	2. (a)	3. (b)		
Passage VII:	1. (c)	2. (a)	3. (b)	4. (a)	5. (b).

MATRIX MATCH

1. (A) → (R), (B) → (Q), (C) → (P)
2. (A) → (Q), (B) → (R), (S) → (R), (D) → (P)
3. (A) → (R), (B) → (Q), (C) → (P)
4. (A) → (P), (B) → (R), (C) → (P)
5. (A) → (Q), (B) → (R), (C) → (P),
6. (A) → (P), (B) → (Q), (C) → (R),
7. (A) → (Q), (B) → (R), (C) → (R),
(D) → (S),(E) → (T),
8. (A) → (T), (B) → (S), (C) → (Q), (D) → (P),
(E) → (R)
9. (A) → (R), (B) → (S), (C) → (T),
(D) → (P), (E) → (Q), (F) → (U)
10. (A) → (R), (B) → (P), (C) → (Q)
11. (A) → (Q), (B) → (P), (C) → (R)

Hints and Solutions

Level I

(RECTANGULAR CARTESIAN CO-ORDINATES)

1. We have,

$$(3, -4) \Rightarrow \left(\sqrt{3^2+(-4)^2}, \tan^{-1}\left(-\frac{4}{3}\right)\right)$$

$$\Rightarrow \quad \left(5, -\tan^{-1}\left(\frac{4}{3}\right)\right)$$

and

$$(-3, 4) \Rightarrow \left(\sqrt{(-3)^2+(4)^2}, \tan^{-1}\left(-\frac{3}{4}\right)\right)$$

$$\Rightarrow \quad \left(5, \pi - \tan^{-1}\left(\frac{4}{3}\right)\right)$$

2.. We have,

$$r^2 = a^2 \cos 2\theta$$

$$\Rightarrow \quad r^2 = a^2 (\cos^2\theta - \sin^2\theta)$$

$$\Rightarrow \quad r^2 = a^2\left(\frac{x^2}{r^2} - \frac{y^2}{r^2}\right)$$

$$r^4 = a^2(x^2 - y^2)$$

$$\Rightarrow \quad (x^2 + y^2)^2 = a^2(x^2 - y^2)$$

3. We have,

$$r = 2a \cos 2\theta$$

$$\Rightarrow \quad r = 2a\left(\frac{x}{r}\right)$$

$$\Rightarrow \quad r^2 = 2ax$$

$$\Rightarrow \quad (x^2 + y^2 = 2ax)$$

4 We have,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow \quad b^2x^2 + a^2y^2 = a^2b^2$$

$$\Rightarrow \quad b^2r^2\cos^2\theta + a^2r^2\sin^2\theta = a^2b^2$$

$$\Rightarrow \quad \frac{r^2\cos^2\theta}{a^2} + \frac{r^2\sin^2\theta}{b^2} = 1$$

$$\Rightarrow \quad \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2} = \frac{1}{r^2}$$

5. We have,

$$2x^2 + 3xy + 2y^2 = 1$$

$$\Rightarrow \quad 2r^2\cos^2\theta + 3r^2\sin\theta\cos\theta + 2r^2\sin^2\theta = 1$$

$$\Rightarrow \quad r^2(2\cos^2\theta + 3\sin\theta\cos\theta + 2\sin^2\theta = 1$$

$$\Rightarrow \quad r^2\left(2 + \frac{3}{2}\sin 2\theta\right) = 1$$

$\Rightarrow \quad r^2 = \dfrac{2}{(4 + 3\sin 2\theta)}$

6. The required distance

$$= \sqrt{a^2(\cos\alpha - \cos\beta)^2 + a^2(\sin\alpha - \sin\beta)^2}$$
$$= a\sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2}$$
$$= a\sqrt{[(\cos^2\alpha + \sin^2\alpha) + (\cos^2\beta + \sin^2\beta) - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)]}$$
$$= a \times \sqrt{2 - 2\cos(\alpha - \beta)}$$
$$= a \times \sqrt{2(1 - \cos(\alpha - \beta))}$$
$$= 2a\left|\sin\left(\frac{\alpha - \beta}{2}\right)\right|$$

7. The required distance

$$= \sqrt{3^2 + 7^2 - 2 \times 3 \times 7 \times \cos\left(\frac{5\pi}{4} - \frac{\pi}{4}\right)}$$
$$= \sqrt{3^2 + 7^2 - 2 \times 3 \times 7 \times \cos(\pi)}$$
$$= \sqrt{9 + 49 + 42} = \sqrt{100} = 10$$

8. Let $P = (x, y)$, $A = (6, -1)$ and $B = (2, 3)$.
We have,
$PA = PB$
$\Rightarrow \quad (x - 6)^2 + (y + 1)^2 = (x - 2)^2 + (y - 3)^2$
$\Rightarrow \quad x^2 - 12x + 36 + y^2 + 2y + 1$
$\qquad = x^2 - 4x + 4 + y^2 - 6y + 9$
$\Rightarrow \quad -12x + 36 + 2y + 1 = -4x + 4 - 6y + 9$
$\Rightarrow \quad -8x + 8y + 37 - 13 = 0$
$\Rightarrow \quad -8x + 8y + 24 = 0$
$\Rightarrow \quad A = -8, B = 8, C = 24$
Hence, the value of $A + B + C + 10$
$\qquad = -8 + 8 + 24 + 10 = 34$

9. We have,

$$AB = \sqrt{(2 + 2)^2 + (-1 - 3)^2} = 4\sqrt{2}$$
$$BC = \sqrt{(2 - 4)^2 + (-1)^2} = \sqrt{5}$$
and $$CA = \sqrt{(4 + 2)^2 + (3)^2} = \sqrt{45} = 3\sqrt{5}$$
Thus, $$\cos A = \frac{(4\sqrt{2})^2 + (3\sqrt{5})^2 - (\sqrt{5})^2}{2 \times 4\sqrt{2} \times 3\sqrt{5}}$$
$$\Rightarrow \quad \cos A = \frac{32 + 45 - 5}{24\sqrt{10}} = \frac{72}{24\sqrt{10}} = \frac{3}{\sqrt{10}}$$

10. Do yourself.

11. $\left(\dfrac{1 \pm \sqrt{3}}{2}, \dfrac{7 \pm 5\sqrt{3}}{2}\right)$

12. Let the point be (x, y).
Thus, $x = \dfrac{4 + 5}{2 + 1} = 3$, $y = \dfrac{6 - 3}{2 + 1} = 1$
Hence, the point is (3, 1).

13. Let the ratio be $m : n$.
Since the point lies on y-axis, so x co-ordinate will be zero.
Thus, $\dfrac{7m - 3n}{m + n} = 0$
$\Rightarrow \quad 7m - 3n = 0$
$\Rightarrow \quad \dfrac{m}{n} = \dfrac{3}{7}$

14. Let the ratio be $\lambda : 1$.
Thus, the point is $\left(\dfrac{1 - 2\lambda}{\lambda + 1}, \dfrac{3\lambda - 2}{\lambda + 1}\right)$.
Since, the point $\left(\dfrac{1 - 2\lambda}{\lambda + 1}, \dfrac{3\lambda - 2}{\lambda + 1}\right)$ lies on the line $3x + 4y = 7$, we get
$$3\left(\frac{1 - 2\lambda}{\lambda + 1}\right) + 4\left(\frac{3\lambda - 2}{\lambda + 1}\right) = 7$$
$\Rightarrow \quad 3(1 - 2\lambda) + 4(3\lambda - 2) = 7\lambda + 7$
$\Rightarrow \quad 12\lambda - 6\lambda - 7\lambda = 8 + 7 - 3$
$\Rightarrow \quad \lambda = -12$
Hence, the ratio is 12 : 1 externally.

15. Let $A = (1, -2)$ and $B = (-3, 4)$ and the line AB is trisected at P and Q, respectively.
Therefore P divides A and B internally in the ratio 1 : 2 and Q divides A and B in the ratio 2 : 1.
Thus, the co-ordinates of P and Q are $\left(-\dfrac{2}{3}, 0\right)$ and $\left(-\dfrac{5}{3}, 2\right)$.

16. Let the co-ordinates of the vertices are (x_1, y_1), (x_2, y_2), and (x_3, y_3).
Therefore, $x_1 + x_2 = 1$, $x_2 + x_3 = 4$, $x_1 + x_3 = 3$
$\Rightarrow \quad x_1 + x_2 + x_3 = 4$
Thus, $x_1 = 0$, $x_2 = 1$, $x_3 = 3$
Also, $y_1 + y_2 = 1$, $y_2 + y_3 = 1$, $y_3 + y_1 = 2$
$\Rightarrow \quad y_1 + y_2 + y_3 = 2$
Thus, $y_1 = 1$, $y_2 = 0$, $y_3 = 1$
Hence, the vertices are (0, 1), (1, 0) and (3, 1).

17. Let the co-ordinates of the third vertex be(x, y).
As we know that the diagonals of a parallelogram bisect each other.
Thus, $\dfrac{x - 1}{2} = \dfrac{1 + 2}{2}$ and $\dfrac{y + 2}{2} = \dfrac{5 + 3}{2}$
$\Rightarrow \quad x = 4, y = 6$
Hence, the fourth vertex is (4, 6).

18. The co-ordinates of the centroid of ΔABC are
$$\left(\frac{2 + 6 + 2}{3}, \frac{4 + 4 + 0}{3}\right) = \left(\frac{10}{3}, \frac{8}{3}\right)$$

19. Let the co-ordinates of the third vertex be (x, y).
Therefore, $\dfrac{x - 1 + 5}{3} = 0 \quad \Rightarrow \quad x = -4$ and
$\dfrac{y + 4 + 2}{3} = -3 \quad \Rightarrow \quad y = -15$
Hence, the co-ordinates of the third vertex be (–4, –15).

20. Let $a = BC$, $b = CA$, $c = AB$ be the lengths of the side of the given ΔABC.

Therefore, $a = \sqrt{(3-2)^2 + (4-3)^2} = \sqrt{2}$,

$b = \sqrt{(3-1)^2 + (4-2)^2} = 2\sqrt{2}$

and $c = \sqrt{(2-1)^2 + (3-2)^2} = \sqrt{2}$

Thus, the incentre are

$$\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$$

$$= \left(\frac{1\cdot\sqrt{2} + 2\cdot 2\sqrt{2} + 3\cdot\sqrt{2}}{\sqrt{2} + 2\sqrt{2} + \sqrt{2}}, \frac{2\cdot\sqrt{2} + 3\cdot 2\sqrt{2} + 4\cdot\sqrt{2}}{\sqrt{2} + 2\sqrt{2} + \sqrt{2}}\right)$$

$$= \left(\frac{8\sqrt{2}}{4\sqrt{2}}, \frac{12\sqrt{2}}{4\sqrt{2}}\right)$$

$= (2, 3)$

21. As we know that the centroid divides the orthocentre and the circumcentre in the ratio 2 : 1.
Thus, the centroid are

$$\left(\frac{2\cdot\frac{3}{2} + 1\cdot 1}{2+1}, \frac{2\cdot\frac{3}{4} + 1\cdot 1}{2+1}\right) = \left(\frac{4}{3}, \frac{5}{6}\right)$$

22. Clearly, it is a right-angled triangle.
As we know that in case of a right angled triangle, the circumcentre is the mid-point of the hypotenuse and the orthocentre is at right angle.

Thus, Circumcentre $= \left(\frac{0+2}{2}, \frac{2+0}{2}\right)$
$= (1, 1)$

and
Orthocentre = (0, 0)

Hence, the required distance $= \sqrt{(1-0)^2 + (1-0)^2}$
$= \sqrt{2}$

23. The required area

$$= \frac{1}{2}\begin{vmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 3-7 & 7+1 \\ -4-5 & 5-10 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} -4 & 8 \\ -9 & -5 \end{vmatrix}$$

$$= \frac{1}{2}(20 + 72) = 46 \text{ sq.u.}$$

24. The required area

$$= \frac{1}{2}\begin{vmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} t - t - 3 & t + 3 - t - 2 \\ t + 2 - \text{t} & t - t - 2 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} -3 & 1 \\ 2 & -2 \end{vmatrix}$$

$$= \frac{1}{2}(6 - 2) = 2 \text{ sq.u.}$$

25. Given area of a triangle is 6.

$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} x-1 & 1-2 \\ y-2 & 2-1 \end{vmatrix} = \pm 6$

$\Rightarrow \quad \begin{vmatrix} x-1 & -1 \\ y-2 & 1 \end{vmatrix} = \pm 12$

$\Rightarrow \quad (x - 1) + (y - 2) = \pm 12$

$\Rightarrow \quad x + y - 3 + 12 = 0, x + y - 3 - 12 = 0$

$\Rightarrow \quad x + y + 9 = 0, x + y - 15 = 0$

Hence, the result.

26. The required area of a quadrilateral

$$= \frac{1}{2}\begin{vmatrix} 1 & 1 \\ 7 & -3 \\ 12 & 2 \\ 7 & 21 \\ 1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\{(-3 + 14 + 252 + 7) - (21 + 14 - 36 + 7)\}$$

$$= \frac{1}{2}(270 - 6) = 132 \text{ sq.u.}$$

27. The required area of a pentagon

$$= \frac{1}{2}\begin{vmatrix} 4 & 3 \\ -5 & 6 \\ 0 & 7 \\ 3 & -6 \\ -7 & -2 \\ 4 & 3 \end{vmatrix}$$

$$= \left|\frac{1}{2}\{(24 - 35 + 0 - 6 - 21) - (-8 + 42 + 21 + 0 + 15)\}\right|$$

$$= \frac{1}{2}(38 + 70) = 54 \text{ sq. u.}$$

28. We have,

$$\frac{1}{2}\begin{vmatrix} x_1 - x_2 & x_2 - x_3 \\ y_1 - y_2 & y_2 - y_3 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} a - b & b - c \\ b + c - c - a & c + a - a - b \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} a - b & b - c \\ b - a & c - b \end{vmatrix}$$

$$= \frac{1}{2}(a - b)(b - c)\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix}$$

$= 0$

Hence, the result.

29. Now,

$$\text{ar}(\Delta DBC) = \left|\frac{1}{2}\begin{vmatrix} x+3 & -3-4 \\ 3x-3 & 5+2 \end{vmatrix}\right|$$

$$= \left|\frac{1}{2}(7x + 21 - 21x - 21)\right|$$

$$= \left|\frac{1}{2}(-14x)\right| = 7x$$

and

$$\text{ar}(\Delta ABC) = \frac{1}{2}\begin{vmatrix} 6+3 & -3-4 \\ 3-5 & 5+2 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 9 & -7 \\ -2 & 7 \end{vmatrix}$$

$$= \frac{1}{2}(63 - 14) = \frac{49}{2}$$

It is given that

$$\frac{7x}{\frac{49}{2}} = \frac{1}{2}$$

$$\Rightarrow \quad x = \frac{7}{4}$$

30. The required area

$$= \frac{1}{2 \times |C_1 C_2 C_3|}\begin{vmatrix} 7 & -2 & 10 \\ 7 & 2 & -10 \\ 9 & 1 & 2 \end{vmatrix}^2,$$

where $C_1 = \begin{vmatrix} 7 & 2 \\ 9 & 1 \end{vmatrix} = -11, \; C_2 = -\begin{vmatrix} 7 & -2 \\ 9 & 1 \end{vmatrix} = -25,$

and $\quad C_3 = -\begin{vmatrix} 7 & -2 \\ 7 & 2 \end{vmatrix} = -28$

$$= \frac{1}{2 \times 11 \times 25 \times 28}(96)^2$$

$$= \frac{9216}{2 \times 11 \times 25 \times 28}$$

$$= \frac{9216}{15400} = \frac{2304}{3850} \text{ sq. u.}$$

31. Let $x = a\cos\theta, y = a\sin\theta$
Then $x^2 + y^2 = a^2\cos^2\theta + a^2\sin^2\theta = a^2$
which is the required locus of the point P.

32. Let $x = at^2$ and $y = 2at$.
Then, $y^2 = 4a^2t^2 = 4a(at^2) = 4ax$
which is the required locus of P.

33. Let the movable point P be (x, y) and the point on y-axis be $(0, y)$.
Given condition is

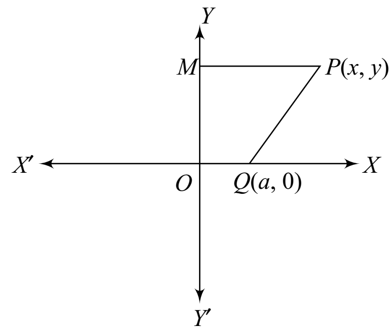
$$\sqrt{x^2 + y^2} = 3x$$

$$\Rightarrow \quad x^2 + y^2 = 9x^2$$

$$\Rightarrow \quad y^2 = 8x^2$$

which is the required locus of P.

34. Let the movable point be (x, y).



Given condition is

$$PQ = PM$$

$$\Rightarrow \quad \sqrt{(x-a)^2 + y^2} = x$$

$$\Rightarrow \quad (x-a)^2 + y^2 = x^2$$

$$\Rightarrow \quad y^2 = x^2 - (x-a)^2 = 2ax - a^2$$

which is the required locus of the given movable point.

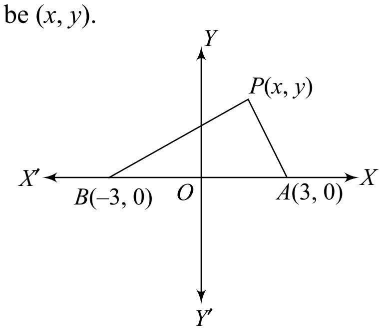
35. Let the co-ordinates of the variable point P be (x, y).

Then $x = t + \frac{1}{t}$ and $y = t - \frac{1}{t}$

$$\Rightarrow \quad x^2 - y^2 = \left(t + \frac{1}{t}\right)^2 - \left(t - \frac{1}{t}\right)^2 = 4$$

which is the required locus of P and represents a rectangular hyperbola.

36. Let P be (x, y).



Given condition is

$$PA + PB = 8.$$

$$\Rightarrow \quad \sqrt{(x)^2 + (y+3)^2} + \sqrt{(x)^2 + (y-3)^2} = 8$$

$$\Rightarrow \quad \sqrt{(x)^2 + (y+3)^2} = 8 - \sqrt{(x)^2 + (y-3)^2}$$

$$\Rightarrow \quad (x)^2 + (y+3)^2 = 64 + ((x)^2 + (y-3)^2) - 16\sqrt{(x)^2 + (y-3)^2}$$

$$\Rightarrow \quad 6y = 64 - 6y - 16\sqrt{(x)^2 + (y-3)^2}$$

$$\Rightarrow \quad 12y - 64 = 16\sqrt{(x)^2 + (y-3)^2}$$

$$\Rightarrow \quad 3y - 16 = 4\sqrt{(x)^2 + (y-3)^2}$$

$$\Rightarrow \quad (3y - 16)^2 = 16((x)^2 + (y-3)^2)$$

$$\Rightarrow \quad 9y^2 - 96y + 256 = 16(x^2 + y^2 - 6y + 9)$$

$$\Rightarrow \quad 16x^2 + 7y^2 = 256 - 144 = 112$$

$$\Rightarrow \quad \frac{x^2}{7} + \frac{y^2}{16} = 1$$

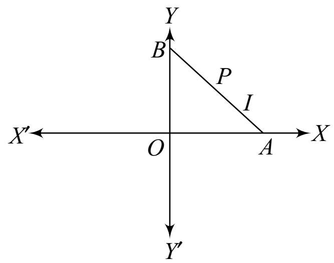
which is the required locus of the given point and represents an ellipse.

37. Let $AB = l$.

Consider the point A lies on x-axis and B lies on y-axis such that

$A = (a, 0)$ and

$B = (0, b)$



Therefore, $AB = l$

$\Rightarrow \quad \sqrt{a^2 + b^2} = l$

$\Rightarrow \quad a^2 + b^2 = l^2 \qquad \ldots\text{(i)}$

Let the mid-point of A and B be (α, β)

Thus, $\alpha = \dfrac{a}{2}, \beta = \dfrac{b}{2}$

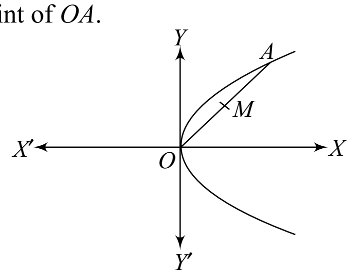
$\Rightarrow \quad a = 2\alpha, b = 2\beta$

From Eq. (i), we get,

$$\alpha^2 + \beta^2 = \frac{l^2}{4}$$

Hence, the locus of (α, β) is $x^2 + y^2 = \dfrac{l^2}{4}$.

38. Let the point M be (α, β) and $A(2\alpha, 2\beta)$ where M is the mid-point of OA.



Since A lies on the curve $y^2 = 8x$, so

$(2\beta)^2 = 8(2\alpha)$

$\Rightarrow \quad 4\beta^2 = 16\alpha$

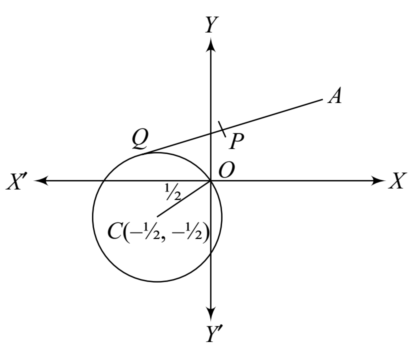
$\Rightarrow \quad \beta^2 = 4\alpha$

Hence, the required locus of M is $y^2 = 4x$.

39. We have

$x^2 + y^2 + x + y = 0$

$\Rightarrow \quad \left(x + \dfrac{1}{2}\right)^2 + \left(y + \dfrac{1}{2}\right)^2 = \left(\dfrac{1}{\sqrt{2}}\right)^2$



Let the point Q be

$$\left(-\frac{1}{2} + \frac{1}{\sqrt{2}}\cos\theta, -\frac{1}{2} + \frac{1}{\sqrt{2}}\sin\theta\right)$$

and P be (α, β)

Then $\alpha = \dfrac{1}{4} + \dfrac{1}{2\sqrt{2}}\cos\theta, \beta = \dfrac{3}{4} + \dfrac{1}{2\sqrt{2}}\sin\theta$

Now, $8\left(\alpha - \dfrac{1}{4}\right)^2 + 8\left(\beta - \dfrac{3}{4}\right)^2$

$= \cos^2\theta + \sin^2\theta = 1$

$\Rightarrow \quad 8\left(\alpha^2 + \beta^2 - \dfrac{\alpha}{2} - \dfrac{3\beta}{2}\right) + \dfrac{1}{2} + \dfrac{9}{2} - 1 = 0$

$\Rightarrow \quad 8\left(\alpha^2 + \beta^2 - \dfrac{\alpha}{2} - \dfrac{3\beta}{2}\right) + 4 = 0$

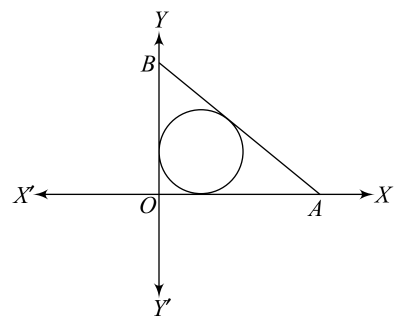
$\Rightarrow \quad 2(\alpha^2 + \beta^2) - \alpha - 3\beta + 1 = 0$

Hence, the locus of (α, β) is

$2(x^2 + y^2) - x - y + 1 = 0$,

40. Let $OA = a$ and $OB = b$.

Since the circle touches both the axes and the line AB, so the inradius of ΔOAB be 2.



From trigonometry, we can write

$r = \dfrac{\Delta}{s}, \qquad \ldots\text{(i)}$

where Δ = area of a triangle

and s = semi-perimeter of the triangle.

Let (x, y) be the circumcentre of ΔOAB, then $x = \dfrac{a}{2}$ and $y = \dfrac{b}{2}$

Thus, $a = 2x$ and $b = 2y$.

Putting the values of a and b in Eq. (i), we get

$$2 = \frac{\frac{1}{2}\cdot(2x)\cdot(2y)}{\left(\dfrac{2x + 2y + \sqrt{4x^2 + 4y^2}}{2}\right)}$$

$\Rightarrow \quad xy = x + y + \sqrt{x^2 + y^2}$

is the required locus.

44. $x^2 + y^2 = 4$

45. $x^2 + y^2 = 3$

46. Given curve is

$2x^2 + y^2 - 3x + 5y - 8 = 0 \qquad \text{(i)}$

Replacing x by $x - 1$ and y by $y + 2$ in Eq. (i), we get

$2(x - 1)^2 + (y + 2)^2 - 3(x - 1) + 5(y + 2) - 8 = 0$

$\Rightarrow \quad 2(x^2 - 2x + 1) + (y^2 + 4y + 4)$

$-3(x - 1) + 5(y + 2) - 8 = 0$

$\Rightarrow \quad 2x^2 + y^2 - 7x + 9y + 11 = 0$

47. Given curve is

$$x^2 + y^2 = 36 \quad \ldots(i)$$

Replacing x by $x - 4$ and y by $y - 5$ in Eq. (i), we get

$$(x - 4)^2 + (y - 5)^2 = 36$$

$\Rightarrow \quad x^2 + y^2 - 8x - 10y + 16 + 25 - 36 = 0$

$\Rightarrow \quad x^2 + y^2 - 8x - 10y + 5 = 0$

which is the required equation of the original axes.

48. Let the point be (x, y).

Thus $x - 1 = -7$, $y + 8 = 3$

$\Rightarrow \quad x = -6, y = -5$

Hence, the point be $(-6, -5)$.

49. Given equation is

$$3x^2 + 2xy + 3y^2 = 2 \quad (i)$$

Replacing x by $(x \cos 45° - y \sin 45°)$ and y by $(x \cos 45° + y \sin 45°)$ in Eq. (i), we get

$$3\left(\frac{x-y}{\sqrt{2}}\right)^2 + 3\left(\frac{x+y}{\sqrt{2}}\right)^2 + 2\left(\frac{x-y}{\sqrt{2}}\right)\left(\frac{x+y}{\sqrt{2}}\right) = 2$$

$\Rightarrow \quad 3(x^2 + y^2) + (x^2 - y^2) = 2$

$\Rightarrow \quad 4x^2 + 2y^2 = 2$

$\Rightarrow \quad 2x^2 + y^2 = 1$

50. Given equation is

$$y^2 - 4x + 4y + 8 = 0 \quad (i)$$

Replacing x by $x + 1$ and y by $y - 2$ in Eq. (i), we get

$$(y - 2)^2 - 4(x + 1) + 4(y - 2) + 8 = 0$$

$\Rightarrow \quad y^2 - 4y + 4 - 4x - 4 + 4y - 8 + 8 = 0$

$\Rightarrow \quad y^2 = 4x$

which is the required transformed equation.

51. If the point $P(1, 2)$ is translated itself 2 units along the positive direction of x-axis, then P becomes $(3, 2)$.

Let $Z = (3, 2) = 3 + 2i$

Thus, the new position of $P = iZ$

$$= i(3 + 2i)$$
$$= -2 + 3i = (-2, 3)$$

52. 12
53. $x^2 - y^2 = a^2$
54. $4x^2 + 2y^2 = 1$
55. $(-3, 2)$
56. $\left(\frac{1-4\sqrt{3}}{2}, \frac{4+\sqrt{3}}{2}\right)$

Level I

(STRAIGHT LINE)

1. Hence, the required slope of $PQ = \frac{y_2 - y_1}{x_2 - x_1} = \frac{10-4}{3-2} = 6$
2. Given slope = 2

$\Rightarrow \quad \frac{7-5}{\lambda - 2} = 2$

$\Rightarrow \quad \lambda - 2 = 1$

$\Rightarrow \quad \lambda = 3$

3. Let m_1 be the line joining the points $(2, -3)$ and $(-5, 1)$ and m_2 be the line joining the points $(7, -1)$ and $(0, 3)$.

Thus $m_1 = \frac{1+3}{-5-2} = -\frac{4}{7}$ and $m_2 = \frac{3+1}{0-7} = -\frac{4}{7}$.

Since, $m_1 = m_2$, so the slopes are parallel.

4. Consider the points $P = (a, b + c)$, $Q = (b, c + a)$, $R = (c, a + b)$

Therefore, $m(PQ) = \frac{c+a-b-c}{b-a} = \frac{a-b}{b-a} = -1$

and $m(QR) = \frac{a+b-c-a}{c-b} = \frac{b-c}{c-b} = -1$

Since, $m(PQ) = m(QR)$

$\Rightarrow$ Thus, the points P, Q and R are collinear.

5. Let m_1 and m_2 be the slopes of the line joining the points $(0, 0)$, $(2, 2)$ and $(2, -2)$, $(3, 5)$.

Therefore, $m_1 = \frac{2-0}{2-0} = 1$ and $m_2 = \frac{5+2}{3-2} = 7$

Let θ be the angle between them

$$\tan\theta = \left|\left(\frac{7-1}{1+7.1}\right)\right|$$

$\Rightarrow \quad \tan\theta = \frac{3}{4}$

$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{3}{4}\right)$

6. Let m be the slope of the other line.

Given,

$$\tan 45° = \left|\frac{m - \frac{1}{2}}{1 + \frac{m}{2}}\right|$$

$\Rightarrow \quad \tan 45° = \left|\frac{2m-1}{2+m}\right|$

$\Rightarrow \quad (2m - 1) = \pm (2 + m)$

$\Rightarrow \quad (2m - 1) = (2 + m)$

and $\quad (2m - 1) = -(2 + m)$

$\Rightarrow \quad m = 3$ and $m = -\frac{1}{3}$

Hence, the slopes of the other line are 3 or $-1/3$.

7. Equation of a line parallel to x-axis is $y = k$.

Which is passing through $(2, -3)$, we have, $k = -3$.

Hence, the equation of the line is $y + 3 = 0$.

8. Equation of a line perpendicular to y-axis is $y = k$ which is passing through the point $(3, 4)$, we have, $k = 4$.

Hence, the equation of the line is $y = 4$.

9. Hence, the equation of a line, which is equidistant from the lines $x = 6$ and $x = 10$ is $x = \frac{6+10}{2} = 8$

10. Given $m = 3$ and $c = 7$.

Hence, the required equation of the line is

$$y = mx + c = 3x + 7$$

11. Here, $m = \tan(135°) = -1$ and $c = 5$.

Hence, the required equation of the given line is

$$y = mx + c = -x + 5$$

12. Given,

$$\theta = \tan^{-1}\left(\frac{3}{5}\right)$$

$$\Rightarrow \quad \tan\theta = \frac{3}{5}$$

$$\Rightarrow \quad m = \frac{3}{5} \text{ and } c = -6$$

Hence, the equation of the given straight line is

$$y = mx + c = \frac{3}{5}x - 6$$

13. Given $m = \pm 1$ and $c = 4$.

Hence, the equation of the line is

$$y = mx + c = \pm x + 4$$

14. Here, $m = \tan(45°) = 1$

and $m = \tan(135°) = 1$ and $c = 0$

Hence, the equation of the bisectors is

$$y = mx + c = \pm x$$

15. Here, $m = \tan(120°) = -\sqrt{3}$ and the point $(x_1, y_1) = (2, 3)$

Hence, the equation of a straight line is

$$y - y_1 = m(x - x_1)\alpha$$

$$\Rightarrow \quad y - 3 = -\sqrt{3}(x - 2)$$

$$\Rightarrow \quad \sqrt{3}x + y = 2\sqrt{3} + 3$$

16. Let the points A and B are (1, 2) and (5,7), respectively.

Therefore, $m(AB) = \dfrac{7-2}{5-1} = \dfrac{5}{4}$

and the mid-point of A and B are

$$\left(\frac{1+5}{2}, \frac{2+7}{2}\right) = \left(3, \frac{9}{2}\right)$$

Hence, the equation of the right (perpendicular) bisector is

$$y - \frac{9}{2} = -\frac{4}{5}(x - 3)$$

$$\Rightarrow \quad 8x + 10y = 69$$

17. Given,

$$\cos\theta = -\frac{3}{5}$$

$$\Rightarrow \quad \tan\theta = -\frac{4}{5}$$

$$\Rightarrow \quad m = -\frac{4}{5}$$

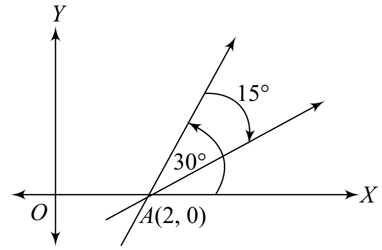
Hence, the equation of the given line is

$$y - 2 = -\frac{4}{5}(x - 1)$$

$$\Rightarrow \quad 4x + 5y = 14$$

18. We have,

$$m = \tan(15°) = (2 - \sqrt{3})$$



Hence, the required equation of the line is

$$y - 0 = (2 - \sqrt{3})(x - 2)$$

$$\Rightarrow \quad (2 - \sqrt{3})x - y - 2(2 - \sqrt{3}) = 0$$

19. Hence, the equation of a line is

$$\begin{vmatrix} x - x_1 & y - y_1 \\ x_1 - x_2 & y_1 - y_2 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x - 1 & y - 2 \\ 3 - 1 & 4 - 2 \end{vmatrix} = 0$$

$$\Rightarrow \quad 2(x - 1) - 2(y - 2) = 0$$

$$\Rightarrow \quad x - y + 1 = 0.$$

20. Here slope of $BC = \dfrac{-1-9}{-2+4} = -\dfrac{10}{2} = -5$

Hence, the equation of the altitude through A is

$$y - 4 = \frac{1}{5}(x - 10)$$

$$\Rightarrow \quad x - 5y + 10 = 0$$

21. The co-ordinates of the mid points of B and C are $\left(\dfrac{2+5}{2}, \dfrac{3+4}{3}\right) = \left(\dfrac{7}{2}, \dfrac{7}{2}\right)$

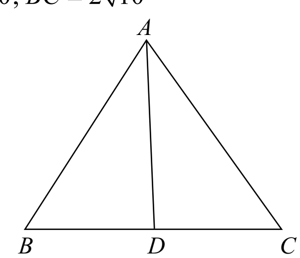
Hence, the required equation of the median through A is

$$\begin{vmatrix} x - 1 & y - 2 \\ \frac{7}{2} - 1 & \frac{7}{2} - 2 \end{vmatrix} = 0$$

$$\Rightarrow \quad 3(x - 1) - 5(y - 2) = 0$$

$$\Rightarrow \quad 3x - 5y + 7 = 0$$

22. Here, $AB = \sqrt{10}$, $AC = \sqrt{10}$, $BC = 2\sqrt{10}$



Let AD is the internal bisector of the angle $\angle BAC$. If AD is the internal bisector of the angle $\angle BAC$, then

$$\frac{BD}{DC} = \frac{AB}{AC} = \frac{\sqrt{10}}{\sqrt{10}} = \frac{1}{1}$$

$$\Rightarrow \quad BD : DC = 1 : 1$$

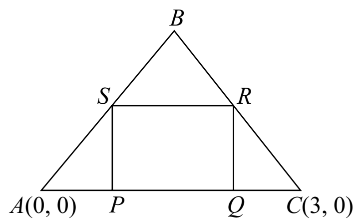
Thus D is the mid-point of B and C, i.e. $D = (4, 4)$

Hence, the equation of the internal bisector is

$$y - 2 = \frac{4-2}{4-5}(x - 5) = -2(x - 5)$$

$$\Rightarrow \quad 2x + y = 12$$

23. Let the square be $PQRS$, whose length of the side is k. Consider the point of P be $(a, 0)$.



Then $Q = (a + k, 0)$, $R = (a + k, k)$ and $S = (a, k)$.
The equation of the line AB is

$$y - 0 = \frac{1-0}{2-0}(x-0)$$

$\Rightarrow \quad x - 2y = 0 \quad \ldots$(i)

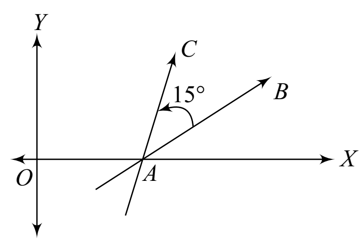
and the equation of the line BC is

$$y - 1 = \frac{0-1}{3-2}(x-2)$$

$\Rightarrow \quad x + y = 3 \quad \ldots$(ii)

Since, the point $S(a, k)$ on Eq. (i), we get,
$a = 2k$
and the point $R(a + k, k)$ on Eq. (ii), we get,
$a + 2k = 3$.
Thus, $k = 3/4$ and $a = 3/2$.
Hence, the co-ordinates of P, Q, R and S are
(3/2, 0), (9/4, 0), (9/4, 3/4) and (3/2, 3/4).

24. Let slope of AC is m. The slope of AB is 1. We have



$$\tan(15°) = \left|\frac{m-1}{1+m}\right|$$

$\Rightarrow \quad (2-\sqrt{3})m + (2-\sqrt{3}) = m - 1$

$\Rightarrow \quad (1-\sqrt{3})m = (-3+\sqrt{3})$

$\Rightarrow \quad (1-\sqrt{3})m = (-3+\sqrt{3}) = \sqrt{3}(1-\sqrt{3})$

$\Rightarrow \quad m = \sqrt{3}$

Hence, the equation of the line AC is

$$y - 0 = \sqrt{3}(x-2)$$

$\Rightarrow \quad x\sqrt{3} - y - 2\sqrt{3} = 0$

25. Let the equation of the line is $\frac{x}{a} + \frac{y}{b} = 1$.
Since, the line makes equal intercepts with the axes, so $a = b$.
Thus $x + y = a$ which is passing through (3, 4).
Therefore $a = 7$.
Hence, the equation of the required line is $x + y = 7$.

26. Let the equation of the line is $\frac{x}{a} + \frac{y}{b} = 1 \quad \ldots$(i)
Given condition is $a = 2b \quad \ldots$(ii)
From Eqs (i) and (ii), we get,
$2x + y = 2b$
Which is passing through the point (2, 3).
Therefore, $2b = 4 + 3 = 7$

$\Rightarrow \quad b = \frac{7}{2}$

Hence, the equation of the line is $2x + y = 7$.

27. Let the equation of the line be $\frac{x}{a} + \frac{y}{b} = 1$,
where $A = (a, 0)$ and $B = (0, b)$.
Consider the given line is bisected at the point $M(2, 3)$.

Thus, $2 = \frac{a}{2} \Rightarrow a = 4$

and $3 = \frac{b}{2} \Rightarrow b = 6$

Hence, the equation of the line is

$$\frac{x}{4} + \frac{y}{6} = 1.$$

$\Rightarrow \quad 3x + 2y = 12$

28. Let the equation of the line be $\frac{x}{a} + \frac{y}{b} = 1$, where $A = (a, 0)$ and $B = (0, b)$
Given condition is $a = -b$
Therefore, $x - y = a$ which is passing through (3, –4)
Thus, $a = 7$.
Hence, the equation of the line is $x - y = 7$.

29. Let the equation of the line is $\frac{x}{a} + \frac{y}{b} = 1$
which is passing through (3, 2), so

$$\frac{3}{a} + \frac{2}{b} = 1 \quad \ldots\text{(i)}$$

Also given condition is $a - b = 2 \quad \ldots$(ii)
From Eq. (i) and (ii), we get
$3b + 2a = ab = b(b + 2)$
$\Rightarrow \quad 3b + 2b + 4 = b^2 + 2b$
$\Rightarrow \quad b^2 - 3b - 4 = 0$
$\Rightarrow \quad (b - 4)(b + 1) = 0$
$\Rightarrow \quad b = -1, 4$
Thus, $a = 1, 6$.
Hence, the equation of the line are
$x - y = 1$, and $2x - 3y = 12$.

30. Let the equation of the line AB is

$$2x + 3y = 6 \quad \Rightarrow \quad \frac{x}{3} + \frac{y}{2} = 1$$

Given lines are $x\,y = 0$
$\Rightarrow \quad x = 0, y = 0$

Hence, the area of a triangle $= \frac{1}{2} \cdot 3 \cdot 2 = 3$ sq. u.

31. $2x + y = 10$.
32. $3x - 8y = 0$, $3x - 2y = 0$.
33. Do yourself
34. Do yourself.

35. Here, $p = 5$ and $\alpha = 60°$

Hence, the equation of the line is

$x\cos\alpha + y\sin\alpha = p$

$\Rightarrow \quad x\cos(60°) + y\sin(60°) = 5$

$\Rightarrow \quad x + \sqrt{3}y = 10$

36. Here, $p = 2$ and $\tan\theta = \dfrac{5}{12}$

$\Rightarrow \quad \sin\theta = \dfrac{5}{13}$ and $\cos\theta = \dfrac{12}{13}$

Hence, the equation of the line is

$x\cos\theta + y\sin\theta = p$

$\Rightarrow \quad 12x + 5y = 26.$

37. Hence, the co-ordinates of the required points are

$(x_1 \pm r\cos\theta, y_1 \pm r\sin\theta)$.

Here $r = 5$, $(x_1, y_1) = (3, 2)$

and $\quad \tan\theta = \dfrac{3}{4} \Rightarrow \cos\theta = \dfrac{4}{5}, \sin\theta = \dfrac{3}{5}$

Thus, the points are

$(x_1 \pm r\cos\theta, y_1 \pm r\sin\theta)$

$= \left(3 \pm 5\cdot\dfrac{4}{5}, 2 \pm 5\cdot\dfrac{3}{5}\right)$

$= (3 \pm 4, 2 \pm 3)$

$= (7, 5), (-1, -1)$

38. Here, $r = 4\sqrt{2}$, $(x_1, y_1) = (-2, 3)$ and $\theta = 45°$.

Hence, the co-ordinates of the points are

$(x_1 \pm r\cos\theta, y_1 \pm r\sin\theta)$

$= \left(-2 \pm 4\sqrt{2}\cdot\dfrac{1}{\sqrt{2}}, 3 \pm 4\sqrt{2}\cdot\dfrac{1}{\sqrt{2}}\right)$

$= (-2 \pm 4, 3 \pm 4)$

$= (2, 7)$ and $(-6, -1)$

39. Given point P is $(3, 4)$.

Here, $r = \sqrt{2}$ and $\theta = 45°$

Let Q be the new position of P.

Hence, the co-ordinates of Q are

$$\left(3 + \sqrt{2}\cdot\frac{1}{\sqrt{2}}, 4 + \sqrt{2}\cdot\frac{1}{\sqrt{2}}\right) = (4, 5)$$

40. Let the new position of B is C.

Here, $r = AB = AC = \sqrt{(3-2)^2 + (1-0)^2} = \sqrt{2}$ and

$\theta = 45° + 15° = 60°$

Thus, the co-ordinates of C are

$$(2 + \sqrt{2}\cos(60°), 0 + \sqrt{2}\sin(60°))$$

$$= \left(2 + \sqrt{2}\cdot\frac{1}{2}, \sqrt{2}\cdot\frac{\sqrt{3}}{2}\right) = \left(2 + \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\right)$$

41. Let the point P be $(1, 2)$ and the line PQ makes an angle θ with the positive direction of x-axis.

Here, $r = \sqrt{\dfrac{2}{3}}$ and $(x_1, y_1) = (1, 2)$

Thus, the co-ordinates of Q are

$$\left(1 + \sqrt{\frac{2}{3}}\cos\theta, 2 + \sqrt{\frac{2}{3}}\sin\theta\right)$$

Since, the point Q lies on $x + y = 4$, so

$$1 + \sqrt{\frac{2}{3}}\cos\theta + 2 + \sqrt{\frac{2}{3}}\sin\theta = 4$$

$\Rightarrow \quad \sqrt{\dfrac{2}{3}}\cos\theta + \sqrt{\dfrac{2}{3}}\sin\theta = 1$

$\Rightarrow \quad \cos\theta + \sin\theta = \sqrt{\dfrac{3}{2}}$

$\Rightarrow \quad \dfrac{1}{\sqrt{2}}\cos\theta + \dfrac{1}{\sqrt{2}}\sin\theta = \dfrac{\sqrt{3}}{2}$

$\Rightarrow \quad \left(\theta - \dfrac{\pi}{4}\right) = \pm\left(\dfrac{\pi}{6}\right)$

$\Rightarrow \quad \theta = \dfrac{\pi}{4} \pm \dfrac{\pi}{6} = 75°, 15°$.

42. $R(-2, -1)$, $Q(-1, 2)$ and $S(1, -2)$

43. $\left(-\dfrac{3}{2}, \dfrac{3}{2}\right)$ and $\left(\dfrac{1}{2}, -\dfrac{3}{2}\right)$

44 $\sqrt{2}$

45. 6

46. $4\sqrt{2}$

47. $\sqrt{5}$

48. 5.

49. $(8, 2)$ and $(0, -4)$

50. Given line is $x + \sqrt{3}y + 4 = 0$

(i) slope intercept form is

$\Rightarrow \quad y = \left(-\dfrac{1}{\sqrt{3}}\right)x + \left(-\dfrac{4}{\sqrt{3}}\right)$

Thus, slope $= -\dfrac{1}{\sqrt{3}}$

and y-intercept $= -\dfrac{4}{\sqrt{3}}$

(ii) Intercept form of $x + \sqrt{3}y + 4 = 0$ is

$$\frac{x}{(-4)} + \frac{y}{\left(-\frac{4}{\sqrt{3}}\right)} = 1$$

where x-intercept = 4 and

y-intercept $= 4/\sqrt{3}$.

(iii) Normal form of $x + \sqrt{3}y + 4 = 0$ is

$-x - \sqrt{3}y = 4$

$\Rightarrow \quad \left(-\dfrac{1}{4}\right)x + \left(\dfrac{-\sqrt{3}}{4}\right)y = 1$

$\Rightarrow x\cos\alpha + y\sin\alpha = 1,$

where $\alpha = \pi + \tan^{-1}(\sqrt{3}) = 240°$

51. (i) Let $(x_1, y_1) = (2, 2)$ and $(x_2, y_2) = (3, 5)$ and $a = 2$, $b = 3$, $c = 4$.

Now, $\dfrac{ax_1 + by_1 + c}{ax_2 + by_2 + c} = \dfrac{2.2 + 3.2 + 4}{2.3 + 3.5 + 4}$

$= \dfrac{14}{25} > 0$

Thus, the points (2, 2) and (3, 5) lie on the same side of the line $2x + 3y + 4 = 0$.

(ii) Do yourself

(iii) Do yourself.

52. As we know that, if $ax_1 + by_1 + c$ and c have the same sign, then the point (x_1, y_1) lies on the origin side of the line $ax_1 + by_1 + c$.

Here, $2 > 0$ and

$ax_1 + by_1 + c = 2.\ 2 - 7 + 2 = 6 - 7 = -1 < 0$

Thus, the point (2, –7) does not lie on the origin side of the line $2x + y + 2 = 0$.

53. Do yourself.

54. Equation of a line parallel to

$3x + 4y + 5 = 0$ is $3x + 4y + k = 0$.

which passes through (2, 3).

Therefore, $6 + 12 + k = 0$

$\Rightarrow \quad k = -18$

Hence, the equation of the line is

$3x + 4y - 18 = 0$.

55 The co-ordinates of the mid-point of the line joining the points (2, 3) and (4, –1) is (3, 1).

Equation of any line parallel to $3x - 4y + 6 = 0$ is

$3x - 4y + k = 0$

which is passing through (3, 1).

Therefore, $9 - 4 + k = 0$

$\Rightarrow \quad k = -5$

Hence, the equation of the line is

$3x - 4y - 5 = 0$

56. The slope of the line joining the points (2, 3) and (3, – 1) is $\dfrac{-1-3}{3-2} = -4$

Equation of the line parallel to the given line and passing through (2, 1) is

$y - y_1 = m(x - x_1)$

$\Rightarrow \quad y - 1 = -4(x - 2)$

$\Rightarrow \quad 4x + y = 9$

57. Do yourself.

58. Equation of any line perpendicular to

$2x + 3y - 2012 = 0$ is

$3x - 2y + k = 0$

which is passing through (3, 4).

Therefore, $9 - 8 + k = 0$

$\Rightarrow \quad k = -1$

Hence, the equation of the line is

$3x - 2y - 1 = 0$.

59. Equation of any line perpendicular to $2x - 3y - 5 = 0$ is $3x + 2y - k = 0$ which is passing through (1, 0).

Therefore, $3 + 0 + k = 0$

$\Rightarrow \quad k = -3$

Hence, the equation of the line is

$3x + 2y + 3 = 0$.

60. The slope of the line joining the points (1, 2) and (3, 5) is $\dfrac{5-2}{3-1} = \dfrac{3}{2}$.

The mid-point of the line joining the points (1, 2) and (3, 5) is $\left(\dfrac{1+3}{2}, \dfrac{2+5}{2}\right) = \left(2, \dfrac{7}{2}\right)$.

Hence, the equation of the line is

$y - \dfrac{7}{2} = -\dfrac{2}{3}(x - 2)$

$\Rightarrow \quad 4x + 6y = 29$

61. Slope of $BC = \dfrac{8-5}{3-4} = -3$

Let the altitude through A is AD.

Therefore, the slope of AD is $\dfrac{1}{3}$.

Hence, the equation of the altitude through A is

$y - 2 = \dfrac{1}{3}(x - 1)$

$\Rightarrow \quad x - 3y + 5 = 0$

62. The point of intersection of $2x + y = 8$ and $x - y = 10$ is (6, – 4).

Equation of any line perpendicular to

$3x + 4y + 2012 = 0$ is

$4x - 3y + k = 0$

which is passing through (6, – 4).

Therefore, $24 + 12 + k = 0 \Rightarrow k = -36$

Hence, the equation of the line is

$4x - 3y - 36 = 0$.

63. Hence, the distance of the point (4, 5) from the straight line $3x - 5y + 7 = 0$ is

$$\left|\frac{3 \times 4 - 5 \times 5 + 7}{\sqrt{3^2 + 5^2}}\right| = \left|\frac{12 - 25 + 7}{\sqrt{34}}\right| = \frac{6}{\sqrt{34}}.$$

64. Let ABC be an equilateral triangle, where A is (2, –1) and BC is $x + y = 2$.

Let AD be the length of perpendicular from A on BC.

Thus, $AD = \left|\dfrac{2 - 1 - 2}{\sqrt{1^2 + 1^2}}\right| = \dfrac{1}{\sqrt{2}}$

Therefore,

$\sin(60°) = \dfrac{AD}{AB}$

$\Rightarrow \quad AB = \dfrac{2}{\sqrt{3}} \times \dfrac{1}{\sqrt{2}} = \sqrt{\dfrac{2}{3}}$

Thus, the length of the side $= \sqrt{\dfrac{2}{3}}$

65. Equation of a straight line, whose intercepts are a and b is $\frac{x}{a}+\frac{y}{b}=1$.

Given, the length of the perpendicular from the origin $= p$.

$$\Rightarrow \quad \left|\frac{0+0-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}\right| = p$$

$$\Rightarrow \quad \frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{p^2}$$

66. Clearly, both the lines are parallel

Hence, the required distance

$$=\left|\frac{\frac{11}{2}+5}{\sqrt{3^2+4^2}}\right|=\left|\frac{11+10}{2.5}\right|=\frac{21}{10}=2\frac{1}{10} \text{ units.}$$

67. Equation of the line L is $\frac{x}{a}+\frac{y}{b}=1$.

After rotation, the line L becomes, $\frac{x}{p}+\frac{y}{q}=1$.

Therefore, the lengths of perpendicular from the origin to both the lines are same. Thus, $\left|\frac{0+0-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}\right|=\left|\frac{0+0-1}{\sqrt{\frac{1}{p^2}+\frac{1}{q^2}}}\right|$

$$\Rightarrow \quad \frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{p^2}+\frac{1}{q^2}$$

Hence, the result.

68. As we know that, area of a parallelogram whose sides are $a_1x+b_1y+d_1=0$, $a_2x+b_2y+c_2=0$ and $a_2x+b_2y+d_2=0$ is

$$=\left|\frac{(c_1-d_1)(c_2-d_2)}{(a_1b_2-a_2b_1)}\right|$$

$$=\left|\frac{(4a+a)\times(7a-3a)}{(1\times(-2)-3\times 3)}\right|=\frac{20a^2}{11}$$

Thus, $m=\frac{20}{11}$.

Now $11m+30=20+30=50$

69. Given four lines are $ax+by+c=0$,

$ax+by-c=0$, $ax-by+c=0$

and $ax-by-c=0$.

Hence, the area $=\left|\frac{(c+c)(-c-c)}{(a\cdot(-b)-b\cdot a)}\right|$

$$=\left|\frac{4c^2}{2ab}\right|=\left|\frac{2c^2}{ab}\right|=\frac{2c^2}{|ab|}$$

70. Hence, the points of intersection of the given lines is (2, 6).

71. As we know that, the three lines $a_ix+b_iy+c_i=0$, $i=1, 2, 3$ are concurrent, if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$=\begin{vmatrix} 2 & -3 & 5 \\ 3 & 4 & -7 \\ 9 & -5 & 8 \end{vmatrix}$$

$= 2(32-35)+3(24+63)+5(-15-36)$

$= -6+261-255$

$= 0$

Hence, the three lines are concurrent.

72. Equation of any line passing through the point of intersection of the lines

$x+3y-8=0$ and $2x+3y+5=0$ is

$(x-3y+8)+\lambda(2x+3y+5)=0$ …(i)

which is passing through (1, 2).

Therefore, $(1-6+8)+\lambda(2+6+5)=0$

$$\Rightarrow \quad \lambda=-\frac{3}{13}$$

From Eq. (i), we get

$$(x-3y+8)-\frac{3}{13}(2x+3y+5)=0$$

$\Rightarrow \quad 13(x-3y+8)-3(2x+3y+5)=0$

$\Rightarrow \quad 7x-48y+89=0$

73. On solving $y=x+1$ and $2x+y=16$, we get,

$x=5$ and $y=6$.

Thus, the point of concurrency is (5, 6)

Since, the given lines are concurrent, so

$6=5m-4$

$\Rightarrow \quad 5m=10$

$\Rightarrow \quad m=2$

Hence, the value of m is 2.

74. Since the given lines $ax+y+1=0$, $x+by+1=0$ and $x+y+c=0$ are concurrent,

$$\text{so} \quad \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix}=0$$

$$\Rightarrow \quad \begin{vmatrix} a-1 & 0 & 1 \\ 0 & b-1 & 1 \\ 1-c & 1-c & c \end{vmatrix}=0$$

$\Rightarrow \quad (a-1)(c(b-1)-(1-c))-(1-c)(b-1)=0$

$\Rightarrow \quad c(a-1)(b-1)-(a-1)(1-c)-(1-c)(b-1)=0$

$\Rightarrow \quad c(1-a)(1-b)+(1-a)(1-c)+(1-c)(1-b)=0$

$$\Rightarrow \quad \frac{c}{1-c}+\frac{1}{1-b}+\frac{1}{1-a}=0$$

$$\Rightarrow \quad 1+\frac{c}{1-c}+\frac{1}{1-b}+\frac{1}{1-a}=1$$

$$\Rightarrow \quad \frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}=1$$

75. Given family of lines are

$$ax + by + c = 0 \quad \ldots\text{(i)}$$

and also the given condition is

$$2a + 3b + 6c = 0 \quad \ldots\text{(ii)}$$

From Eqs (i) and (ii), we get

$$ax + by - \frac{a}{3} - \frac{b}{2} = 0$$

$$\Rightarrow \quad a\left(x - \frac{1}{3}\right) + b\left(y - \frac{1}{2}\right) = 0$$

$$\Rightarrow \quad \left(x - \frac{1}{3}\right) + \frac{b}{a}\left(y - \frac{1}{2}\right) = 0$$

$$\Rightarrow \quad \left(x - \frac{1}{3}\right) = 0 \text{ and } \left(y - \frac{1}{2}\right) = 0$$

$$\Rightarrow \quad x = \frac{1}{3} \text{ and } y = \frac{1}{2}$$

78. We have $4a^2 + 9b^2 - c^2 + 12ab = 0$

$\Rightarrow \quad (4a^2 + 9b^2 + 12ab) - c^2 = 0$

$\Rightarrow \quad (2a^2 + 3b^2) - c^2 = 0$

$\Rightarrow \quad (2a + 3b + c)(2a + 3b - c) = 0$

$\Rightarrow \quad (2a + 3b + c) = 0$ and $(2a + 3b - c) = 0$

Given family of lines are

$$ax + by + c = 0 \quad \ldots\text{(i)}$$

and also thegiven conditions are

$$(2a + 3b + c) = 0 \quad \ldots\text{(ii)}$$

and $\quad (2a + 3b - c) = 0 \quad \ldots$(iii)

From Eqs (i) and (ii), we get

$$ax + by - 2a - 3b = 0$$

$\Rightarrow \quad a(x - 2) + b(y - 3) = 0$

$$\Rightarrow \quad (x - 2) + \frac{b}{a}(y - 3) = 0$$

$\Rightarrow \quad x = 2, y = 3$

$\Rightarrow \quad m = 2, n = 3$

From Eqs (i) and (iii), we get

$$ax + by + 2a + 3b = 0$$

$\Rightarrow \quad a(x + 2) + b(y + 3) = 0$

$$\Rightarrow \quad (x + 2) + \frac{b}{a}(y + 3) = 0$$

$\Rightarrow \quad x = -2, y = -3$

$\Rightarrow \quad p = -2, q = -3$

Thus, the value of

$m + n + p + q + 10 = 2 + 3 - 2 - 3 + 10 = 10$

81. Let m be the slope of the given line and the slope of $3x - y + 5 = 0$ is 3.

Therefore,

$$\tan 45° = \left|\frac{m - 3}{1 + 3m}\right|$$

$$\Rightarrow \quad \frac{m - 3}{1 + 3m} = \pm 1$$

$$\Rightarrow \quad \frac{m - 3}{1 + 3m} = 1 \text{ and } \frac{m - 3}{1 + 3m} = -1$$

$\Rightarrow \quad 2m = -4$ and $4m = 2$

$$\Rightarrow \quad m = -2 \text{ and } m = \frac{1}{2}$$

82. Therefore,

$$\tan(45°) = \left|\frac{m - \frac{1}{2}}{1 + \frac{m}{2}}\right|$$

$$\Rightarrow \quad \left|\frac{2m - 1}{2 + m}\right| = 1$$

$$\Rightarrow \quad \frac{2m - 1}{2 + m} = 1 \text{ and } \frac{2m - 1}{2 + m} = -1$$

$$\Rightarrow \quad m = 3 \text{ and } m = -\frac{1}{3}$$

Hence, the equation of the lines are

$$y - 2 = 3(x - 3) \text{ and } y - 2 = -\frac{1}{3}(x - 3)$$

$\Rightarrow \quad 3x - y - 7 = 0$ and $x + 3y - 9 = 0$

83. Let ABC be an equilateral triangle, where A is (2, 3) and BC is $x + y = 2$.

Equation of any line passing through (2, 3) is

$$y - 3 = m(x - 2).$$

The slope of the line $x + y = 2$ is (– 1).

Therefore,

$$\tan(60°) = \left|\frac{m + 1}{1 - m}\right|$$

$$\Rightarrow \quad \frac{m + 1}{1 - m} = \pm\sqrt{3}$$

$$\Rightarrow \quad \frac{m + 1}{1 - m} = \sqrt{3} \text{ and } \frac{m + 1}{1 - m} = -\sqrt{3}$$

$$\Rightarrow \quad m(\sqrt{3} + 1) = (\sqrt{3} - 1) \text{ and } m(\sqrt{3} + 1) = (\sqrt{3} - 1)$$

$$\Rightarrow \quad m = \left(\frac{\sqrt{3} - 1}{\sqrt{3} + 1}\right) \text{ and } m = \left(\frac{\sqrt{3} + 1}{\sqrt{3} - 1}\right)$$

$$\Rightarrow \quad m = 2 - \sqrt{3} \text{ and } m = 2 + \sqrt{3}$$

Hence, the equations of the line are

$$y - 3 = (2 - \sqrt{3})(x - 2)$$

$$\text{and } y - 3 = (2 + \sqrt{3})(x - 2)$$

$$\Rightarrow \quad (2 + \sqrt{3})x - y - (2\sqrt{3} + 1) = 0 \text{ and}$$

$$(2 - \sqrt{3})x - y + (2\sqrt{3} - 1) = 0$$

84. Let AB: $4x + y = 1$,

AC: $3x - 4y + 1 = 0$

and slope of AC is m.

The slope of $AB = -4$ and the slope of $AC = 3/4$.

Let $\angle ABC = \theta = \angle ACB$

Thus, $\left(\dfrac{\frac{3}{4}-m}{1+\frac{3}{4}m}\right)=\left(\dfrac{-4-\frac{3}{4}}{1-4\cdot\frac{3}{4}}\right)$

$\Rightarrow \quad \left(\dfrac{3-4m}{4+3m}\right)=\left(\dfrac{-16-3}{4-12}\right)$

$\Rightarrow \quad \left(\dfrac{3-4m}{4+3m}\right)=\dfrac{19}{8}$

$\Rightarrow \quad 57m + 76 = 24 - 32m$

$\Rightarrow \quad 89m = 24 - 76 = -52$

$\Rightarrow \quad m=-\dfrac{52}{89}$

Hence, the equation of AC is

$$y+7=-\frac{52}{89}(x-2)$$

$\Rightarrow \quad 52x + 89y + 519 = 0$

85. Let PQ line intersects the line $4x + 3y = 12$ at A and $4x + 3y = 3$ at B, respectively.

Equation of any line PQ passing through (–2, –7) is

$y + 7 = m(x + 2)$ …(i)

Thus, $A=\left(\dfrac{33-6m}{4+3m},\dfrac{20m-28}{4+3m}\right)$

and $B=\left(\dfrac{24-6m}{4+3m},\dfrac{11m-28}{4+3m}\right)$

Given, $AB = 3 \quad \Rightarrow AB^2 = 9$

$\Rightarrow \quad \dfrac{81}{(4+3m)^2}+\dfrac{81m^2}{(4+3m)^2}=9$

$\Rightarrow \quad 9 + 9m^2 = 16 + 24m + 9m^2$

$\Rightarrow \quad 24m = -7$

$\Rightarrow \quad m = -7/24$ and $m = \infty$

Hence, the line PQ can be $x + 2 = 0$ and

$\Rightarrow \quad x + 2 = 0$ and $7x + 24y + 182 = 0$

86. Equation of any line passing through (2, 3) is

$y - 3 = m(x - 2)$ …(i)

Equation (i) is equally inclined with the lines

$3x - 4y = 7$ …(ii)

and $12x - 5y + 6 = 0$ …(iii)

Therefore, $\left(\dfrac{m-\frac{3}{4}}{1+\frac{3}{4}m}\right)=-\left(\dfrac{m-\frac{12}{5}}{1+\frac{12}{5}m}\right)$

$\Rightarrow \quad \left(\dfrac{m-\frac{3}{4}}{1+\frac{3}{4}m}\right)=-\left(\dfrac{m-\frac{12}{5}}{1+\frac{12}{5}m}\right)$

$\Rightarrow \quad \left(\dfrac{4m-3}{4+3m}\right)=\left(\dfrac{12-5m}{5+12m}\right)$

$\Rightarrow \quad 63m^2 - 32m - 63 = 0$

$\Rightarrow \quad 63m^2 - 81m + 49m - 63 = 0$

$\Rightarrow \quad 9m(7m - 9) + 7(7m - 9) = 0$

$\Rightarrow \quad (7m - 9)(9m + 7) = 0$

$\Rightarrow \quad m=\dfrac{9}{7},-\dfrac{7}{9}$

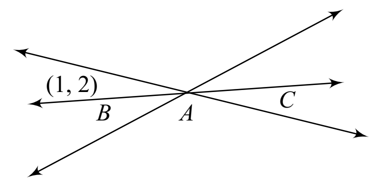
From Eqs (i), we get,

$$y-3=\frac{9}{7}(x-2) \text{ and } y-3=-\frac{7}{9}(x-2)$$

$\Rightarrow \quad 9x - 7y + 3 = 0$ and $7x + 9y = 41$

which is the equations of the lines.

87.



Let AB: $3x + 4y = 5$ …(i)

and $4x - 3y = 15$ …(ii)

On solving Eqs (i) and (ii), we get

$x = 3$ and $y = 1$

Thus, the co-ordinates of the point A be (3, 1).

Equation of any line BC which passes through the point (1, 2) is $y - 2 = m(x - 1)$ …(iii)

On solving Eqs (i) and (iii), we get the co-ordinate B, i.e.

$$B=\left(\frac{4m-3}{4m+3},\frac{2m+6}{4m+3}\right).$$

On solving Eqs (ii) and (iii), we get the co-ordinate C, i.e.

$$C=\left(\frac{21-3m}{4-3m},\frac{11m+8}{4-3m}\right).$$

Given condition is $AB = AC$

$\Rightarrow \quad AB^2 = AC^2$

$\Rightarrow \quad \left(\dfrac{21-3m}{4-3m}-3\right)^2+\left(\dfrac{8+11m}{4-3m}+1\right)^2$

$$=\left(\frac{4m-3}{4m+3}-3\right)^2+\left(\frac{6+2m}{4m+3}+1\right)^2$$

$\Rightarrow \quad \dfrac{(9+6m)^2+(12+8m)^2}{(4-3m)^2}$

$$=\frac{(-8m-12)^2+(6m+9)^2}{(4m+3)^2}$$

$\Rightarrow \quad (4m + 3)^2(100m^2 + 300m + 225)$

$= (4 - 3m)^2(100m^2 + 300m + 225)$

$\Rightarrow \quad (100m^2 + 300m + 225)$

$[(4m + 3)^2 - (4 - 3m)^2] = 0$

$\Rightarrow \quad (4m^2 + 12m + 9)[(4m + 3)^2 - (4 - 3m)^2] = 0$

$\Rightarrow \quad (2m + 3)^2(m + 7)(7m - 1) = 0$

$\Rightarrow \quad m=-\dfrac{3}{2},-7,\dfrac{1}{7}$

Put $m = -\frac{3}{2}$ in Eq. (iii), we get

$$y - 2 = -\frac{3}{2}(x - 1)$$

$\Rightarrow \quad 3x + 2y - 7 = 0$

Clearly, it passes through $A(3, -1)$.

Thus, it can not be the equation of BC.

Put $m = -7$ in Eq. (iii), we get,

$y = -7(x - 1) = -7x + 7$

Thus, the equation of BC is $7x + y = 7$.

Also, put $m = 1/7$ in Eq. (iii), we get

$$y - 2 = \frac{1}{7}(x - 1)$$

$\Rightarrow \quad x - 7y + 13 = 0$

Hence, the equations of BC can be $x - 7y + 13 = 0$ and $7x + 7 = 7$.

88. $3x + y + 7 = 0$ and $x - 3y = 31$.
Do yourself.

89. Hence, the equations of the bisectors are

$$\left|\frac{3x - 4y + 7}{\sqrt{3^2 + 4^2}}\right| = \left|\frac{12x + 5y - 2}{\sqrt{12^2 + 5^2}}\right|$$

$$\Rightarrow \left(\frac{3x - 4y + 7}{5}\right) = \pm\left(\frac{12x + 5y - 2}{13}\right)$$

$\Rightarrow \quad 13(3x - 4y + 7) = 5(12x + 5y - 2)$

$\Rightarrow \quad 13(3x - 4y + 7) = 5(12x + 5y - 2)$

and $\quad 13(3x - 4y + 7) = -5(12x + 5y - 2)$

$\Rightarrow \quad 21x + 77y = 101$ and $3x - y + 3 = 0$

90. Hence, the equation of the bisector, containing the origin is

$$\left(\frac{-4x - 3y + 6}{\sqrt{4^2 + 3^2}}\right) = \left(\frac{5x + 12y + 9}{\sqrt{5^2 + 12^2}}\right)$$

$$\Rightarrow \left(\frac{-4x - 3y + 6}{5}\right) = \left(\frac{5x + 12y + 9}{13}\right)$$

$\Rightarrow \quad 13(-4x - 3y + 6) = 5(5x + 12y + 9)$

$\Rightarrow \quad 7x + 9y = 3$

91. The given lines are $-2x - y + 6 = 0$
and $\quad 2x - 4y + 7 = 0$.
Therefore, $-2.\ 1 - 2 + 6 = -4 + 6 = 2$
and $\quad 2.\ 1 - 4.\ 2 + 7 = 9 - 8 = 1$
Thus, positive one is the equation of the bisector.
Hence, the equation of bisector is

$$\left(\frac{-2x - y + 6}{\sqrt{2^2 + 1^2}}\right) = \left(\frac{2x - 4y + 7}{\sqrt{2^2 + 4^2}}\right)$$

$$\Rightarrow \left(\frac{-2x - y + 6}{\sqrt{5}}\right) = \left(\frac{2x - 4y + 7}{2\sqrt{5}}\right)$$

$\Rightarrow \quad 2(-2x - y + 6) = (2x - 4y + 7)$

$\Rightarrow \quad 6x - 2y = 5$

92. The given lines are $3x - 4y + 7 = 0$
and $\quad -12x - 5y + 2 = 0$.
Now, $a_1a_2 + b_1b_2 = -36 + 20 = -16 < 0$.
Therefore, negative one is the obtuse-angle bisector.
Hence, the equation of the bisector of the obtuse angle is

$$\left(\frac{3x - 4y + 7}{\sqrt{3^2 + 4^2}}\right) = -\left(\frac{-12x - 5y + 2}{\sqrt{12^2 + 5^2}}\right)$$

$$\Rightarrow \left(\frac{3x - 4y + 7}{5}\right) = -\left(\frac{-12x - 5y + 2}{13}\right)$$

$\Rightarrow \quad 21x + 77y = 101$

93. The given lines are $-x - y + 3 = 0$ and $7x - y + 5 = 0$.
Now, $a_1a_2 + b_1b_2 = -7 + 1 = -6 < 0$.
Therefore, positive one is the acute-angle bisector.
Hence, the equation of the acute-angle bisector is

$$\left(\frac{-x - y + 3}{\sqrt{1^2 + 1^2}}\right) = \left(\frac{7x - y + 5}{\sqrt{7^2 + 1^2}}\right)$$

$$\Rightarrow \left(\frac{-x - y + 3}{\sqrt{2}}\right) = \left(\frac{7x - y + 5}{5\sqrt{2}}\right)$$

$\Rightarrow \quad 5(-x - y + 3) = (7x - y + 5)$

$\Rightarrow \quad 6x - 2y = 5$

94. The lengths of perpendicular from any point on the line $7x - 9\ y + 10 = 0$ to the lines $3x + 4y = 5$ and $12x + 5y = 7$ are same if $7x - 9y + 10 = 0$ is the bisector of
$3x + 4y = 5$ and $12x + 5y = 7$.
Hence, the equation of bisectors is

$$\left(\frac{-3x - 4y + 5}{\sqrt{3^2 + 4^2}}\right) = \left(\frac{-12x - 5y + 7}{\sqrt{12^2 + 5^2}}\right)$$

$\Rightarrow \quad 13(-3x - 4y + 5) = 5(-12x - 5y + 7)$

$\Rightarrow \quad 21x - 27y + 30 = 5$

$\Rightarrow \quad 7x - 9y + 10 = 0$

Hence, the result.

95. Let ABC be a triangle,
where AB: $x + 1 = 0$, BC: $3x - 4y = 5$ and CA: $5\ x + 12y = 27$.

Case I: Acute-angle bisector between AB and AC

The given lines are $x + 1 = 0$ and $-3x + 4y + 5 = 0$.
Now, $a_1a_2 + b_1b_2 = 1.(-3) + 0 = -3 < 0$.
Thus, positive one is the acute-angle bisector.
Hence, the acute-angle bisector is

$$\left(\frac{x + 1}{\sqrt{1^2}}\right) = \left(\frac{-3x + 4y + 5}{\sqrt{3^2 + 4^2}}\right)$$

$$\Rightarrow \left(\frac{x + 1}{1}\right) = \left(\frac{-3x + 4y + 5}{5}\right)$$

$\Rightarrow \quad 5(x + 1) = (-3x + 4y + 5)$

$\Rightarrow \quad 2x - y = 5 \qquad \ldots$(i)

Case II: The acute-angle bisector between BC and AC.

The given lines are $-3x + 4y + 5 = 0$

and $-5x - 12y + 27 = 0$.
Now, $a_1a_2 + b_1b_2 = 15 - 48 = -23 < 0$
Thus, the positive one is the acute-angle bisector.
Hence, the acute-angle bisector is

$$\left(\frac{-3x+4y+5}{\sqrt{3^2+4^2}}\right) = \left(\frac{-5x-12y+27}{\sqrt{5^2+12^2}}\right)$$

$$\Rightarrow \left(\frac{-3x+4y+5}{5}\right) = \left(\frac{-5x-12y+27}{13}\right)$$

$$\Rightarrow x - 8y = -5 \quad \ldots\text{(ii)}$$

On solving Eqs (i) and (ii), we get

$$x = \frac{1}{3}, y = \frac{2}{3}.$$

Hence, the in-centre is $\left(\frac{1}{3}, \frac{2}{3}\right)$.

96. The given lines are $\sqrt{3}x - y + 3 = 0$ and $x - \sqrt{3}y + 3\sqrt{3} = 0$
Hence, the equation of bisectors are

$$\left|\left(\frac{\sqrt{3}x - y + 3}{\sqrt{3+1}}\right)\right| = \left|\left(\frac{x - \sqrt{3}y + 3\sqrt{3}}{\sqrt{1+3}}\right)\right|$$

and $(\sqrt{3}x - y + 3) = -(x - \sqrt{3}y + 3\sqrt{3})$

$$(\sqrt{3}-1)x + (\sqrt{3}-1)y = 3(\sqrt{3}-1)$$

$$\Rightarrow \text{ and } (\sqrt{3}+1)x + (\sqrt{3}+1)y = -3(\sqrt{3}+1)$$

$\Rightarrow$ $x + y = 3$ and $x + y = -3$
Thus, the point P is $(3, 0)$ and the point Q is $(-3, 0)$
Hence, the length of $PQ = 6$.

97. Give lines are $AB: x + y = 1$ …(i)
and $CD: x + y = 5$ …(ii)
Slope of $AB = -1 = \tan(135°)$
Since, AB makes an angle of 45° with AC, therefore it will make an angle of (135° – 45°) or (135° + 45°), i.e. 90° or 180° with the positive direction of x-axis.
Thus, AB is parallel to x-axis or y-axis.
Hence, $x - 2 = 0$ and $y + 1 = 0$.
Case I: Consider the lines are $-x - y + 1 = 0$
and $-x + 2 = 0$
Now, $a_1a_2 + b_1b_2 = 1 + 0 = 1 > 0$
Thus negative one is the acute-angle bisector.
Hence, the equation of the acute-angle bisector of $\angle BAC$ is

$$\left(\frac{-x-y+1}{\sqrt{1^2+1^2}}\right) = -\left(\frac{-x+2}{\sqrt{1^2}}\right)$$

$$\Rightarrow (\sqrt{2}+1)x + y = (2\sqrt{2}+1) \quad \ldots\text{(iii)}$$

Case II: Consider the lines are $-x - y + 1 = 0$ and $y + 1 = 0$
Now, $a_1a_2 + b_1b_2 = 0 - 1 = -1 < 0$
Thus, positive one is the acute-angle bisector.
Hence, the equation of the acute-angle bisector of $\angle BAC$ is

$$\left(\frac{-x-y+1}{\sqrt{1^2+1^2}}\right) = \left(\frac{y+1}{\sqrt{1^2}}\right)$$

$$\Rightarrow x + (\sqrt{2}+1)y = (1-\sqrt{2}) \quad \ldots\text{(iv)}$$

On solving Eqs (i) and (iii), we get

$$x = 6 + 2\sqrt{2},\ y = -1 - 2\sqrt{2}$$

Thus, the co-ordinates of C can be

$$(6 + 2\sqrt{2}, 1 - 2\sqrt{2})$$

Again, solving Eqs (i) and (iv), we get

$$x = 2 - 2\sqrt{2},\ y = 3 + 2\sqrt{2}$$

Thus, The co-ordinates of C may be

$$(2 - 2\sqrt{2}, 3 + 2\sqrt{2})$$

98. (–1/10, 37/10)

99. Let the image of the point (–8, 12) be (x_2, y_2).

Then $$\frac{x_2+8}{4} = \frac{y_2-12}{7} = -\frac{2(4\cdot -8 + 7\cdot 12 + 13)}{(4^2+7^2)}$$

$$= \frac{2(-32+84+13)}{65}$$

$$= \frac{2\times 65}{65} = 2$$

$$\Rightarrow \frac{x_2+8}{4} = 2 \text{ and } \frac{y_2-12}{7} = 2$$

$\Rightarrow$ $x_2 = 0$ and $y_2 = 28$
Hence, the image of the point (–8, 12) is (0, 28).

100. Hence, the image of the point (3, 4) with respect to the line $y = x$ is (4, 3).

101. Let the point (4, 2) be P and the point (–2, 6) be Q.
Now the mid-point of P and Q is $M(1, 4)$.
Equation of a line passing through (4, 2) and (–2, 6) is

$$y - 2 = \frac{6-2}{-2-4}(x-4) = -\frac{2}{3}(x-2)$$

$\Rightarrow$ $3y - 6 = -2x + 4$
$\Rightarrow$ $2x + 13y = 10$ …(i)
Equation of any line perpendicular to Eq. (i) is
$3x - 2y + k = 0$
which is passing through $M(1, 4)$.
So $k = 8 - 3 = 5$.
Hence, the equation of the line is
$3x - 2y + 5 = 0$.

102. Let the point M be (4, 1).
The image of the point $M(1, 4)$ with respect to the line $y = x$ be $P(4, 1)$.
Now, if the point P be translated about the line $x = 2$, then the new position of P is (4 + 2, 1).
Hence, the co-ordinates of Q is (6, 1).

103. Let the point P be (3, 2).
The image of the point P with respect to the line $x = 4$ is $Q(2.4 - 3, 2)$ i.e. $Q(5, 2)$.
Let $Z = 5 + 2i$ and R be Z_1
Now, by rotation theorem of complex number,

$$\Rightarrow \frac{Z_1 - 0}{Z - 0} = \frac{|Z-0|}{|Z_1-0|}e^{i\frac{\pi}{4}}$$

$$\Rightarrow \frac{Z_1}{Z} = \frac{|Z_1-0|}{|Z-0|}e^{i\frac{\pi}{4}} = e^{i\frac{\pi}{4}}$$

$\Rightarrow \quad Z_1 = Z \times e^{i\frac{\pi}{4}} = (5+2i)\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$

$\Rightarrow \quad Z_1 = \frac{1}{\sqrt{2}}(5+5i+2i-2) = \frac{1}{\sqrt{2}}(3+7i)$

Hence, the co-ordinates of R are $\left(\frac{3}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$.

104.

105. Let PM: $x - 2y + 3 = 0$ be the incident ray and MR be the reflected ray. Produced MQ in such a way that $PM = MQ$. Now we shall find the image of Q w.r.t. AB is R.
Co-ordinates of M are $(1, -1)$.
Let co-ordinates of Q be $(3, 0)$ and R be (h, k).
Thus, R is the image of Q with respect to $3x - 2y = 5$.

Now $\frac{h-3}{3} = \frac{k-0}{-2} = -\frac{2(3.3+2.0-5)}{(3^2+2^2)} = -\frac{8}{13}$

$\Rightarrow \quad h = \frac{15}{13}$ and $k = \frac{16}{13}$

Thus, the co-ordinates of R be $\left(\frac{15}{13}, \frac{16}{13}\right)$.

Slope of $MR = \frac{29}{2}$

Therefore, the equation of MR is

$$y + 1 = \frac{29}{2}(x-1)$$

$\Rightarrow \quad 29x - 2y = 31$

Hence, the equation of the reflected ray is
$29x - 2y = 31$

106. Let N be the image of M with respect to x-axis. Thus N is $(5, -3)$

Slope of $NQ = \frac{-3-2}{5-1} = -\frac{5}{4}$

Therefore, the equation QN is

$$y - 2 = -\frac{5}{4}(x-1)$$

$\Rightarrow \quad 5x + 4y = 13$

Thus, the co-ordinates of P be $\left(\frac{13}{5}, 0\right)$.

Hence, the abscissa of the point P is $\frac{13}{5}$.

107. Let PM be the incident ray and QM be the reflected ray.
Let $P(1, 5)$ be any point on the line $x = 1$
Produce PM in such a way that $PM = RM$,
where $M = (1, 0)$
Clearly, $R = (1, -5)$, since, the incident ray is parallel to y-axis.
Now, image of R with respect to the line $x + y = 1$ is Q.
Let the co-ordinate of Q be (h, k).

Now, $\frac{h-1}{1} = \frac{k+5}{1} = \frac{-2(1-5-1)}{(1+1)} = 5$

$\Rightarrow \quad h = 6, k = 0$

Thus, the co-ordinates of Q is $(6, 0)$.
Since, the co-ordinates of M is $(1, 0)$ and the co-ordinates of Q is $(6, 0)$, so the equation of the reflected ray is $y = 0$.

108. Let PM: $x - 6y = 8$ be the incident ray
AB: $x + y = 1$ is the mirror, MQ is the reflected ray and MS is the refracted ray.
Clearly, M is $(2, -1)$

Now, $\tan 15° = \left|\frac{m - \frac{1}{6}}{1 + \frac{m}{6}}\right|$

$\Rightarrow \quad (2 - \sqrt{3}) = \pm\left(\frac{6m-1}{m+6}\right)$

$\Rightarrow \quad \left(\frac{6m-1}{m+6}\right) = 2 - \sqrt{3}$ or $\sqrt{3} - 2$

$\Rightarrow \quad m = \frac{70 - 37\sqrt{3}}{13}$ or $\frac{37\sqrt{3} - 70}{13}$

Let the angle between $x + y = 1$ and the line through $M(2, -1)$ with the slope

$$m = \frac{70 - 37\sqrt{3}}{13} \text{ be } \alpha.$$

Then $\tan(\alpha) = \left|\frac{\frac{70-37\sqrt{3}}{13} + 1}{1 - \frac{70-37\sqrt{3}}{13}}\right| = \left|\frac{83 - 37\sqrt{3}}{37\sqrt{3} - 57}\right|$

$$= \frac{83 - 37\sqrt{3}}{37\sqrt{3} - 57}$$

and the angle between $x + y = 1$ and the line through $M(2, -1)$ with slope $m = \frac{37\sqrt{3} - 70}{13}$ be β.

Then $\tan(\beta) = \left|\frac{\frac{37\sqrt{3}-70}{13} + 1}{1 - \frac{37\sqrt{3}-70}{13}}\right| = \left|\frac{37\sqrt{3} - 9}{131 - 37\sqrt{3}}\right|$

$$= \frac{37\sqrt{3} - 9}{131 - 37\sqrt{3}}$$

Here, $\tan(\alpha) > \tan(\beta)$

$\Rightarrow \quad \alpha > \beta$

Therefore, the slope of the refracted ray is $\frac{70 - 37\sqrt{3}}{13}$.

Hence, the equation of the refracted ray is

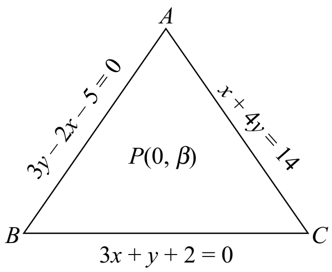
$$(y+1) = \left(\frac{70 - 37\sqrt{3}}{13}\right)(x-2)$$

$\Rightarrow \quad 13y + 13 = (70 - 37\sqrt{3})x - 140 + 74\sqrt{3}$

$\Rightarrow \quad (70 - 37\sqrt{3})x - 13y - 153 + 74\sqrt{3} = 0$

Level III

1. On solving, we get, the co-ordinates of A, B and C.



Thus, $A = (2, 3)$,
$B = (-1, 1)$
and $C = (-2, 4)$.

Now, the points P and A lie on the same side of the line
$$3x + y + 2 = 0.$$

So, $$\frac{3.0 + \beta + 2}{6 + 3 + 2} > 0 \Rightarrow \beta > -2 \quad \ldots\text{(i)}$$

Also, the points P and B on the same side of the line
$$x + 4y = 14$$

So, $$\frac{0 + 4\beta - 14}{-1 + 4 - 14} > 0$$

$\Rightarrow$ $4\beta - 14 < 0$

$\Rightarrow$ $$\beta < \frac{7}{2} \quad \ldots\text{(ii)}$$

Again, the points P and C lie on the same side of the line
$$3y - 2x = 5$$

So, $$\frac{3\beta - 5}{12 + 4 - 5} > 0$$

$\Rightarrow$ $$\beta > \frac{5}{3} \quad \ldots\text{(iii)}$$

From Eqs (i), (ii) and (iii), we get,
$$\frac{5}{3} < \beta < \frac{7}{2}$$

2. We have,
$$3x + 4y = 9$$

$\Rightarrow$ $$8 = \frac{9 - 3x}{4}$$

and $y = mx + 1$

$\Rightarrow$ $$\frac{9 - 3x}{4} = mx + 1$$

$\Rightarrow$ $9 - 3x = 4mx + 4$

$\Rightarrow$ $(4m + 3)x = 5$

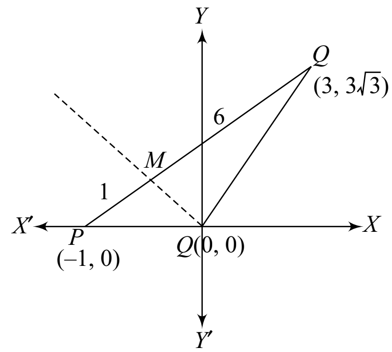
$\Rightarrow$ $$x = \frac{5}{(4m + 3)}$$

When $m = -1$, then x is -5

When $m = -2$, then x is -1

Hence, the number of integral values of m is 2 for which x is also an integer.

3.



Since OM is the internal bisector of the angle PQR.
$$\frac{QM}{PM} = \frac{OQ}{OP} = \frac{6}{1}$$

Thus, the co-ordinates of $M = \left(-\frac{3}{7}, \frac{3\sqrt{3}}{7}\right)$

Therefore, the equation of OM is
$$y = -\sqrt{3}\,x$$

$\Rightarrow$ $y + \sqrt{3}\,x = 0$

4. Equation of any line passing through (2, 2) is
$$y - 2 = m(x - 2)$$

Let the points A and B are
$$A = \left(\frac{2m - 2}{m + \sqrt{3}}, -\sqrt{3}\left(\frac{2m - 2}{m + \sqrt{3}}\right)\right)$$

and $$B = \left(\frac{2m - 2}{m - \sqrt{3}}, \sqrt{3}\left(\frac{2m - 2}{m - \sqrt{3}}\right)\right)$$

Since the ΔOAB is equilateral, so
$$OA = OB = AB$$

$\Rightarrow$ $$\left(\frac{2m - 2}{(m + \sqrt{3})}\right)^2 + 3\left(\frac{2m - 2}{(m + \sqrt{3})}\right)^2 = \left(\frac{2m - 2}{(m - \sqrt{3})}\right)^2 + 3\left(\frac{2m - 2}{(m - \sqrt{3})}\right)^2$$

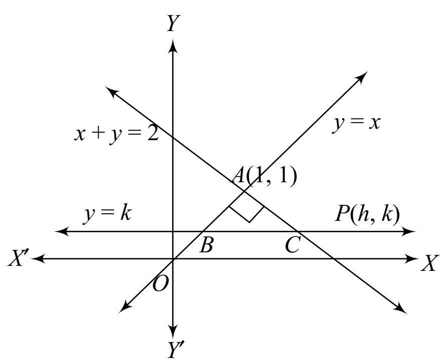
$\Rightarrow$ $$\frac{4}{(m + \sqrt{3})^2} = \frac{4}{(m - \sqrt{3})^2}$$

$\Rightarrow$ $(m + \sqrt{3})^2 = (m - \sqrt{3})^2$

$\Rightarrow$ $m = 0$

Hence the required equation of the line is $y = 2$.

5.



The co-ordinates of A, B and C are (1, 1), (k, k) and $(2-k, k)$.

Given $\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ k & k & 1 \\ 2-k & k & 1 \end{vmatrix} = 4h^2$

$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ k & 0 & 1-k \\ 2-k & 2k-2 & k-1 \end{vmatrix} = 4h^2 \begin{pmatrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{pmatrix}$

$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} 0 & 1-k \\ 2k-2 & k-1 \end{vmatrix} = 4h^2$

$\Rightarrow \quad \begin{vmatrix} 0 & 1-k \\ k-1 & k-1 \end{vmatrix} = 4h^2$

$\Rightarrow \quad (k-1)^2 = 4h^2$

$\Rightarrow \quad (k-1) = \pm 2h$

Hence, the locus of P is $2x - y + 1 = 0$.

6. Given lines are $x + y = |a|$ and $ax - y = 1$
Since two rays intersect in the first quadrant, so the lines should be $x + y = a$ and $ax - y = 1$.
On solving, we get, the point of intersection is $(1, a-1)$
Clearly, $a_0 = 1$.

7. Let the variable line be $\frac{x}{a} + \frac{y}{b} = 1$ and the co-ordinates of O, A, B are (0, 0), $(a, 0)$, $(0, b)$.
Let the centroid be $G(\alpha, \beta)$.
Thus, $\alpha = \frac{a}{3}, \beta = \frac{b}{3}$

$\Rightarrow \quad a = 3\alpha, b = 3\beta$

It is given that

$OM = p$

$\Rightarrow \quad \left|\frac{0+0-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}\right| = p$

$\Rightarrow \quad \frac{1}{a^2}+\frac{1}{b^2} = \frac{1}{p^2}$

$\Rightarrow \quad \frac{1}{(3\alpha)^2}+\frac{1}{(3\beta)^2} = \frac{1}{p^2}$

$\Rightarrow \quad \frac{1}{\alpha^2}+\frac{1}{\beta^2} = \frac{9}{p^2}$

Hence, the locus of $G(\alpha, \beta)$ is

$$\frac{1}{x^2}+\frac{1}{y^2} = \frac{9}{p^2}$$

8. Let $z_1 = A(3, 0)$, $z_2 = B(5, 2)$ and $z_3 = C$
We have

$$\frac{z_3 - z_1}{z_2 - z_1} = \left|\frac{z_3 - z_1}{z_2 - z_1}\right| \times e^{i\frac{\pi}{4}}$$

$\Rightarrow \quad \frac{z_3 - z_1}{z_2 - z_1} = \frac{|z_3 - z_1|}{|z_2 - z_1|} \times e^{i\frac{\pi}{4}}$

$\Rightarrow \quad (z_3 - z_1) = (z_2 - z_1)e^{i\frac{\pi}{4}}$

$\Rightarrow \quad z_3 = z_1 + (5 + 2i - 3) \times \frac{1}{\sqrt{2}}(i + \mathrm{i})$

$\Rightarrow \quad z_3 = z_1 + (2 + 2i) \times \frac{1}{\sqrt{2}}(i + i)$

$\Rightarrow \quad z_3 = 3 + \frac{2}{\sqrt{2}}(1+i)^2 = 3 + \sqrt{2}(1+i)^2$

$\Rightarrow \quad z_3 = 3 + 2\sqrt{2}\,i$

$\Rightarrow \quad C = (3, 2\sqrt{2})$

It is given that D is the image of C w.r.t. y-axis
Thus, $D = (-3, 2\sqrt{2})$
So, $\quad x = -3, y = 2\sqrt{2}$
Hence, the value of

$x + y + 7$

$= -3 + 2\sqrt{2} + 7$

$= 2(2 + \sqrt{2})$

9. Let the line L: $y = mx + c$ be equally inclined to L_1: $3x - 4y = 7$ and L_2: $5y - 12x = 6$.
Since the line $L = 0$ is equally inclined with the lines $L_1 = 0$ and $L_2 = 0$, so we have

$$\frac{\frac{3}{4} - m}{1 + \frac{3}{4}m} = \frac{m - \frac{12}{5}}{1 + \frac{12}{5}m}$$

$\Rightarrow \quad \frac{3-4m}{4+3m} = \frac{5m-12}{5+12m}$

$\Rightarrow \quad 15 + 16m - 48m^2 = 15m^2 - 16m - 48$

$\Rightarrow \quad 63m^2 - 32m - 63 = 0$

$\Rightarrow \quad 63m^2 - 81m + 49m - 63 = 0$

$\Rightarrow \quad 9m(7m - 9) + 7(7m - 9) = 0$

$\Rightarrow \quad (7m - 9)(9m + 7) = 0$

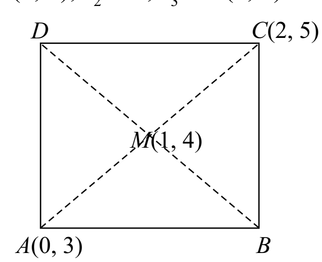
$\Rightarrow \quad m = \frac{9}{7}, -\frac{7}{9}$

Hence, the equation of the line is

$$y - 5 = \frac{9}{7}(x-4) \text{ or } y - 5 = -\frac{7}{9}(x-4)$$

$\Rightarrow \quad 9x - 7y = 1$ or $7x + 9y = 73$

10. Let $z_1 = A(3, 0)$; $z_2 = B$; $z_3 = C(2, 5)$ and $z_4 = D$



Now, $\dfrac{z_3 - z_2}{z_1 - z_2} = \left|\dfrac{z_3 - z_2}{z_1 - z_2}\right| \times e^{-i\frac{\pi}{2}}$

$\Rightarrow \quad \dfrac{z_3 - z_2}{z_1 - z_2} = \dfrac{|z_3 - z_2|}{|z_1 - z_2|} \times e^{-i\frac{\pi}{2}} = -i$

$\Rightarrow \quad z_3 - z_2 = -iz_1 + iz_2$

$\Rightarrow \quad (1 + i)z_2 = z_3 + iz_1$

$\Rightarrow \quad z_2 = \dfrac{z_3 + iz_1}{1+i} = \dfrac{2 + 5i + 3i}{1+i}$

$\Rightarrow \quad z_2 = \dfrac{2 + 5i + 3i}{1+i} = \dfrac{2(1+4i)}{1+i}$

$\Rightarrow \quad z_2 = \dfrac{2(1+4i)}{1+i} \times \dfrac{(1-i)}{(1-i)}$

$\Rightarrow \quad z_2 = \dfrac{2(1+4i)(1-i)}{2}$

$\Rightarrow \quad z_2 = 1 + 3i + 4 = 5 + 3i$

Thus, $B = (5, 3)$

Let $\quad D = (\alpha, \beta)$

Thus, $\dfrac{\alpha + 5}{2} = 1, \dfrac{\beta + 3}{2} = 4$

$\Rightarrow \quad \alpha = -3, \beta = 5$

Therefore, $D = (-3, 5)$

11. First we find the equations of the sides of the triangle ABC, i.e. AB, BC and CA.

AB: $\quad y - 3 = \dfrac{3-0}{-2-0}(x - 0)$

$\Rightarrow \quad -2y + 6 = 3x$

$\Rightarrow \quad 3x + 2y = 6$

AC: $\quad y - 3 = \dfrac{3-1}{6-0}(x - 0)$

$\Rightarrow \quad 2y - 6 = x$

$\Rightarrow \quad x - 2y + 6 = 0$

BC: $\quad y - 0 = \dfrac{1-0}{6+2}(x + 2)$

$\Rightarrow \quad 8y = x + 2$

$\Rightarrow \quad x - 8y + 2 = 0$

Case I: The points P and A lie on the same side of the line BC

So, $\quad \dfrac{m - 8(m+1) + 2}{0 - 24 + 2} > 0$

$\Rightarrow \quad \dfrac{-7m - 6}{-22} > 0$

$\Rightarrow \quad 7m + 6 > 0$

$\Rightarrow \quad m > -\dfrac{6}{7}$

Case II: When the points P and B lie on the same side of AC

So, $\quad \dfrac{m - 2(m+1) + 6}{-2 - 0 + 6} > 0$

$\Rightarrow \quad \dfrac{-m + 4}{4} > 0$

$\Rightarrow \quad m < 4$

Case III: When the points P and C lie on the same side of AB.

So, $\quad \dfrac{3m + 2(m+1) - 6}{18 + 2 - 6} > 0$

$\Rightarrow \quad \dfrac{5m - 4}{14} > 0$

$\Rightarrow \quad m > \dfrac{4}{5}$

Hence, the value of m lies in

$$\frac{4}{5} < m < 4$$

12. Let the line be $\dfrac{x}{a} + \dfrac{y}{b} = 1$

It intersects the x-axis at $A(a, 0)$ and y-axis at $B(0, b)$.Clearly, the point $M(h, k)$ is the mid-point of A and B.

$\Rightarrow \quad h = \dfrac{a+0}{2}, k = \dfrac{0+b}{2}$

$\Rightarrow \quad a = 2h, b = 2k$

Hence, the equation of the line is

$$\frac{x}{2h} + \frac{y}{2k} = 1$$

13. Since the points $(m, 3)$ and $(0, 0)$ lie on the opposite sides $3x + 2y - 6 = 0$ and $x - 4y + 16 = 0$,

so, $\quad \dfrac{3m + 6 - 6}{0 + 0 - 6} < 0$

$\Rightarrow \quad 3m > 0$

$\Rightarrow \quad m > 0$

Also, $\quad \dfrac{m - 12 + 16}{0 - 0 + 16} < 0$

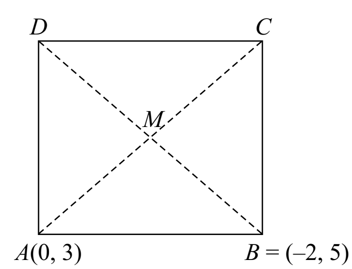
$\Rightarrow \quad m + 4 < 0$

$\Rightarrow \quad m < -4$

Hence, the values of m lies in

$$R - (-4, 0)$$

14. Let $z_1 = A(0, 3)$; $z_2 = B(-2, 5)$; $z_3 = C$ and $z_4 = D$



Now, $\dfrac{z_3 - z_2}{z_1 - z_2} = \left|\dfrac{z_3 - z_2}{z_1 - z_2}\right| \times e^{-i\frac{\pi}{2}}$

$\Rightarrow \quad \dfrac{z_3 - z_2}{z_1 - z_2} = \dfrac{|z_3 - z_2|}{|z_1 - z_2|} \times e^{-i\frac{\pi}{2}} = -i$

$\Rightarrow \quad z_3 - z_2 = -iz_1 + iz_2$

$\Rightarrow \quad z_3 = z_2 + -iz_2 - iz_1$

$\Rightarrow \quad z_3 = -2 + 5i - 2i - 5 + 3 = -4 + 3i$

Thus, $C = (4, 3)$

and $\quad M = (-2, 3)$

Let $\quad D = (\alpha, \beta)$

Thus, $\dfrac{\alpha - 2}{2} = -2, \dfrac{\beta + 5}{2} = 3$

$\Rightarrow \quad \alpha = -2, \beta = 1$

Therefore, $D = (-2, 1)$

15 Hence, the co-ordinates of the points are

$(x_1 \pm r \cos \beta, y_1 \pm r \sin \theta)$

$= (2 \pm \sqrt{8} \cos(135°), 1 \pm \sqrt{8} \sin(135°))$

$= \left(2 \pm \sqrt{8}\left(-\dfrac{1}{\sqrt{2}}\right), 1 \pm \sqrt{8}\left(\dfrac{1}{\sqrt{2}}\right)\right)$

$= (2 \mp 2, 1 \pm 2)$

$= (0, 3)$ and $(4, -1)$

16. Given family of lines be

$(a + b)x + (2a - b)y = 0$

$\Rightarrow \quad a(x + 2y) + b(x - y) = 0$

$\Rightarrow \quad (x + 2y) + \dfrac{b}{a}(x - y) = 0$

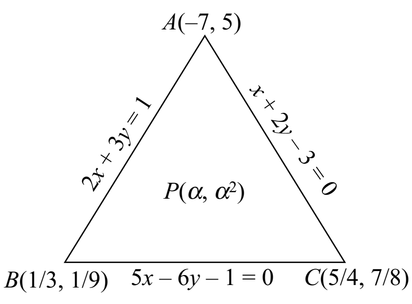
$\Rightarrow \quad (x + 2y) + \lambda(x - y) = 0$

$\Rightarrow \quad (x + 2y) = 0, (x - y) = 0$

On solving, we get the co-ordinates of the fixed point is (0, 0).

Hence, the family of lines passes through a fixed point is (0, 0).

17.



On solving the equations, we get the co-ordinates of A, B and C, respectively.

Now, A and P lie on the same side of the line

$5x - 6y - 1 = 0$

Thus, $\dfrac{5\alpha - 6\alpha^2 - 1}{5(-7) - 6(5) - 1} > 0$

$\Rightarrow \quad 6\alpha^2 - 5\alpha + 1 > 0$

$\Rightarrow \quad (3\alpha - 1)(2\alpha - 1) > 0$

$\Rightarrow \quad \alpha < \dfrac{1}{3}$ or $\alpha > \dfrac{1}{2}$ ...(i)

Again, the points P and B lie on the same side of the line $x + 2y - 3 = 0$.

Thus, $\dfrac{\alpha + 2\alpha^2 - 3}{\dfrac{1}{3} + \dfrac{2}{9} - 3} > 0$

$\Rightarrow \quad 2\alpha^2 + \alpha - 3 < 0$

$\Rightarrow \quad (2\alpha + 3)(\alpha - 1) < 0$

$\Rightarrow \quad -\dfrac{3}{2} < \alpha < 1$ ...(ii)

Finally, the points P and C lie on the same side of the line $2x - 3y - 1 = 0$

Thus, $\dfrac{2\alpha + 3\alpha^2 - 1}{2\left(\dfrac{5}{4}\right) + 3\left(\dfrac{7}{8}\right) - 1} > 0$

$\Rightarrow \quad 3\alpha^2 + 2\alpha - 1 > 0$

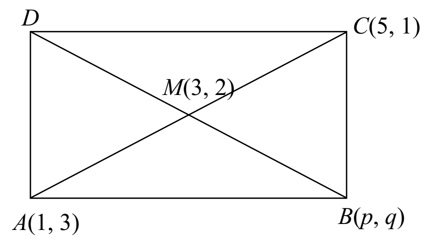
$\Rightarrow \quad (3\alpha - 1)(\alpha + 1) > 0$

$\Rightarrow \quad \alpha < -1$ or $\alpha > \dfrac{1}{3}$ ...(iii)

From Eqs (i), (ii) and (iii), we get

$\alpha \in \left(-\dfrac{3}{2}, -1\right) \cup \left(\dfrac{1}{2}, 1\right)$

18.



Clearly, the mid-point (3, 2) of (1, 3) and (5, 1) lies on the diagonal.

So, $\quad 2 = 6 + c$

$\Rightarrow \quad c = -4$

Equation of one diagonal is $y = 2x - 4$.

Let $\quad B = (p, q)$

Now, $m(BC) \times m(AB) = -1$

$\Rightarrow \quad \left(\dfrac{q - 1}{p - 5}\right) \times \left(\dfrac{q - 3}{p - 1}\right) = -1$

$\Rightarrow \quad (q - 1)(q - 3) = -(p - 1)(p - 5)$

$\Rightarrow \quad q^2 - 4q + 3 = -p^2 + 6p - 5$

$\Rightarrow \quad p^2 + q^2 - 6p - 4q + 8 = 0$ ...(i)

Also B lies on the line $y = 2x - 4$.

So, $\quad q = 2p - 4$ ...(ii)

On solving Eqs (i) and (ii), we get,

$p = 2, 4$ and $q = 0, 4$

Hence, the co-ordinates of the other vertices are (2, 0) and (4, 4).

19. Clearly, the slope of the given line is –1.

Thus, $\theta = 135°$.

Therefore, the equation of the other two sides are

$$y - 3 = \tan(135° \pm 60°)(x - 2)$$

$\Rightarrow \quad y - 3 = \tan(195°)(x - 2)$

or $\quad y - 3 = \tan(75°)(x - 2)$

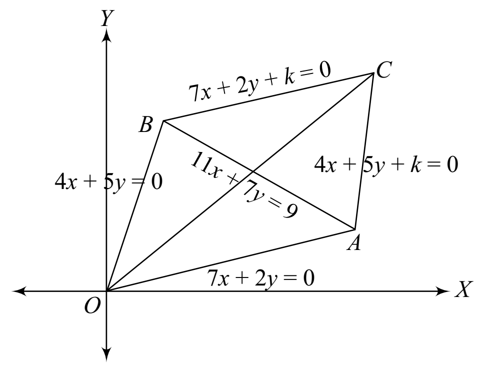
$\Rightarrow \quad y - 3 = \tan(15°)(x - 2)$

or $\quad y - 3 = \cot(15°)(x - 2)$

$\Rightarrow \quad y - 3 = (2 - \sqrt{3})(x - 2)$

or $\quad y - 3 = (2 + \sqrt{3})(x - 2)$

20.



On solving OA and AB, we get

$$A = \left(-\frac{2}{3}, \frac{7}{3}\right)$$

On solving OB and AB, we get

$$B = \left(\frac{5}{3}, -\frac{7}{3}\right)$$

Therefore, the equations of BC and AC are

$$7x + 2y = 9, 4x + 5y = 9$$

On solving BC and AC, we get,

$$C = (1, 1).$$

Hence, the equation of the other diagonal is $y = x$.

21. The reflection of the point (1, 2) w.r.t. x-axis is (–1, 2)
Thus, the equation of line containing the reflected ray is

$$y - 2 = \frac{3 - 2}{5 + 1}(x + 1)$$

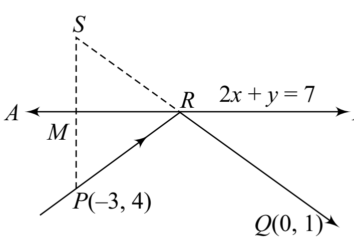
$\Rightarrow \quad y - 2 = \frac{1}{6}(x + 1)$

$\Rightarrow \quad 6y - 12 = x + 1$

$\Rightarrow \quad x - 6y + 13 = 0$

Hence, the point A is (–13, 0).

22.



Given equation is line AB is

$$2x + y = 7 \qquad \text{…(i)}$$

Let S be the image of the point $P(-3, 4)$.
Equation of PM is $x - 2y + k = 0$ which is passing through $P(-3, 4)$

So, $\quad -3 - 8 + k = 0$

$\Rightarrow \quad k = 11$

Thus, PM is $x - 2y + 11 = 0 \qquad$ …(ii)

Solving Eqs (i) and (ii), we get,

$$x = \frac{3}{5}, y = \frac{29}{5}$$

Thus, $M = \left(\frac{3}{5}, \frac{29}{5}\right)$

Let $\quad S = (a, b)$

Then $\frac{3}{5} = \frac{a - 3}{2}$ and $\frac{29}{5} = \frac{b + 4}{2}$

$\Rightarrow \quad a = \frac{21}{5}$ and $b = \frac{38}{5}$

Thus, $S = \left(\frac{21}{5}, \frac{38}{5}\right)$

Therefore, the equation of SQ is

$$y - 1 = \frac{\frac{38}{5} - 1}{\frac{21}{5} - 0}(x - 0)$$

$\Rightarrow \quad y - 1 = \frac{33}{21}x$

$\Rightarrow \quad 33x - 21y - 21 = 0$

$\Rightarrow \quad 11x - 7y - 7 = 0 \qquad$ …(iii)

On solving Eqs (ii) and (iii), we get,

$$x = \frac{42}{25}, y = \frac{91}{25}$$

Thus, $R = \left(\frac{42}{25}, \frac{91}{25}\right)$

23. The incident ray intersects the mirror at $A(6, 0)$.
Let $B(0, -4)$ be a point on the incident ray.
The reflection of the point B w.r.t. x-axis is $C(0, 4)$.
Hence, the equation of the reflected ray is

$$\frac{x}{6} + \frac{y}{4} = 1$$

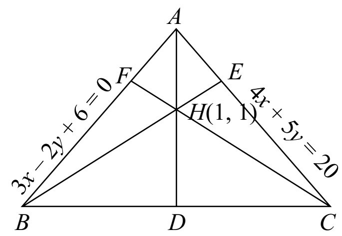
$\Rightarrow \quad 2x + 3y = 12$

24. The lines $|x + y| = 4$ are

$$x + y = 4, x + y = -4$$

Clearly, both are parallel lines.
Now a line $y = x$ intersects both the lines at (–2, –2) and (2, 2)
Thus, if the point (a, a) lies between the lines, so, $a > -2$ and $a < 2$.
Hence, $-2 < a < 2$.

25.



Here, altitude AD passes through the intersection of AB and AC.

Let AD: $(3x - 2y + 6) + l(4x + 5y - 20) = 0$ which is passing through $H(1, 1)$

$\Rightarrow \quad (3 - 2 + 6) + \lambda(4 + 5 - 20) = 0$

$\Rightarrow \quad 7 - 11\lambda = 0$

$\Rightarrow \quad \lambda = \dfrac{7}{11}$

Thus, $(3x - 2y + 6) + \dfrac{7}{11}(4x + 5y - 20) = 0$

$\Rightarrow \quad 61x + 13y - 74 = 0$

Consider BC: $13x - 61y + k = 0$

Now, altitude BE:

$$\frac{3x - 2y + 6}{3.4 - 2.5} = \frac{13x - 61y + \text{k}}{13.4 - 61.5}$$

which is passing through $H(1, 1)$

So, $\dfrac{7}{2} = \dfrac{k - 48}{52 - 305}$

$\Rightarrow \quad \dfrac{7}{2} = \dfrac{k - 48}{-253}$

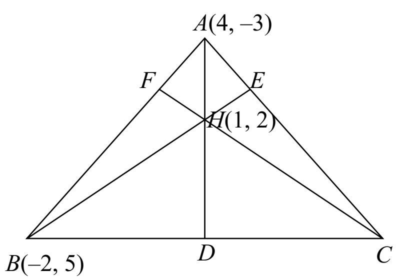
$\Rightarrow \quad k = 48 - \dfrac{7}{2} \times 253 = -\dfrac{1675}{2}$

Hence, the equation of the BC is

$$13x - 61y - \frac{1675}{2} = 0$$

$\Rightarrow \quad 26x - 122y - 1675 = 0$

26.



Let $C = (a, b)$

Now, $m(AB) \times m(CF) = -1$

$\Rightarrow \quad \dfrac{5 + 3}{-2 - 4} \times \dfrac{2 - b}{1 - a} = -1$

$\Rightarrow \quad \dfrac{2 - b}{1 - a} = \dfrac{3}{4}$

$\Rightarrow \quad 8 - 4b = 3 - 3a$

$\Rightarrow \quad 3a - 4b = -5$ …(i)

Also, $m(AC) \times m(BE) = -1$

$\Rightarrow \quad \dfrac{b + 3}{a - 4} \times \dfrac{2 - 5}{1 + 2} = -1$

$\Rightarrow \quad \dfrac{b + 3}{a - 4} = 1$

$\Rightarrow \quad a - 4 = b + 3$

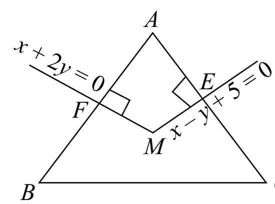
$\Rightarrow \quad a - b = 7$ …(ii)

Solving Eqs (i) and (ii), we get,

$a = 33, b = 26$

Hence, the third vertex is $C(33, 26)$.

27.



Let the co-ordinates of B and C are (x_1, y_1) and (x_2, y_2) respectively.

Clearly the mid-point of A and B and A and C lie on the perpendicular bisectors $x + 2y = 0$ and $x - y + 5 = 0$.

So, $\dfrac{x_1 + 1}{2} + 2\left(\dfrac{y_1 - 2}{2}\right) = 0$

$\Rightarrow \quad x_1 + 2y_1 - 3 = 0$ …(i)

and $\left(\dfrac{x_2 + 1}{2}\right) - \left(\dfrac{y_2 - 2}{2}\right) + 5 = 0$

$\Rightarrow \quad x_2 + y_2 + 13 = 0$ …(ii)

Also, $\left(\dfrac{y_1 + 2}{x_1 - 1}\right)\left(-\dfrac{1}{2}\right) = -1$

$\Rightarrow \quad 2x_1 - y_1 - 4 = 0$

and $\left(\dfrac{y_2 - 2}{x_2 - 1}\right)(1) = -1$

$\Rightarrow \quad x_2 + y_2 + 1 = 0$ …(iv)

Solving Eqs (i) and (iii), we get the co-ordinates of B,

i.e. $B = \left(\dfrac{11}{5}, \dfrac{2}{5}\right)$

Solving Eqs (ii) and (iv), we get the co-ordinates of C, i.e. $C = (-7, 6)$

Now, $m(BC) = \dfrac{6 - \dfrac{2}{5}}{-7 - \dfrac{11}{5}} = -\dfrac{28}{46} = -\dfrac{14}{23}$

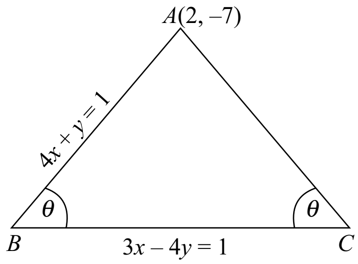
Therefore, the equation of BC is

$$y - 6 = -\frac{14}{23}(x + 7)$$

$\Rightarrow \quad 23y - 138 = -14x - 98$

$\Rightarrow \quad 14x + 23y - 40 = 0$

28.



Let m be the slope of AC.
It is given that $AB = AC$.
Clearly BC is equally inclined with AB and AC

So, $$\frac{\frac{3}{4}+4}{1+\frac{3}{4}\times(-4)} = \frac{m-\frac{3}{4}}{1+\frac{3}{4}\cdot m}$$

$$\Rightarrow \quad \frac{19}{-8} = \frac{4m-3}{3m+4}$$

$$\Rightarrow \quad m = -\frac{52}{89}$$

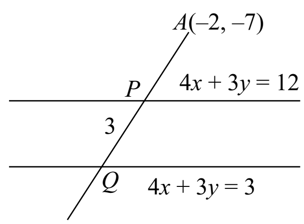
Hence, the equation of AC is

$$y+7 = -\frac{52}{89}(x-2)$$

$\Rightarrow \quad 89y + 623 = -52x + 104$

$\Rightarrow \quad 52x + 89y + 519 = 0$

29.



Equation of any line passing through $A(-2, -7)$ is $y + 7 = m(x + 2)$.
Clearly, it is equally inclined to parallel lines with slope –4/3.

So, $$\tan(\pm\theta) = \frac{m-\left(-\frac{4}{3}\right)}{1+m\cdot\left(-\frac{4}{3}\right)}$$

$$\Rightarrow \quad \pm\frac{3}{4} = \frac{3m+4}{3-4m}.$$

$$\Rightarrow \quad m = -\frac{7}{24} \text{ and } m = \infty$$

Hence, the equation of the required lines are

$$\frac{y+7}{x+2} = \infty,\ y+7 = -\frac{7}{24}(x+2)$$

$$\Rightarrow \quad \frac{x+2}{y+7} = \frac{1}{\infty},\ 24y+168 = -7x-14$$

$\Rightarrow \quad x + 2 = 0$ and $7x + 24y + 182 = 0$

30. Given points are $A(1, p^2)$, $B(0, 1)$, $C(p, 0)$

$$\text{Area of the triangle} = \frac{1}{2}\begin{vmatrix} 1 & p^2 \\ 0 & 1 \\ p & 0 \\ 1 & p^2 \end{vmatrix}$$

$$\Delta = \frac{1}{2}(1 + p^3 - p)$$

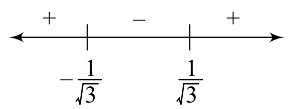
$$\frac{d\Delta}{dp} = \frac{1}{2}(3p^2 - 1)$$

For maximum or minimum,

$$\frac{d\Delta}{dp} = 0$$

$$\Rightarrow \quad \frac{1}{2}\left(3p^2 - 1\right) = 0$$

$$\Rightarrow \quad p = \pm\frac{1}{\sqrt{3}}$$



Clearly, its area is minimum at $p = \frac{1}{\sqrt{3}}$.

31. Clearly, the point A is $(3, -1)$
Equation of BC is $y - 2 = m(x - 1)$

Thus, $$\left|\frac{m+\frac{3}{4}}{1-\frac{3}{4}m}\right| = \tan 45^\circ$$

$$\Rightarrow \quad \left|\frac{4m+3}{4-3m}\right| = 1$$

$$\Rightarrow \quad \left(\frac{4m+3}{4-3m}\right) = \pm 1$$

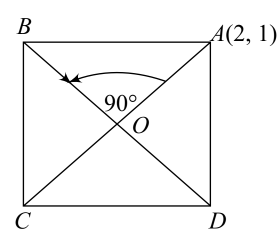
$$\Rightarrow \quad m = -7, \frac{1}{7}$$

Hence, the equation of the line BC is

$$y - 2 = -7(x-1) \text{ or } y - 2 = \frac{1}{7}(x-1)$$

$x - 7y + 13 = 0$ or $7x + y - 9 = 0$

32. Given $A = (2, 1)$
So, $z_A = 2 + i$
Then $z_B = i(2 + i)$
$= -1 + 2i$
$B = (-1, 2)$
Therefore, $C = (-2, -1)$ and $D = (1, -2)$, since O is the origin.

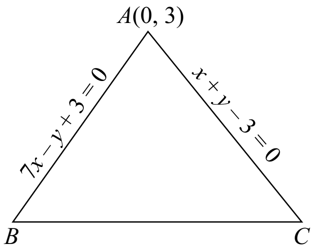


33. Two given lines are $7x - y + 3 = 0$ and $x + y - 3 = 0$.
So the point of intersection is $(0, 3)$

Let $m(B) = m$

$m(AB) = 7$, $m(AC) = -1$

Clearly, $\angle ABC = \angle ACB$



Thus, $\left|\frac{m-7}{1+7m}\right| = \left|\frac{m+1}{1-m}\right|$

$\Rightarrow 6m^2 + 16m - 6 = 0$

$\Rightarrow 3m^2 + 8m - 2 = 0$

$\Rightarrow m = -3, \frac{1}{3}$

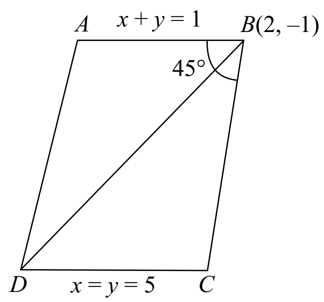
Hence, the equation of the BC is

$$y + 10 = -3(x-1) \text{ or } y + 10 = \frac{1}{3}(x-1)$$

$$3x + y - 7 = 0 \text{ or } x - 3y - 31 = 0$$

34. Given lines are $x + y - 1 = 0$ and $x + y - 5 = 0$.

Let the slope of BC be m.



So,

$$\tan(\pm 45°) = \frac{m+1}{1-m}$$

$$\frac{m+1}{1-m} = \pm 1$$

$$m = 0, \infty$$

Thus, the equation of BC may be $y + 1 = 0$ or $x - 2 = 0$.

Now, the equation of the bisector of $\angle ABC$ are

$$\frac{y+1}{1} = \pm\frac{x+y-1}{\sqrt{2}}$$

or $$\frac{x-2}{1} = \pm\frac{x+y-1}{\sqrt{2}}$$

The equation of the bisectors of the acute angle in two cases are

$$x + (\sqrt{2}+1)y = (1-\sqrt{2})$$

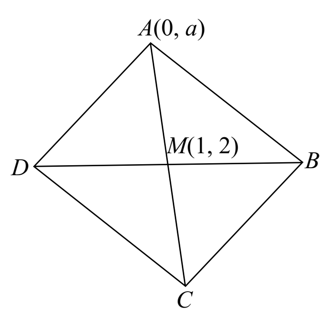
and $$(\sqrt{2}+1)x + y = (1+2\sqrt{2})$$

On solving, we get the co-ordinates of D be

$(6 + 2\sqrt{2}, -1 - 2\sqrt{2})$ or $(2 - 2\sqrt{2}, 3 + 2\sqrt{2})$.

35. Let the co-ordinates of A be $(0, a)$.

As the sides of the rhombus are parallel to the line $y = x + 2$ and $y = 7x + 3$, the diagonals of the rhombus are parallel to the bisectors of the angles between the given lines.



Equation of the bisectors of the angles between the lines

$$\frac{x-y+2}{\sqrt{1+1}} = \pm\frac{7x-y-3}{\sqrt{49+1}}$$

$$\frac{x-y+2}{\sqrt{2}} = \pm\frac{7x-y-3}{5\sqrt{2}}$$

$$5(x - y + 2) = \pm(7x - y - 3)$$

$$2x + 4y = 7,\ 12x - 6y + 13 = 0$$

Thus, slope of MA is either 2 or $-1/2$

$\Rightarrow \quad \frac{a-2}{0-1} = 2 \text{ or } -\frac{1}{2}$

So, $\quad a = 0 \text{ or } \frac{5}{2}$

Hence, the co-ordinates of A may be

$$(0, 0) \text{ or } \left(0, \frac{5}{2}\right)$$

Level IV

1. Sides are $x - 3y = 3$, $3x - 2y = 16$
   Diagonals are $x + 4y = 10$, $5x - 8y + 6 = 0$
2. Clearly $B = (-1, 4)$, since the equation of the bisector in the first quadrant is $y = x$
   Thus, $AB = \sqrt{25+25} = \sqrt{50} = 5\sqrt{2}$
3. We have,

$$\frac{z_3 - z_1}{z_2 - z_1} = \left|\frac{z_3 - z_1}{z_2 - z_1}\right| \times e^{i\frac{\pi}{4}}$$

$$\Rightarrow \quad \frac{z_3 - z_1}{z_2 - z_1} = e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}}(1+i)$$

$$\Rightarrow \quad z_3 = z_1 + (z_2 - z_1)\frac{1}{\sqrt{2}}(1+i)$$

$$\Rightarrow \quad z_3 = 2 + (2 + \sqrt{2} + i - 2)\frac{1}{\sqrt{2}}(1+i)$$

$$\Rightarrow \quad z_3 = 2 + \frac{1}{\sqrt{2}}(\sqrt{2}+i)(1+i)$$

$$\Rightarrow \quad z_3 = 2 + \frac{1}{\sqrt{2}}(\sqrt{2} + i + i\sqrt{2} - 1)$$

$$\Rightarrow \quad z_3 = 2 + \frac{1}{\sqrt{2}}[(\sqrt{2}-1) + i(\sqrt{2}+1)]$$

$$\Rightarrow \quad z_3 = 2 + \frac{(\sqrt{2}-1)}{\sqrt{2}} + i\frac{(\sqrt{2}+1)}{\sqrt{2}}$$

$$\Rightarrow \quad z_3 = \frac{3\sqrt{2}-1}{\sqrt{2}} + i\frac{(\sqrt{2}+1)}{\sqrt{2}}$$

$$\Rightarrow \quad C = \left(\frac{3\sqrt{2}-1}{\sqrt{2}}, \frac{(\sqrt{2}+1)}{\sqrt{2}}\right)$$

4. Let $A = (1, -2)$.
   Thus, the image of A w.r.t x-axis is, say, $B(1, 2)$.
   Thus, the new position of B is $C(4, 2)$.

5. Clearly, the slope of the new line, $m = \tan(45°) = 1$.
Hence, the equation of the new line is
$$y - 0 = 1(x - 2)$$
$$\Rightarrow \quad y = x - 2$$
$$\Rightarrow \quad x - y = 2$$

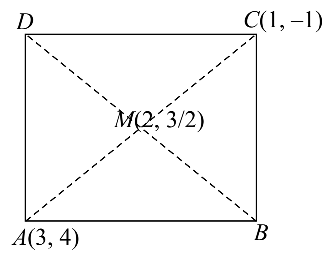
6. Clearly, $A = (0, 1)$ and
$$m = \tan(45° + 105°) = \tan(150°) = -\frac{1}{\sqrt{3}}$$
Hence, the equation of the new line is
$$\Rightarrow \quad y - 1 = -\frac{1}{\sqrt{3}}(x - 0)$$
$$\Rightarrow \quad \sqrt{3}y - \sqrt{3} = -x$$
$$\Rightarrow \quad x + \sqrt{3}y = \sqrt{3}$$

7. Clearly, $\tan\theta = 2$
$$\Rightarrow \quad \sin\theta = \frac{2}{\sqrt{5}}, \cos\theta = \frac{1}{\sqrt{5}}$$
Hence, the required point
$$= (x_1 + r\cos\theta\, y_1 + r\sin\theta)$$
$$= \left(1 + 1\cdot\frac{1}{\sqrt{5}}, 1 + 1\cdot\frac{2}{\sqrt{5}}\right)$$
$$= \left(1 + \frac{1}{\sqrt{5}}, 1 + \frac{2}{\sqrt{5}}\right)$$

8. Thus, the point of intersection of the two given lines is $(2, -1)$
Let Q the point where P moving 2 units along the line $x + y = 1$
$$\text{So,} \quad Q = \left(2 - 2\cdot\frac{1}{\sqrt{2}}, -1 + 2\cdot\frac{1}{\sqrt{2}}\right)$$
$$= (2 - \sqrt{2}, \sqrt{2} - 1)$$
and R be the point where the point P moves 5 units along the line $x - 2y = 4$.
$$\text{So,} \quad R = \left(2 + 5\cdot\frac{2}{\sqrt{5}}, -1 + 5\cdot\frac{1}{\sqrt{5}}\right)$$
$$= (2 + 2\sqrt{5}, \sqrt{5} - 1)$$
Hence, the required distance
$$= QR$$
$$= \sqrt{(-\sqrt{2} - 2\sqrt{5})^2 + (\sqrt{2} - \sqrt{5})^2}$$
$$= \sqrt{(\sqrt{2} + 2\sqrt{5})^2 + (\sqrt{2} - \sqrt{5})^2}$$
$$= \sqrt{2 + 20 + 2 + 5}$$
$$= \sqrt{29}$$

9. Let $A = (a, 0)$, $B = (b, 6b)$
and $M(h, k)$ is the mid-point of AB, where
$$h = \frac{a + b}{2}, k = 3b$$
$$\text{Thus,} \quad a = 2h - \frac{k}{3}, b = \frac{k}{3}$$
We have, $AB = 2l$
$$\Rightarrow \quad AB^2 = 4\lambda^2$$
$$\Rightarrow \quad (a - b)^2 + 36b^2 = 4l^2$$
$$\Rightarrow \quad \left(2h - \frac{k}{3} - \frac{k}{3}\right)^2 + 36\left(\frac{k}{3}\right)^2 = 4l^2$$
$$\Rightarrow \quad \left(2h - \frac{2k}{3}\right)^2 + 4k^2 = 4l^2$$
$$\Rightarrow \quad \left(h - \frac{k}{3}\right)^2 + k^2 = l^2$$
Hence, the locus of $M(h, k)$ is
$$\left(x - \frac{y}{3}\right)^2 + k^2 = l^2$$
$$\Rightarrow \quad 9x^2 - 6xy + 10y^2 = 9l^2$$

10.



Let $z_1 = A(3, 4)$, $z_2 = B$ and $z_3 = C(1, -1)$
$$\text{Now,} \quad \frac{z_3 - z_2}{z_1 - z_2} = \left|\frac{z_3 - z_2}{z_1 - z_2}\right| \times e^{-i\frac{\pi}{2}}$$
$$\Rightarrow \quad \frac{z_3 - z_2}{z_1 - z_2} = \frac{|z_3 - z_2|}{|z_1 - z_2|} \times e^{-i\frac{\pi}{2}}$$
$$\Rightarrow \quad \frac{z_3 - z_2}{z_1 - z_2} = e^{-i\frac{\pi}{2}} = -i$$
$$\Rightarrow \quad z_3 - z_2 = -iz_1 + iz_2$$
$$\Rightarrow \quad (1 + i)z_2 = z_3 + iz_1$$
$$\Rightarrow \quad z_2 = \frac{z_3 + iz_1}{(1 + i)} = \frac{1 - i + i(3 + 4i)}{(1 + i)}$$
$$\Rightarrow \quad z_2 = \frac{-3 + 2i}{(1 + i)}$$
$$\Rightarrow \quad z_2 = \frac{(-3 + 2i)(1 - i)}{2}$$
$$\Rightarrow \quad z_2 = \frac{-3 + 3i + 2i + 2}{2} = \frac{-1 + 5i}{2}$$
Thus, the co-ordinates of B are $\left(-\frac{1}{2}, \frac{5}{2}\right)$

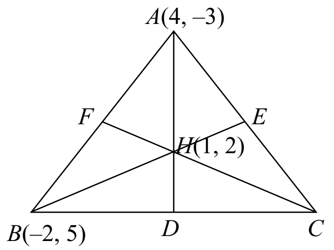
Let the co-ordinates of D be (α, β).

The mid-point of $AC = M = \left(2, \frac{3}{2}\right)$
$$\text{Now,} \quad \frac{\alpha - \frac{1}{2}}{2} = 2 \Rightarrow \alpha = \frac{9}{2}$$

and $\dfrac{\beta + \dfrac{5}{2}}{2} = \dfrac{3}{2} \Rightarrow \beta = \dfrac{1}{2}$

Hence, the co-ordinates of D are $\left(\dfrac{9}{2}, \dfrac{1}{2}\right)$.

11.



First we find the equation of BC and AC.

$$m(AD) = \frac{2+3}{1-4} = -\frac{5}{3}$$

Now, $m(BC) = \dfrac{3}{5}$

Thus, the equation of BC is

$$y - 5 = \frac{3}{5}(x+2)$$

$$5y - 25 = 3x + 6$$

$$3x - 5y + 31 = 0 \quad \ldots\text{(i)}$$

Also, $m(BE) = \dfrac{2-5}{1+2} = -\dfrac{3}{3} = -1$

$$m(AC) = 1$$

Thus, the equation of AC is

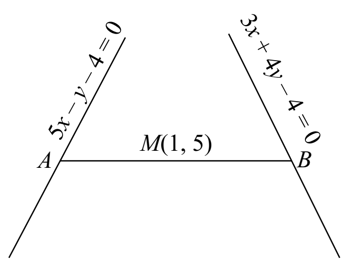
$$y + 3 = (x + 4)$$

$$x - y + 1 = 0 \quad \ldots\text{(ii)}$$

On solving Eqs (i) and (ii), we get the co-ordinates of C are (13, 14).

Thus, the third vertex is (13, 14).

12.



Let the points A and B be (x_1, y_1) and (x_2, y_2), respectively.

Thus, $5x_1 - y_1 - 4 = 0$ and $3x_2 + 4y_2 - 4 = 0$

Also, $x_1 + x_2 = 2$ and $y_1 + y_2 = 10$

Therefore,

$$y_1 + y_2 = 10$$

$$\Rightarrow \quad 5x_1 - 4 + \frac{4 - 3x_2}{4} = 10$$

$$\Rightarrow \quad 20x_1 - 3x_2 = 52$$

On solving, $20x_1 - 3x_2 = 52$ and $x_1 + x_2 = 2$, we get,

$$x_1 = \frac{58}{23} \text{ and } x_2 = -\frac{12}{23}$$

and so $y_1 = \dfrac{158}{23}, y_2 = \dfrac{32}{23}$

Hence, the points A and B are

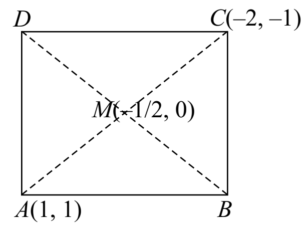
$$\left(\frac{58}{23}, \frac{158}{23}\right) \text{ and } \left(-\frac{12}{23}, \frac{32}{23}\right)$$

Now, slope of AB is = $\dfrac{83}{35}$

Hence, the equation of the line is

$$83x - 35y + 92 = 0$$

13.



Let $z_1 = A(1, 1)$, $z_2 = B$, $z_3 = C(-2, -1)$

Now, $\dfrac{z_3 - z_2}{z_1 - z_2} = \left|\dfrac{z_3 - z_2}{z_1 - z_2}\right| \times e^{-i\frac{\pi}{2}}$

$$\Rightarrow \quad \frac{z_3 - z_2}{z_1 - z_2} = \frac{|z_3 - z_2|}{|z_1 - z_2|} \times e^{-i\frac{\pi}{2}} = -i$$

$$\Rightarrow \quad z_3 - z_2 = iz_1 + iz_2$$

$$\Rightarrow \quad (1 + i)z_2 = z_3 + iz_1$$

$$\Rightarrow \quad (1 + i)z_2 = -2 - i + i(1 + i)$$

$$\Rightarrow \quad (1 + i)z_2 = -3$$

$$\Rightarrow \quad z_2 = \frac{-3}{(1+i)} = \frac{-3(1-i)}{2}$$

$$\Rightarrow \quad z_2 = \frac{-3+3i}{2} = \left(-\frac{3}{2}, \frac{3}{2}\right)$$

Thus, the co-ordinates of B are $\left(-\dfrac{3}{2}, \dfrac{3}{2}\right)$.

Let the co-ordinates of D be (α, β).

Now, $\dfrac{\alpha - \dfrac{3}{2}}{2} = -\dfrac{1}{2} \Rightarrow \alpha = \dfrac{1}{2}$

and $\dfrac{\beta + \dfrac{3}{2}}{2} = 0 \Rightarrow \beta = -\dfrac{3}{2}$

Therefore, the co-ordinates of D be $\left(\dfrac{1}{2}, -\dfrac{3}{2}\right)$.

Hence, the equation of the other diagonal is

$$y+\frac{3}{2}=-\frac{3}{2}\left(x-\frac{1}{2}\right)$$

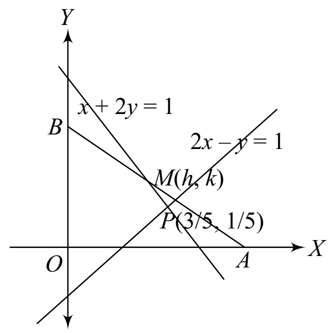
$$\Rightarrow \quad y+\frac{3}{2}=-\frac{3}{2}x+\frac{3}{4}$$

$$\Rightarrow \quad \frac{3}{2}x+y+\frac{3}{2}-\frac{3}{4}=0$$

$$\Rightarrow \quad \frac{3}{2}x+y+\frac{3}{4}=0$$

$$\Rightarrow \quad 6x+4y+3=0$$

14.



Equation of any line passing through the point of intersection of the lines $x + 2y = 1$ and $2x - y = 1$ is

$$y-\frac{1}{5}=m\left(x-\frac{3}{5}\right)$$

Clearly, the co-ordinates of A and B are

$$\left(\frac{3m-1}{5m}, 0\right) \text{ and } \left(0, \frac{1-3m}{5}\right)$$

Let $M(h, k)$ be the mid-point of AB

Thus, $h=\dfrac{3m-1}{10m}$ and $k=\dfrac{1-3m}{10}$

$$\Rightarrow \quad m=\frac{1}{3-10h}, m=\frac{1-10k}{3}$$

$$\Rightarrow \quad m=\frac{1}{3-10h}, \frac{1}{m}=\frac{3}{1-10k}$$

Eliminating m, we get

$$\frac{1}{3-10h}\times\frac{3}{1-10k}=1$$

$$\Rightarrow \quad (3-10h)(1-10k)=3$$

Hence, the locus of the mid-point $M(h, k)$ is

$$(3-10x)(1-10y)=3$$

15. The point of intersection of the lines

$$\frac{x}{a}+\frac{y}{b}=1 \text{ and } \frac{x}{b}+\frac{y}{a}=1 \text{ is } \left(\frac{ab}{a+b}, \frac{ab}{a+b}\right)$$

Thus, the equation of the line passing through $\left(\dfrac{ab}{a+b}, \dfrac{ab}{a+b}\right)$ is

$$y-\frac{ab}{a+b}=m\left(x-\frac{ab}{a+b}\right)$$

Therefore, the co-ordinates of A and B are

$$\frac{ab}{a+b}\left(1-\frac{1}{m}\right) \text{ and } \frac{ab}{a+\text{b}}(1-m)$$

Let the mid-point be $M(h, k)$.

Thus, $h=\dfrac{ab}{2(a+b)}\left(1-\dfrac{1}{m}\right)$ and $k=\dfrac{ab}{2(a+b)}(1-m)$

$$\Rightarrow \quad \frac{2h(a+b)}{ab}=\left(1-\frac{1}{m}\right) \text{ and } \frac{2k(a+b)}{ab}=(1-m)$$

$$\Rightarrow \quad \left(\frac{2h(a+b)}{ab}-1\right)=-\frac{1}{m} \text{ and } \left(\frac{2k(a+b)}{ab}-1\right)=-m$$

Eliminating m, we get

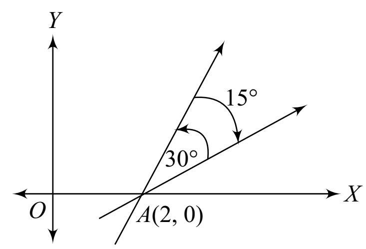
$$\Rightarrow \quad \left(\frac{2h(a+b)}{ab}-1\right)\left(\frac{2k(a+b)}{ab}-1\right)=1$$

Hence, the locus of $M(h, k)$ is

$$\left(2x-\frac{ab}{(a+b)}\right)\left(2y-\frac{ab}{(a+b)}\right)=\frac{ab}{(a+b)}$$

16. We have,

$$m=\tan(15°)=(2-\sqrt{3})$$



Hence, the equation of the line is

$$y-0=(2-\sqrt{3})(x-2)$$

$$\Rightarrow \quad (2-\sqrt{3})x-y-2(2-\sqrt{3})=0$$

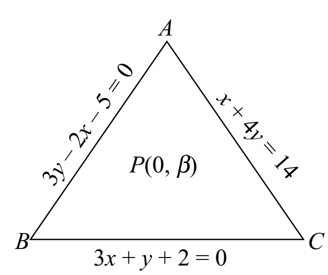
17. Clearly, $\dfrac{-3+8-7}{9-32-7}=\dfrac{-2}{-30}=\dfrac{1}{15}>0$

Thus, the points $(-1, -1)$ and $(3, 7)$ lie on the same side of the line.

18. On solving, we get the co-ordinates of A, B and C.

Thus, $A = (2, 3)$, $B = (-1, 1)$ and $C = (-2, 4)$.

Now, the points P and A lie on the same side of the line $3x + y + 2 = 0$.



So, $\dfrac{3.0+\beta+2}{6+3+2}>0 \Rightarrow \beta>-2$ …(i)

Also, the points P and B on the same side of the line $x + 4y = 14$

So, $\dfrac{0+4\beta-14}{-1+4-14} > 0$

$\Rightarrow \quad 4\beta - 14 < 0$

$\Rightarrow \quad \beta < \dfrac{7}{2}$ …(ii)

Again, the points P and C lie on the same side of the line $3y - 2x = 5$

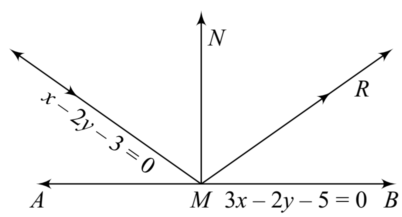
So, $\dfrac{3\beta - 5}{12+4-5} > 0$

$\Rightarrow \quad \beta > \dfrac{5}{3}$ …(iii)

From Eqs (i), (ii) and (iii), we get

$$\frac{5}{3} < \beta < \frac{7}{2}$$

19.



Let incident ray = IM and reflected ray = MR and MN be the normal.

On solving, we get

$M = (1, -2)$

Now, slope of $IM = 1/2$ and slope of $AB = 3/2$ and slope of $MN = -2/3$.

As we know that the normal is equally inclined with the incident ray as well as reflected ray.

Thus, $\dfrac{\frac{1}{2}+\frac{2}{3}}{1-\frac{1}{2}\cdot\frac{2}{3}} = \dfrac{-\frac{2}{3}-m}{1-\frac{2}{3}\cdot m}$

$\Rightarrow \quad \dfrac{3+4}{6-2} = \dfrac{-2-3m}{3-2m} = \dfrac{3m+2}{2m-3}$

$\Rightarrow \quad \dfrac{7}{4} = \dfrac{2+3m}{2m-3}$

$\Rightarrow \quad 14m - 21 = 8 + 12m$

$\Rightarrow \quad m = \dfrac{29}{2}$

Hence, the equation of the reflected ray

$$y + 2 = \frac{29}{2}(x-1)$$

$\Rightarrow \quad 2y + 4 = 29x - 29$

$\Rightarrow \quad 29x - 2y - 33 = 0$

20. Equation of any line parallel to $3x - 4y - 2 = 0$ is $3x - 4y + k = 0$

Given distance between parallel lines = 4

$\Rightarrow \quad \left|\dfrac{k+2}{\sqrt{9+16}}\right| = 4$

$\Rightarrow \quad \left|\dfrac{\mathrm{k}+2}{5}\right| = 4$

$\Rightarrow \quad k + 2 = \pm 20$

$\Rightarrow \quad k = -2 \pm 20$

$\Rightarrow \quad k = 18, -22$

Hence, the line is $3x - 4y + 18 = 0$ and $3x - 4y - 22 = 0$.

21. Given lines are $x + y + 1 = 0$ and $x + y - 1 = 0$

Clearly both are parallel.

Distance between them is $\sqrt{2}$.

Thus, the line must be perpendicular to the given lines.

Equation of any line perpendicular to $x + y + 1 = 0$ is $x - y + k = 0$ which is passing through $(-5, 4)$.

Thus, $-5 - 4 + k = 0$

$\Rightarrow \quad k = 9$

Hence, the equation of the line is $x - y + 9 = 0$.

22. Given $\Delta RPQ = 7$

$\Rightarrow \quad \dfrac{1}{2}\begin{vmatrix} x & y & 1 \\ 3 & 1 & 1 \\ 6 & 5 & 1 \end{vmatrix} = 7$

$\Rightarrow \quad \begin{vmatrix} x & y & 1 \\ 3 & 1 & 1 \\ 6 & 5 & 1 \end{vmatrix} = 14$

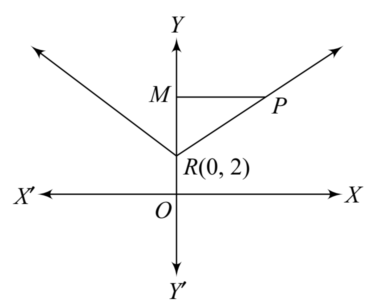
$\Rightarrow \quad x(1-5) - y(3-6) + 1(15-6) = \pm 14$

$\Rightarrow \quad -4x + 3y + 9 = \pm 14$

$\Rightarrow \quad -4x + 3y - 5 = 0, -4x + 3y + 23 = 0$

Clearly, infinite number of points satisfy the point R.

23.



Clearly, the co-ordinates of P are

$$\frac{x-0}{\cos 60^\circ} = \frac{y-2}{\sin 60^\circ} = 5$$

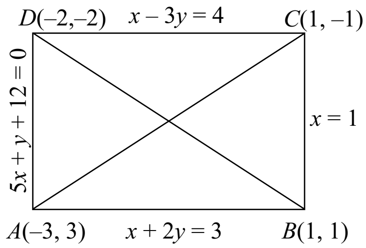
$$x = \frac{5}{2},\ y = \frac{4+5\sqrt{3}}{2}$$

Thus, the co-ordinates of M are

$$\left(0, \frac{4+5\sqrt{3}}{2}\right)$$

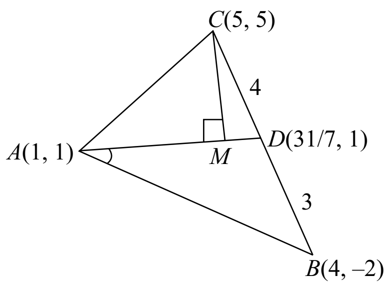
24.

25.



Clearly, the diagonal AC on the line $y = -x$ and the diagonal BD on the line $y = x$. Thus, the angle between $y = x$ and $y = -x$ is right angled.

26.



Here AD is the internal bisector of the angle $\angle BAC$.

Thus, $\dfrac{AB}{AC} = \dfrac{BD}{DC}$

$$\Rightarrow \quad \frac{BD}{DC} = \frac{AB}{AC} = \frac{3\sqrt{2}}{4\sqrt{2}} = \frac{3}{4}$$

Therefore, the co-ordinates of $D = \left(\dfrac{31}{7}, 1\right)$.

Hence, the equation of AD is $y - 1 = 0$

Thus, the equation of the perpendicular from C on the internal bisector AD is $x = 5$.

27. Let the foot of perpendicular be (h, k).

Then $\dfrac{h-2}{1} = \dfrac{k-4}{1} = -\left(\dfrac{2+4-1}{1+1}\right)$

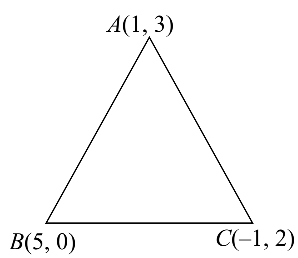
$$\Rightarrow \quad \frac{h-2}{1} = \frac{k-4}{1} = -\frac{5}{2}$$

$$\Rightarrow \quad h = 2 - \frac{5}{2}, k = 4 - \frac{5}{2}$$

$$\Rightarrow \quad h = -\frac{1}{2}, k = \frac{3}{2}$$

Hence, the foot of perpendicular is $\left(-\dfrac{1}{2}, \dfrac{3}{2}\right)$.

28. All the points inside the triangle will lie in the half plane determined by $ax + by + c \geq 0$ ($\leq$) if all the vertices lie in it.



For the first,

$3x + 2y > 0$,
$3(1) + 2(3) = 9 \geq 0$
$3(5) + 2(0) = 15 \geq 0$
$3(-1) + 2(2) = 1 \geq 0$

Thus, all the vertices lie in the half plane determined by $3x + 2y > 0$.

For the second,

$2x + y - 13 > 0$
$2(1) + 3 - 13 = -8 \leq 0$
$2(5) + 0 - 13 = -3 \leq 0$
$2(-1) + 2 - 13 = -13 \leq 0$

Thus, all the vertices do not lie in the half plane determined by $2x + y - 13 > 0$.

For the third,

$2x - 3y - 12 < 0$
$2(1) - 3(3) - 12 = -19 \leq 0$
$2(5) + 3(0) - 12 = -2 \leq 0$
$2(-1) - 3(2) - 12 = -20 \leq 0$

Thus, all the vertices lie in the half plane determined by $2x - 3y - 12 < 0$

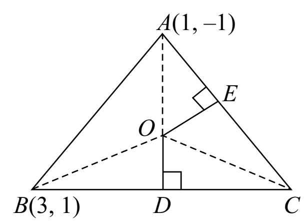
For the fourth,

$-2x + y \leq 0$
$-2(1) + 3 = 1 \geq 0$
$-2(5) + 0 = -10 \leq 0$
$-2(-1) + 2 = 4 \geq 0$

Thus, all the vertices do not lie in the half plane determined by $-2x + y \leq 0$.

Hence, all points inside the triangle satisfy the inequalities $3x + 2y > 0$ and $2x - 3y - 12 < 0$.

29. Given OE: $3x - 2y + 8 = 0$



AC is perpendicular to OE.

$AC : 2x + 3y + k = 0$

which is passing through $A(1, -1)$

Thus, $2 - 3 + k = 0$

$\Rightarrow \quad k = 1$

Hence, the equation of AC is $2x + 3y + 1 = 0$.

Let the co-ordinates of E be (a, b)

Thus, $2a + 3b + 1 = 0$

and $\quad 3a - 2b + 8 = 0$

On solving, we get,

$a = -2$ and $b = 1$

Therefore, $E = (-2, 1)$

Now, E is the mid-point of A and C.

Thus, $C = (-5, 3)$

Hence, the equation of BC is

$$\Rightarrow \quad y - 1 = \frac{1-3}{3+5}(x - 3)$$

$$\Rightarrow \quad y - 1 = -\frac{1}{4}(x - 3)$$

$$\Rightarrow \quad x + 4y - 7 = 0$$

Now, $D = (-1, 2)$

OD is perpendicular to BC

OD: $4x - y + k = 0$

which is passing through $D(-1, 2)$

Thus, $-4 - 2 + k = 0$

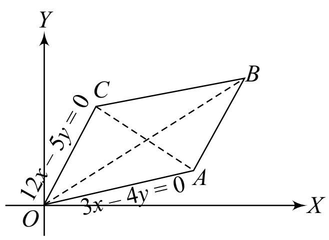
$\Rightarrow \quad k = 6$

Hence, the equation of OD is $4x - y + 6 = 0$

Now, we solve the equations $4x - y + 6 = 0$ and $3x - 2y + 8 = 0$, we get,

$$\text{circumcentre is } \left(-\frac{4}{5}, \frac{14}{5}\right).$$

30.



Clearly OB is the angle bisector of $\angle AOC$.

Thus, OB: $7y - 9x = 0$

It is given that $OB = 12$

Let the co-ordinates of B be (h, k).

Now slope of $OB = \tan\theta = \dfrac{9}{7}$

$$\frac{\sin\theta}{9} = \frac{\cos\theta}{7} = \frac{1}{\sqrt{130}}$$

$$\text{Now, } B = \left(0 + 12\cdot\frac{7}{\sqrt{130}}, 0 + 12\cdot\frac{9}{\sqrt{130}}\right)$$

$$B = \left(\frac{84}{\sqrt{130}}, \frac{108}{\sqrt{130}}\right)$$

Here BC is parallel to OA

BC: $3x - 4y + k = 0$

which is passing through B.

Thus, $k = -\dfrac{180}{\sqrt{130}}$

Hence, the equation of BC is

$$4y - 3x = \frac{180}{\sqrt{130}}$$

Similarly, we can easily find that the equation of AB is

$$5y - 12x + \frac{480}{\sqrt{130}} = 0$$

31. The equations of the line which are equally inclined with the axes are

$x + y = a, x - y = a$

It is given that,

$$\frac{3 - 4 - a}{\sqrt{2}} = \pm\frac{1 + 2 - a}{\sqrt{2}}$$

$\Rightarrow \quad -1 - a = \pm(3 - a)$

$\Rightarrow \quad -1 - a = (3 - a), -1 - a = -3 + a$

$\Rightarrow \quad -1 - a = -3 + a$

$\Rightarrow \quad 2a = 2$

$\Rightarrow \quad a = 1$

Hence, the equation of the line is $x - y - 1 = 0$

32. Given,

$\text{ar}(\Delta) = 8$

$$\Rightarrow \frac{1}{2}\begin{vmatrix} 2a & a & 1 \\ a & 2a & 1 \\ a & a & 1 \end{vmatrix} = 8$$

$$\Rightarrow \begin{vmatrix} 2a & a & 1 \\ a & 2a & 1 \\ a & a & 1 \end{vmatrix} = 16$$

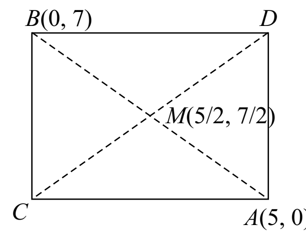
$\Rightarrow \quad 2a(2a - a) + (a^2 - 2a^2) = 16$

$\Rightarrow \quad 2a^2 - a^2 = 16$

$\Rightarrow \quad a^2 = 16$

$\Rightarrow \quad a = \pm 14$

33. Clearly slope of $CD = \dfrac{5}{7}$



$\tan\theta = \dfrac{5}{7}$

$$\Rightarrow \frac{\sin\theta}{5} = \frac{\cos\theta}{7} = \frac{1}{\sqrt{74}}$$

Thus, $MD = \dfrac{\sqrt{74}}{2}$

$$\text{Now, } D = \left(\frac{5}{2} + \frac{\sqrt{74}}{2}\cos\theta, \frac{7}{2} + \frac{\sqrt{74}}{2}\sin\theta\right)$$

$$= \left(\frac{5}{2} + \frac{\sqrt{74}}{2}\times\frac{7}{\sqrt{74}}, \frac{7}{2} + \frac{\sqrt{74}}{2}\times\frac{5}{\sqrt{74}}\right)$$

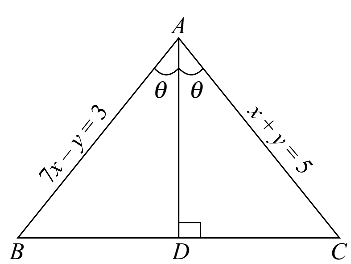
$$= \left(\frac{5}{2} + \frac{7}{2}, \frac{7}{2} + \frac{5}{2}\right) = (6, 6)$$

$$\text{Now, } C = \left(\frac{5}{2} - \frac{\sqrt{74}}{2}\cos\theta, \frac{7}{2} - \frac{\sqrt{74}}{2}\sin\theta\right)$$

$$= \left(\frac{5}{2} - \frac{\sqrt{74}}{2}\times\frac{7}{\sqrt{74}}, \frac{7}{2} - \frac{\sqrt{74}}{2}\times\frac{5}{\sqrt{74}}\right)$$

$$= \left(\frac{5}{2} - \frac{7}{2}, \frac{7}{2} - \frac{5}{2}\right) = (-1, 1)$$

34.



On solving $x + y = 5$ and $7x - y = 3$, we get the co-ordinates of A, i.e. $A = (1, 4)$

Let $AB = AC = m$

Thus, $BC = 2m\sin\theta$

Now, $\text{ar}(\Delta ABC) = 5$

$$\Rightarrow \quad \frac{1}{2}\cdot 2m\sin\theta\cdot m\cos\theta = 5$$

$$\Rightarrow \quad m^2\sin\theta = 10$$

$$\Rightarrow \quad m^2 = \frac{10}{\sin 2\theta} = \frac{10}{4/5} = \frac{25}{2}$$

$$\Rightarrow \quad m = \frac{5}{\sqrt{2}}$$

Now, slope of $AC = \tan\varphi = 7$

$$\Rightarrow \quad \cos\varphi = \frac{1}{\sqrt{50}}, \sin\varphi = \frac{7}{\sqrt{50}}$$

$$\text{Thus, } C = \left(1 \pm \frac{5}{\sqrt{2}}\cdot\frac{1}{\sqrt{50}}, 4 \pm \frac{5}{\sqrt{2}}\cdot\frac{7}{\sqrt{50}}\right)$$

$$\Rightarrow \quad C = \left(1 \pm \frac{1}{2}, 4 \pm \frac{7}{2}\right)$$

$$\Rightarrow \quad C = \left(\frac{3}{2}, \frac{15}{2}\right) \text{ or } \left(\frac{1}{2}, \frac{1}{2}\right)$$

Also, Slope of $AB = \tan\psi = -1$

$$\cos\psi = -\frac{1}{\sqrt{2}} \text{ and } \sin\psi = \frac{1}{\sqrt{2}}$$

$$\text{Thus, } D = \left(1 \pm \frac{5}{\sqrt{2}}\cdot\left(-\frac{1}{\sqrt{2}}\right), 4 \pm \frac{5}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\right)$$

$$= \left(1 \mp \frac{5}{2}, 4 \pm \frac{5}{2}\right)$$

$$= \left(-\frac{3}{2}, \frac{13}{2}\right) \text{ or } \left(\frac{7}{2}, \frac{3}{2}\right)$$

Therefore, the four possible equations of BC are

$$y - \frac{15}{2} = \frac{\frac{15}{2} - \frac{3}{2}}{\frac{3}{2} - \frac{7}{2}}\left(x - \frac{3}{2}\right) = -\frac{3}{2}\left(x - \frac{3}{2}\right),$$

$$= \frac{\frac{15}{2} - \frac{3}{2}}{\frac{3}{2} + \frac{3}{2}}\left(x - \frac{3}{2}\right) = 4\left(x - \frac{3}{2}\right),$$

$$\Rightarrow \quad y - \frac{1}{2} = \frac{\frac{1}{2} - \frac{3}{2}}{\frac{1}{2} - \frac{7}{2}}\left(x - \frac{1}{2}\right) = \frac{1}{3}\left(x - \frac{1}{2}\right)$$

$$\text{and} \quad y - \frac{1}{2} = \frac{\frac{1}{2} - \frac{13}{2}}{\frac{1}{2} + \frac{3}{2}}\left(x - \frac{1}{2}\right) = -3\left(x - \frac{1}{2}\right)$$

35. Let (h, k) be the image of $(4, 1)$ w.r.t. the line $y = x - 1$, i.e. $x - y - 1 = 0$

$$\text{Thus, } \frac{h-4}{1} = \frac{k-1}{-1} = -2\left(\frac{4-1-1}{1^2+1^2}\right)$$

$$\Rightarrow \quad \frac{h-4}{1} = \frac{k-1}{-1} = -2$$

$$\Rightarrow \quad h = 2, k = 3$$

Thus, the required image is $(2, 3)$.

(ii) Now $(2, 3)$ becomes $(2 + 1, 3)$ i.e., $(3, 3)$

(iii) Let $z_1 = 3 + 3i$

Then $z_2 = z_1 \times e^{i\frac{\pi}{4}}$

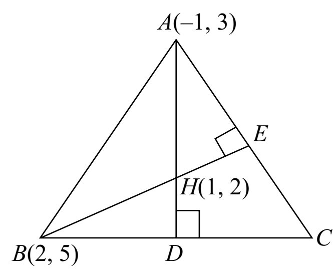
$$\Rightarrow \quad z_2 = (3+3i)\times\frac{1}{\sqrt{2}}(1+i)$$

$$= \frac{1}{\sqrt{2}}(3+3i)\times(1+i)$$

$$= \frac{3}{\sqrt{2}}(1+i)^2 = \frac{3}{\sqrt{2}}i$$

$$= \frac{3}{\sqrt{2}}i = \left(0, \frac{3}{\sqrt{2}}\right)$$

36.



To find the equations of BC and AC.

$$BC: \quad m(AD) = \frac{2-3}{1+1} = -\frac{1}{2}$$

$$\Rightarrow \quad m(BC) = 2$$

Equation of BC is

$$y - 5 = 2(x - 2)$$

$$\Rightarrow \quad 2x - y + 1 = 0 \qquad \text{...(i)}$$

$$AC: \quad m(BE) = \frac{2-5}{1-2} = 3$$

$$\Rightarrow \quad m(AC) = -\frac{1}{3}$$

Equation of AC is

$$y - 3 = -\frac{1}{3}(x+1)$$

$$\Rightarrow \quad 3y - 9 = -x - 1$$

$$\Rightarrow \quad x + 3y = 8 \qquad \text{...(ii)}$$

Solving Eqs (i) and (ii), we get

$$x = \frac{5}{7}, y = \frac{17}{7}$$

Thus, the co-ordinates of the third vertex are $\left(\frac{5}{7}, \frac{17}{7}\right)$.

Integer Type Questions

1. Let the line be $ax + by + c = 0$ ...(i)

Let p_1, p_2 and p_3 are the perpendicular distances from the line (i).

Clearly, $p_1 + p_2 + p_3 = 0$

$\Rightarrow \quad \dfrac{3a+c}{\sqrt{a^2+b^2}} + \dfrac{3b+c}{\sqrt{a^2+b^2}} + \dfrac{2a+2b+c}{\sqrt{a^2+b^2}} = 0$

$\Rightarrow \quad \dfrac{3a+c+3b+c+2a+2b+c}{\sqrt{a^2+b^2}} = 0$

$\Rightarrow \quad \dfrac{5a+5b+3c}{\sqrt{a^2+b^2}} = 0$

$\Rightarrow \quad 5a + 5b + 3c = 0$

$\Rightarrow \quad \dfrac{5}{3}a + \dfrac{5}{3}b + c = 0$

which passes through a fixed point $\left(\dfrac{5}{3}, \dfrac{5}{3}\right)$.

Thus, $p = \dfrac{5}{3}, q = \dfrac{5}{3}$.

Hence, the value of

$3(p + q) - 2$

$= 3\left(\dfrac{5}{3} + \dfrac{5}{3}\right) - 2$

$= 10 - 2 = 8$

2. Clearly, slope of the line $2x - 2y + 5 = 0$ is 1, i.e. $\theta = \dfrac{\pi}{4}$

Equation of any line passing through (2, 3) and making an angle $\dfrac{\pi}{4}$ is

$\dfrac{x-2}{\cos\left(\dfrac{\pi}{4}\right)} = \dfrac{y-3}{\sin\left(\dfrac{\pi}{4}\right)} = r$

$\Rightarrow \quad \dfrac{x-2}{\frac{1}{\sqrt{2}}} = \dfrac{y-3}{\frac{1}{\sqrt{2}}} = r$

$\Rightarrow \quad (x-2) = (y-3) = \dfrac{r}{\sqrt{2}}$ ...(i)

Thus, any point on the line (i) is

$\left(2 + \dfrac{r}{\sqrt{2}}, 3 + \dfrac{r}{\sqrt{2}}\right)$

which lies on $2x - 3y + 9 = 0$

So, $2\left(2 + \dfrac{r}{\sqrt{2}}\right) - 3\left(3 + \dfrac{r}{\sqrt{2}}\right) + 9 = 0$

$\Rightarrow \quad \left(\dfrac{2r}{\sqrt{2}} - \dfrac{3r}{\sqrt{2}}\right) + 4 = 0$

$\Rightarrow \quad -\dfrac{r}{\sqrt{2}} = -4$

$\Rightarrow \quad r = 4\sqrt{2}$

Clearly, $d\sqrt{2} = 4\sqrt{2}$

$\Rightarrow \quad d = 4$

$\Rightarrow \quad d + 2 = 6$

3. Equation of any line passing through (2, 3) is

$y - 3 = m(x - 2)$

$\Rightarrow \quad mx - y = 2m - 3$

$\Rightarrow \quad \dfrac{x}{\left(\dfrac{2m-3}{m}\right)} + \dfrac{y}{(3-2m)} = 1$

It is given that,

$\dfrac{1}{2} \times \left(\dfrac{2m-3}{m}\right) \times (3 - 2m) = \pm 12$

$\Rightarrow \quad (3 - 2m)^2 = \mp 24m$

$\Rightarrow \quad 9 - 12m + 4m^2 = \mp 24m$

$4m^2 + 12m + 9 = 0, 4m^2 - 36m + 9 = 0$

$\Rightarrow \quad (2m + 3)^2 = 0, 4m^2 - 36m + 9 = 0$

$\Rightarrow \quad (2m + 3) = 0, 4m^2 - 36m + 9 = 0$

$\Rightarrow \quad m = -3/2, D > 0$

So, the number of lines is 3.

4. We have,

$3x + 4(mx + 2) = 9$

$\Rightarrow \quad (3 + 4m)x = 1$

$\Rightarrow \quad x = \dfrac{1}{(3+4m)}$

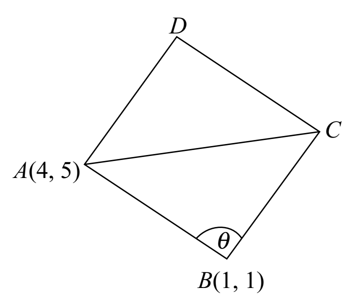
when $m = -1$, then $x = -1$

Thus, the number of integral values of m is 1.

5. Hence, the required area of the parallelogram

$= \left|\dfrac{(1-0)(1-0)}{\begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix}}\right| = \left|\dfrac{1}{(-2+1)}\right| = 1$ sq. u.

6.



We have

$AB = \sqrt{(4-1)^2 + (5-1)^2} = \sqrt{9+16} = \sqrt{25} = 5$

Area of a rhombus = 2ar (ΔABC)

$= 2\left(\dfrac{1}{2} \times 5 \times 5 \times \sin\theta\right)$

$= 25 \sin\theta$

Maximum area of a rhombus is 25.

Thus, $5m = 25$

$\Rightarrow \quad m = 5$

7. Here, the rhombus represented by the lines

$$x + 2y + 2 = 0$$
$$x + 2y - 2 = 0$$
$$x - 2y + 2 = 0$$
$$x - 2y - 2 = 0$$

Hence, the required area of the rhombus

$$= \left| \frac{(2-(-2))(2-(-2))}{\begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix}} \right|$$

$$= \frac{16}{4} = 4 \text{ sq. u.}$$

8. Number of latice point $= \frac{8(8+1)}{2} = 36$

Clearly, $m(m + 5) = 36$

$\Rightarrow \quad m^2 + 5m - 36 = 0$

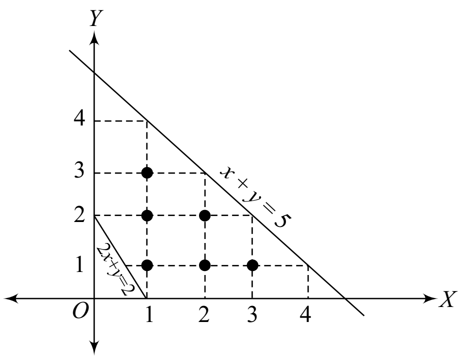
$\Rightarrow \quad (m + 9)(m - 4) = 0$

$\Rightarrow \quad m = 4, -9$

Hence, the value of m is 4.

9. Given lines are

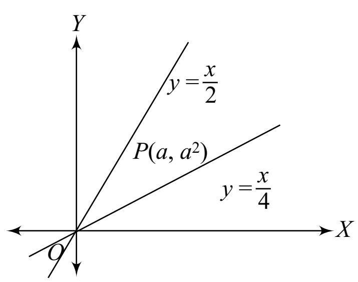
$2x + y = 6$, $x + y = 9$, $x = 0$ and $y = 0$.



From the above figure, it is clear that the number of IIT points is 6.

10. Given lines are $2y = x$ and $4y = x$

$\Rightarrow \quad y = \frac{x}{2}$ and $y = \frac{x}{4}$



We have $a - \frac{a^2}{4} > 0$ and $a - \frac{a^2}{2} < 0$

$\Rightarrow \quad (4a - a^2) < 0$ and $(2a - a^2) > 0$

$\Rightarrow \quad (a^2 - 4a) < 0$ and $(a^2 - 2a) > 0$

$\Rightarrow \quad a(a - 4) < 0$ and $a(a - 2) > 0$

$\Rightarrow \quad 0 < a < 4$ and $a < 0$, $a > 2$

$\Rightarrow \quad 2 < a < 4$

$\Rightarrow \quad a \in (2, 4) = (p, q)$

Hence, the value of $(p + q + 1)$

$= 2 + 4 + 1$

$= 7$

Previous Years' JEE-Advanced Examinations

1. Let the third vertex be (x, y) i.e. $(x, x + 3)$.

It is given that $\left| \frac{1}{2} \begin{vmatrix} 2 & 1 & 1 \\ 3 & -2 & 1 \\ x & x+3 & 1 \end{vmatrix} \right| = 5$

$\Rightarrow \quad \frac{1}{2}(4x - 4) = \pm 5$

$\Rightarrow \quad (4x - 4) = \pm 10$

$\Rightarrow \quad 4x = 4 \pm 10$

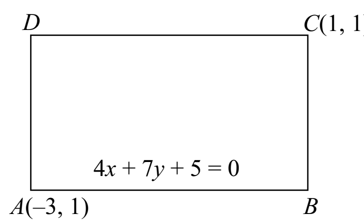
$\Rightarrow \quad 4x = 14, -6$

$\Rightarrow \quad x = \frac{7}{2}, -\frac{3}{2}$

Thus, the co-ordinates of the third vertex be

$$\left(\frac{7}{2}, \frac{13}{2}\right) \text{ or } \left(-\frac{3}{2}, \frac{3}{2}\right)$$

2. Let the equation of DC is $4x + 7y + k = 0$ which is passing through $(1, 1)$.Thus, $k = -11$. Hence, the equation of DC is $4x + 7y - 11 = 0$



Clearly BC is perpendicular to AB.

Let the equation of BC is $7x - 4y + k = 0$ which is passing through $(1, 1)$.

So, $k = -3$

Hence, the equation of DC is $7x - 4y - 3 = 0$.

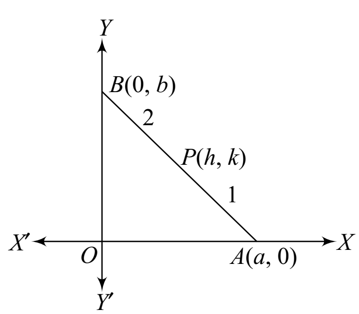
Also, AD is parallel to DC

Let the equation of DC is $7x - 4y + k = 0$ which is passing through $(-3, 1)$

So, $k = 25$

Thus, the equation of DC is $7x - 4y + 25 = 0$.

3.



Here, $h = \frac{2a+0}{2+1} = \frac{2a}{3}, k = \frac{b+0}{2+1} = \frac{b}{3}$

$\Rightarrow \quad a = \frac{3h}{2}, b = 3k$

It is given that, $AB = l$

$\Rightarrow \quad \sqrt{a^2+b^2} = l$

$\Rightarrow \quad a^2 + b^2 = l^2$

$\Rightarrow \quad \frac{9}{4}h^2 + 9k^2 = l^2$

$\Rightarrow \quad 9h^2 + 36k^2 = 4l^2$

Hence, the locus of (h, k) is

$$9x^2 + 36y^2 = 4l^2$$

4.

5. Here, $a_1 = 1, b_1 = -2$ and $a_2 = 4, b_2 = -3$

Now, $a_1a_2 + b_1b_2 = 4 + 16 = 10$

Thus, the obtuse-angle bisector is

$$\frac{x-2y+4}{\sqrt{1+4}} = \frac{4x-3y+2}{\sqrt{16+9}}$$

$\Rightarrow \quad \frac{x-2y+4}{\sqrt{5}} = \frac{4x-3y+2}{5}$

$\Rightarrow \quad \frac{x-2y+4}{1} = \frac{4x-3y+2}{\sqrt{5}}$

$\Rightarrow \quad (4x-3y+2) = \sqrt{5}(x-2y+4)$

$\Rightarrow \quad (4-\sqrt{5})x - (3-2\sqrt{5})y + (2-4\sqrt{5}) = 0$

6. Let $A = (-a, -b), B = (0, 0), C = (a, b)$

Now, $m(AB) = \frac{b}{a} = m(BC)$

Thus, the points A, B and C are collinear.

7. Let $A = (-a, -b), B = (0, 0), C = (a, b)$ and $D = (a^2, ab)$

Now, $m(AB) = \frac{b}{a} = m(BC) = m(BC)$

Thus, the points A, B, C and D are collinear.

8. Equation of any line perpendicular to $5x - y = 1$ is

$$x + 5y - k = 0$$

Let it intersects the x-axis at $(k, 0)$ and y-axis at $\left(0, \frac{k}{5}\right)$

Thus, $\frac{1}{2} \times k \times \frac{k}{5} = 5$

$\Rightarrow \quad k^2 = 50$

$\Rightarrow \quad k = \pm 5\sqrt{2}$

Hence, the equation of the line L is

$$x + 5y = \pm 5\sqrt{2}$$

9. Now, consider the lines $x + 2y = 3$, $2x + 3y = 4$ and $4x + 5y = 6$.

Now, $\begin{vmatrix} 1 & 2 & -3 \\ 2 & 3 & -4 \\ 4 & 5 & -6 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -3 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{vmatrix} \quad \begin{pmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_2 \end{pmatrix}$

$= 0$

Thus, the above three lines are concurrent.

10. Reflection of the point $(4, 1)$ w.r.t. the line $y = x$ is $(1, 4)$.

After transformation of 2 units along the positive direction of x-axis, it becomes $(3, 4)$.

Let $z_1 = 3 + 4i$ and the final position of the point is z_2.

Thus, $z_2 = z_1 \times e^{i\frac{\pi}{4}} = z_1 \times \left(\frac{1}{\sqrt{2}} + i \cdot \frac{1}{\sqrt{2}}\right)$

$\Rightarrow \quad z_2 = (3+4i) \times \frac{1}{\sqrt{2}}(1+i)$

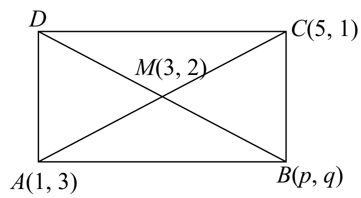
$\Rightarrow \quad z_2 = \frac{1}{\sqrt{2}} \times (3+4i) \times (1+i)$

$\Rightarrow \quad z_2 = \frac{1}{\sqrt{2}} \times (3+4i+3i-4)$

$\Rightarrow \quad z_2 = \frac{1}{\sqrt{2}} \times (-1+7i)$

$\Rightarrow \quad z_2 = \left(-\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$

11.



Clearly, the mid-point of $(1, 3)$ and $(5, 1)$ is $(3, 2)$ lies on the diagonal.

So, $\quad 2 = 6 + c$

$\Rightarrow \quad c = -4$

Equation of one diagonal is $y = 2x - 4$.

Let $B = (p, q)$

Now, $m(BC) \times m(AB) = -1$

$\Rightarrow \quad \left(\frac{q-1}{p-5}\right) \times \left(\frac{q-3}{p-1}\right) = -1$

$\Rightarrow \quad (q-1)(q-3) = -(p-1)(p-5)$

$\Rightarrow \quad q^2 - 4q + 3 = -p^2 + 6p - 5$

$\Rightarrow \quad p^2 + q^2 - 6p - 4q + 8 = 0 \quad \ldots(i)$

Also B lies on the line $y = 2x - 4$.

So, $\quad q = 2p - 4 \quad \ldots(ii)$

On solving Eqs (i) and (ii), we get,

$$p = 2, 4 \text{ and } q = 0, 4$$

Hence, the co-ordinates of the other vertices are $(2, 0)$ and $(4, 4)$.

12. Given line is $ax + by + c = 0 \quad \ldots(i)$

and $\quad 2a + 3b + 4c = 0$

$\Rightarrow \quad \frac{1}{2}a + \frac{3}{4}b + c = 0 \quad \ldots(ii)$

Subtracting Eqs (ii) from (i), we get

$$a\left(x - \frac{1}{2}\right) + b\left(y - \frac{3}{4}\right) = 0$$

$\Rightarrow \quad \left(x - \frac{1}{2}\right) + \frac{b}{a}\left(y - \frac{3}{4}\right) = 0$

$\Rightarrow \quad \left(x - \frac{1}{2}\right) = 0, \left(y - \frac{3}{4}\right) = 0$

$\Rightarrow \quad x = \frac{1}{2}, y = \frac{3}{4}$

Thus, the point is concurrent at $\left(\frac{1}{2}, \frac{3}{4}\right)$.

13. Clearly, the given lines intersect at (2, –2), (–2, 2) and (1, 1).

Let $P = (2, -2)$, $Q = (-2, 2)$ and $R = (1, 1)$

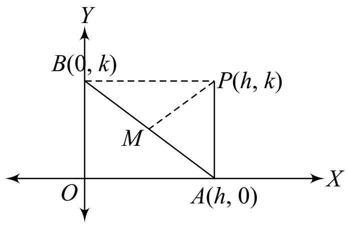
$\therefore \quad PQ = \sqrt{16+16} = 4\sqrt{2}$

Thus, $PR = \sqrt{1+9} = \sqrt{10}$

and $QR = \sqrt{9+1} = \sqrt{10}$.

So, P, Q and R form an isosceles triangle.

14. Given $AB = c$



Let $\quad AB: \frac{x}{h} + \frac{y}{k} = 1 \qquad$ …(i)

and $\quad PM: y - k = \frac{h}{k}(x - h) \qquad$ …(ii)

On solving Eqs (i) and (ii), we get

$$x = \frac{h^3}{h^2 + k^2} = \frac{h^3}{c^2} \text{ and } y = k + \frac{k}{h}\left(\frac{h^3}{c^2} - h\right)$$

Thus, $x = \frac{h^3}{c^2}$ and $y = \frac{k^3}{c^2}$

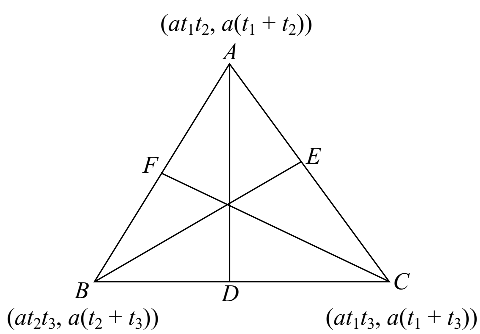
Now, $x^{2/3} + y^{2/3} = \left(\frac{h^3}{c^2}\right)^{2/3} + \left(\frac{k^3}{c^2}\right)^{2/3}$

$\Rightarrow \quad x^{2/3} + y^{2/3} = \frac{h^2 + k^2}{c^{4/3}}$

$\Rightarrow \quad x^{2/3} + y^{2/3} = \frac{c^2}{c^{4/3}}$

$\Rightarrow \quad x^{2/3} + y^{2/3} = c^{2/3}$

15.



First we find the equations of AD and BE.

Now, $m(BC) = \frac{1}{t_3}$

$\Rightarrow \quad m(AD) = t_3$

Thus, AD is $y - a(t_1 + t_2) = -t_3 x + at_1t_2t_3$

Similarly, BE is $y - a(t_2 + t_3) = -t_1 x + at_1t_2t_3$

On solving the equations of AD and BE, we get

$x = -a, y = a(t_1 + t_2 + t_3 + t_1t_2t_3)$

Thus, the required orthocentre is

$(-a, a(t_1 + t_2 + t_3 + t_1t_2t_3))$.

16. The point of intersection of the lines $x + 2y = 10$ and $2x + y + 5 = 0$ is $\left(-\frac{20}{3}, \frac{25}{3}\right)$.

Clearly, the line $5x + 4y = 0$ passes through the point $\left(-\frac{20}{3}, \frac{25}{3}\right)$.

17. Given the co-ordinates of A, B, C and P are (6, 3), (–3, 5), (4, –2) and (x, y).

Now, ar (ΔPBC)

$$= \frac{1}{2}\begin{vmatrix} x & y & 1 \\ -3 & 5 & 1 \\ 4 & -2 & 1 \end{vmatrix}$$

$$= \frac{1}{2}|7x + 7y - 14|$$

$$= \frac{7}{2}|x + y - 2|$$

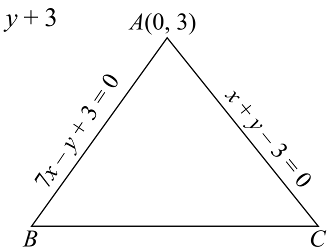
$$\text{ar}(\Delta ABC) = \frac{1}{2}\begin{vmatrix} 6 & 3 & 1 \\ -3 & 5 & 1 \\ 4 & -2 & 1 \end{vmatrix}$$

$$= \frac{1}{2}|42 + 21 - 14| = \frac{49}{2}$$

Thus, $\frac{\text{ar}(\Delta PBC)}{\text{ar}(\Delta ABC)} = \frac{\frac{7}{2}|x + y - 2|}{\frac{49}{2}}$

$$= \frac{|x + y - 2|}{7}$$

18. Two given lines are $7x - y + 3 = 0$ and $x + y - 3 = 0$. So the point of intersection is (0, 3).



Let $m(BC) = m$

$m(AB) = 7$,

$m(AC) = -1$

Clearly, $\angle ABC = \angle ACB$

Thus, $\left|\frac{m - 7}{1 + 7m}\right| = \left|\frac{m + 1}{1 - m}\right|$

$\Rightarrow \quad 6m^2 + 16m - 6 = 0$

$\Rightarrow \quad 3m^2 + 8m - 2 = 0$

$\Rightarrow \quad m = -3, \dfrac{1}{3}$

Hence, the equation of the BC is

$$y + 10 = -3(x - 1) \text{ or } y + 10 = \frac{1}{3}(x - 1)$$

$\Rightarrow \quad 3x + y - 7 = 0$ or $x - 3y - 31 = 0$.

19. Given a, b, c are in AP.

$\Rightarrow \quad 2b = a + c$

Now, $ax + by + c = 0$

$\Rightarrow \quad 2ax + 2by + 2c = 0$

$\Rightarrow \quad 2ax + (a + c)y + 2c = 0$

$\Rightarrow \quad a(2x + y) + c(y + 2) = 0$

$\Rightarrow \quad (2x + y) + \dfrac{c}{a}(y + 2) = 0$

$\Rightarrow \quad (2x + y) = 0, (y + 2) = 0$

$\Rightarrow \quad x = -1, y = -2$

Thus, the given straight line passes through a fixed point is (1, –2).

20. Three given lines are concurrent, if

$$\begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix} = 0$$

$\Rightarrow \quad -(p^3 + q^3 + r^3 - 3pqr) = 0$

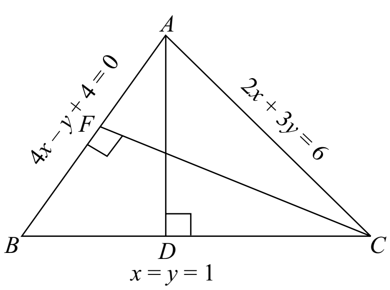
$\Rightarrow \quad (p^3 + q^3 + r^3 - 3pqr) = 0$

$\Rightarrow \quad (p + q + r)(p^2 + q^2 + r^2 - pq - qr - rp) = 0$

$\therefore \quad (p + q + r) = 0, (p^2 + q^2 + r^2 - pq - qr - rp) = 0$

$\Rightarrow \quad (p + q + r) = 0, p^2 + q^2 + r^2 = pq + qr + rp) = 0$

21.



On solving the given equation, we get

$$A = \left(-\frac{3}{7}, \frac{16}{7}\right), B = \left(-\frac{3}{5}, \frac{4}{5}\right) \text{ and } C = (-3, 4)$$

Equation of AD is

$$y - \frac{16}{7} = \left(x + \frac{3}{7}\right)$$

$\Rightarrow \quad 7x - 7y + 19 = 0$

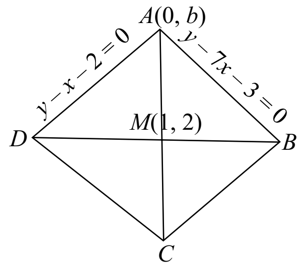
Equation of BC is

$$y - 4 = \frac{1}{4}(x + 3)$$

$\Rightarrow \quad x + 4y - 13 = 0$

On solving AD and BC, we get the orhocentre is $\left(\dfrac{3}{7}, \dfrac{22}{7}\right)$, which lies in the first quadrant.

22. Since the sides of the rhombus are parallel to the lines $y = x + 2$ and $y = 7x + 3$, so its diagonals are parallel to the bisectors of the angles between these lines.



Thus, $\left(\dfrac{y - x - 2}{\sqrt{1 + 1}}\right) = \pm\left(\dfrac{y - 7x - 3}{\sqrt{1 + 49}}\right)$

$\Rightarrow \quad \left(\dfrac{y - x - 2}{\sqrt{2}}\right) = \pm\left(\dfrac{y - 7x - 3}{5\sqrt{2}}\right)$

$\Rightarrow \quad 5(y - x - 2) = \pm(y - 7x - 3)$

$\Rightarrow \quad 2x + 4y - 7 = 0$ or $12x - 6y + 13 = 0$

Therefore, the slope of AM is 2 or –1/2

Thus, $\dfrac{b - 2}{0 - 1} = 2$ or $-\dfrac{1}{2}$

$\Rightarrow \quad b = 0$ or $\dfrac{5}{2}$

Hence, the possible co-ordinates of A are

$$(0, 0) \text{ or } \left(0, \frac{5}{2}\right)$$

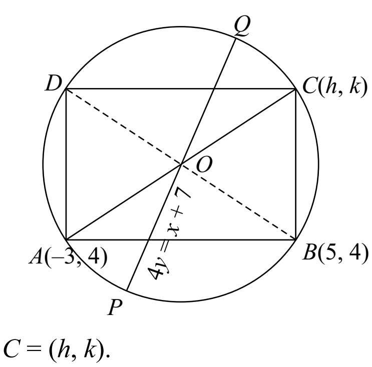
23. Given $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$

$\Rightarrow \quad \dfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \dfrac{1}{2}\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$

Thus, areas are the same.

But it is not sure that it will be congruent.

24.



Let $\quad C = (h, k)$.

Slope of $AB = 0$ and BC is perpendicular to AB.

Thus, $h - 5 = 0$

$\Rightarrow \quad h = 5$

Clearly the mid-point of AC lies on the given diameter PQ.

Thus, $4\left(\frac{k+4}{2}\right) = \frac{h-3}{2} + 7 = 8$

$\Rightarrow \quad k = 0$

Therefore C is $(5, 0)$

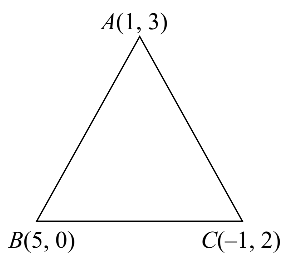
So, $\quad BC = 4$ and $AB = 8$

Hence, the area of the rectangle $ABCD$ is

$= AB \times BC = 8 \times 4 = 32$ sq. u.

25.

26. All the points inside the triangle will lie in the half plane determined by $ax + by + c \geq 0$ if all the vertices lie in it.



For the first,

$3x + 2y > 0,$

$3(1) + 2(3) = 9 \geq 0$

$3(5) + 2(0) = 15 \geq 0$

$3(-1) + 2(2) = 2 \geq 0$

Thus, all the vertices lie in the half plane determined by $3x + 2y > 0$.

For the second,

$2x + y - 13 > 0$

$2(1) + 3 - 13 = -8 \leq 0$

$2(0) + 0 - 13 = -3 \leq 0$

$2(-1) + 2 - 13 = -13 \leq 0$

Thus, all the vertices do not lie in the half plane determined by $2x + y - 13 > 0$.

For the third,

$2x - 3y - 12 < 0$

$2(1) - 3(3) - 12 = -19 \leq 0$

$2(5) - 3(0) - 12 = -2 \leq 0$

$2(-1) - 3(2) - 12 = -20 \leq 0$

Thus, all the vertices lie in the half plane determined by $2x - 3y - 12 \leq 0$.

For the fourth,

$-2x + y \leq 0$

$-2(1) + 3 = 1 \geq 0$

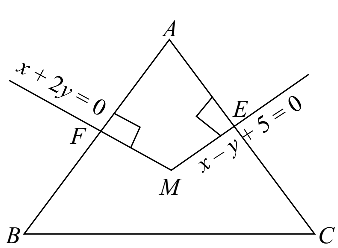
$-2(5) + 0 = -10 \leq 0$

$-2(-1) + 2 = 4 \geq 0$

Thus, all the vertices do not lie in the half plane determined by $-2x + y \leq 0$.

Hence, all points inside the triangle satisfy the inequalities $3x + 2y \geq 0$ and $2x - 3y - 12 \leq 0$.

27.



Let the co-ordinates of B and C are (x_1, y_1) and (x_2, y_2), respectively.

Clearly the mid-point of A and B and A and C lie on the perpendicular bisectors $x + 2y = 0$ and $x - y + 5 = 0$.

So, $\quad \frac{x_1+1}{2} + 2\left(\frac{y_1-2}{2}\right) = 0$

$\Rightarrow \quad x_1 + 2y_1 - 13 = 0 \qquad \ldots\text{(i)}$

and $\quad \left(\frac{x_2+1}{2}\right) - \left(\frac{y_2-2}{2}\right) + 5 = 0$

$\Rightarrow \quad x_2 - y_2 + 13 = 0 \qquad \ldots\text{(ii)}$

Also, $\left(\frac{y_1+2}{x_1-1}\right)\left(-\frac{1}{2}\right) = -1$

$\Rightarrow \quad 2x_1 - y_1 - 4 = 0$

and $\quad \left(\frac{y_2-2}{x_2-1}\right)(1) = -1$

$\Rightarrow \quad x_2 + y_2 + 1 = 0 \qquad \ldots\text{(iv)}$

Solving Eqs (i) and (iii), we get the co-ordinates of B,

i.e. $B = \left(\frac{11}{5}, \frac{2}{5}\right)$

Solving Eqs (ii) and (iv), we get the co-ordinates of C.

i.e. $C = (-7, 6)$

Now, $m(BC) = \dfrac{6 - \frac{2}{5}}{-7 - \frac{11}{5}} = -\frac{28}{46} = -\frac{14}{23}$.

Therefore, the equation of BC is

$$y - 6 = -\frac{14}{23}(x + 7)$$

$\Rightarrow \quad 23y - 138 = 14x - 98$

$\Rightarrow \quad 14x + 23y - 40 = 0$

28 The slope of

$$\left(0, \frac{8}{3}\right) \text{ and } (1, 3) \text{ and } (1, 3) \text{ and } (82, 30)$$

are same. So the given points are collinear.

No questions asked in 1987.

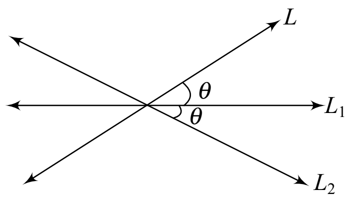
29. We know that two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ cut the co-ordinate axes in concyclic points if $a_1a_2 = b_1b_2$.

Here, $a_1a_2 = -\frac{19}{2} \times \frac{17}{9} = -\frac{243}{18}$

and $\quad b_1b_2 = -\frac{19}{3} \times \frac{17}{6} = -\frac{243}{18}$

Thus, the given lines cut the co-ordinate axes in concyclic points.

30. According to the question, the required line passes through the point of intersection of L_1 and L_2.



Equation of any line passing through the point of intersection of L_1 and L_2 is

$L_1 + \lambda L_2 = 0$

$\Rightarrow \quad ax + by + c + \lambda(lx + my + n) = 0$

Here, L_1 is equally inclined with L and L_2, thus

$$\frac{m_3 - m_1}{1 + m_1 m_3} = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\Rightarrow \quad \frac{\frac{a + \lambda l}{b + \lambda m} - \frac{a}{b}}{1 + \left(\frac{a + \lambda l}{b + \lambda m}\right)\left(\frac{a}{b}\right)} = \frac{\frac{a}{b} - \frac{l}{m}}{1 + \left(\frac{a}{b}\right)\left(\frac{l}{m}\right)}$$

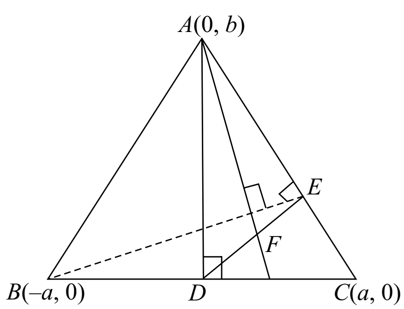
$$\Rightarrow \quad \frac{am - bl}{al + bm} = \frac{\lambda(bl - am)}{a^2 + b^2 + \lambda(bm + al)}$$

$$\Rightarrow \quad \lambda = -\frac{a^2 + b^2}{al + bm + 1}$$

Hence, the equation of the required line is

$$ax + by + c - \frac{a^2 + b^2}{al + bm + 1}(lx + my + n) = 0$$

31.



Let the co-ordinates of A, B and C are $(0, b)$, $(-a, 0)$ and $(a, 0)$, respectively and D as origin.

Equation of AC is $\frac{x}{a} + \frac{y}{b} = 1$

Consider the co-ordinates of E be $\left(t, b\left(1 - \frac{t}{a}\right)\right)$.

$$\text{Now, } m(DE) = \frac{b\left(1 - \frac{t}{a}\right)}{t} = \frac{b(a - t)}{at}$$

$$\text{and} \quad m(AC) = -\frac{b}{a}$$

Since $DE \perp AC$

$$\Rightarrow \quad \frac{b(a - t)}{at} \times \left(-\frac{b}{a}\right) = -1$$

$\Rightarrow \quad b^2(a - t) = a^2 t$

$$\Rightarrow \quad t = \frac{ab^2}{a^2 + b^2}$$

Thus, the co-ordinates of E are

$$\left(\frac{ab^2}{a^2 + b^2}, \frac{a^2 b}{a^2 + b^2}\right)$$

and the co-ordinates of F are

$$\left(\frac{ab^2}{2(a^2 + b^2)}, \frac{a^2 b}{2(a^2 + b^2)}\right)$$

Now, slope of $AF = -\left(\frac{a^2 + b^2}{ab}\right)$

and slope of $BE = -\left(\frac{ab}{a^2 + b^2}\right)$

Clearly, $AF \perp BE$.

32. Clearly, the point A is $(3, -1)$.

Equation of BC is $y - 2 = m(x - 1)$

$$\text{Thus, } \left|\frac{m + \frac{3}{4}}{1 - \frac{3}{4}m}\right| = \tan 45°$$

$$\Rightarrow \quad \left|\frac{4m + 3}{4 - 3m}\right| = 1$$

$$\Rightarrow \quad \left(\frac{4m + 3}{4 - 3m}\right) = \pm 1$$

$$\Rightarrow \quad m = -7, \frac{1}{7}$$

Hence, the equation of the line BC is

$$y - 2 = -7(x - 1) \text{ or } y - 2 = \frac{1}{7}(x - 1)$$

$$x - 7y + 13 = 0 \text{ or } 7x + y - 9 = 0.$$

33. Let $\quad L: \frac{x}{a} + \frac{y}{b} = 1$

After rotation through a given angle is

$$L: \frac{x}{p} + \frac{y}{q} = 1$$

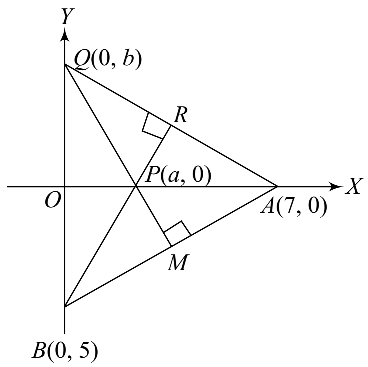
So, the distance from the origin of both the lines will be the same.

$$\text{Thus, } \left|\frac{0 + 0 - 1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}\right| = \left|\frac{0 + 0 - 1}{\sqrt{\frac{1}{p^2} + \frac{1}{q^2}}}\right|$$

$$\Rightarrow \quad \sqrt{\frac{1}{a^2} + \frac{1}{b^2}} = \sqrt{\frac{1}{p^2} + \frac{1}{q^2}}$$

$$\Rightarrow \quad \left(\frac{1}{a^2} + \frac{1}{b^2}\right) = \left(\frac{1}{p^2} + \frac{1}{q^2}\right)$$

34.



Equation of AB is $\frac{x}{7}+\frac{y}{-5}=1$

Equation of PQ is $\frac{x}{a}+\frac{y}{b}=1$

Here, PQ is perpendicular to AB.

Thus, $-\frac{b}{a}\times\frac{5}{7}=-1$

$\Rightarrow \quad \frac{b}{7}=\frac{a}{5}=k$ (say)

Therefore, the equation of BP is

$$\frac{x}{a}-\frac{y}{5}=1$$

$\Rightarrow \quad \frac{x}{5k}-\frac{y}{5}=1$

$\Rightarrow \quad x-ky=5k \qquad \ldots(i)$

Similarly, the equation of AQ is

$$\frac{x}{7}+\frac{y}{b}=1$$

$\Rightarrow \quad \frac{x}{7}+\frac{y}{7k}=1$

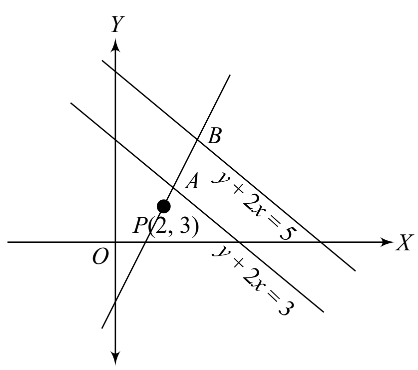
$\Rightarrow \quad kx+y=7k \qquad \ldots(ii)$

From Eqs (i) and (ii), we get

$$\left(\frac{x}{y+5}\right)x+y=7\left(\frac{x}{y+5}\right)$$

$\Rightarrow \quad x^2+y^2-7x+5y=0$

35.



Equation of any line through P is

$$y-3=m(x-2) \qquad \ldots(i)$$

The line (i) intersects the given lines at A and B, respectively.

Thus, the co-ordinates of A and B are

$$A=\left(\frac{2m}{m+2},\frac{6-m}{m+2}\right)$$

and $\quad B=\left(\frac{2m+2}{m+2},\frac{m-6}{m+2}\right)$

It is given that, $AB=2$

$\Rightarrow \quad AB^2=4$

$\Rightarrow \quad \left(\frac{2m+2-2m}{m+2}\right)^2+\left(\frac{m-6-6+m}{m+2}\right)^2=4$

$\Rightarrow \quad \left(\frac{2}{m+2}\right)^2+\left(\frac{2m-12}{m+2}\right)^2=4$

$\Rightarrow \quad \left(\frac{1}{m+2}\right)+\left(\frac{m-6}{m+2}\right)^2=1$

$\Rightarrow \quad 1+(m-6)^2=(m+2)^2$

$\Rightarrow \quad 1+m^2-12m+36=m^2+4m+4$

$\Rightarrow \quad 16m=33 \qquad \Rightarrow m=\frac{33}{16}$

Hence, the equation of the line is

$$y-3=\frac{33}{16}(x-2)$$

$\Rightarrow \quad 16y-48=33x-6$

$\Rightarrow \quad 33x-16y=66-48=18$

36. Let the variable line be $ax+by+c=0$.

Given $\frac{(2a+c)}{\sqrt{a^2+b^2}}+\frac{(2b+c)}{\sqrt{a^2+b^2}}+\frac{(a+b+c)}{\sqrt{a^2+b^2}}=0$

$\Rightarrow \quad (2a+c)+(2b+c)+(a+b+c)=0$

$\Rightarrow \quad 3(a+b+c)=0$

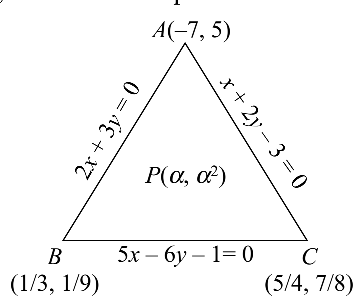
$\Rightarrow \quad (a+b+c)=0$

Thus, the line $ax+by+c=0$ passes through the point (1, 1).

37. $|x|+|y|=1$

Clearly, the locus is a square.

38.



On solving the equations, we get the co-ordinates of A, B and C, respectively.

Now, A and P lie on the same side of the line

$$5x-6y-1=0$$

Thus, $\dfrac{5\alpha - 6\alpha^2 - 1}{5(-7) - 6(5) - 1} > 0$

$\Rightarrow \quad 6\alpha^2 - 5\alpha + 1 > 0$

$\Rightarrow \quad (3\alpha - 1)(2\alpha - 1) > 0$

$\Rightarrow \quad \alpha < \dfrac{1}{3}$ or $\alpha > \dfrac{1}{2}$ …(i)

Again, the points P and B lie on the same side of the line $x + 2y - 3 = 0$.

Thus, $\dfrac{\alpha + 2\alpha^2 - 3}{\frac{1}{3} + \frac{2}{9} - 3} > 0$

$\Rightarrow \quad 2\alpha^2 + \alpha - 3 < 0$

$\Rightarrow \quad (2\alpha + 3)(\alpha - 1) < 0$

$\Rightarrow \quad -\dfrac{3}{2} < \alpha < 1$ …(ii)

Finally, the points P and C lie on the same side of the line $2x - 3y - 1 = 0$

Thus, $\dfrac{2\alpha + 3\alpha^2 - 1}{2\left(\frac{5}{4}\right) + 3\left(\frac{7}{8}\right) - 1} > 0$

$\Rightarrow \quad 3\alpha^2 + 2\alpha - 1 > 0$

$\Rightarrow \quad (3\alpha - 1)(\alpha + 1) > 0$

$\Rightarrow \quad \alpha < -1$ or $\alpha > \dfrac{1}{3}$ …(iii)

From Eqs (i), (ii) and (iii), we get

$$\alpha \in \left(-\frac{3}{2}, -1\right) \cup \left(\frac{1}{2}, 1\right)$$

39. Let BD is the angle bisector of the $\angle ABC$.

Now, $\dfrac{AD}{DC} = \dfrac{BA}{BC} = \dfrac{\sqrt{36 + 64}}{\sqrt{16 + 9}} = \dfrac{10}{5} = \dfrac{2}{1}$

Thus, the co-ordinates of D are $\left(\dfrac{1}{3}, \dfrac{1}{3}\right)$

Therefore, the equation of BD is

$$(y - 1) = \frac{1}{7}(x - 5)$$

$\Rightarrow \quad x - 7y + 2 = 0$

40. Equation of any line passing through A is

$$\frac{x + 5}{\cos\theta} = \frac{y + 4}{\sin\theta} = r_1, r_2, r_3$$

Let $AB = r_1$, $AC = r_2$, $AD = r_3$

Clearly, the point $(r_1\cos\theta - 5, r_1\sin\theta - 4)$ lies on the line $x + 3y + 2 = 0$

So, $\quad (r_1\cos\theta - 5) + 3(r_1\sin\theta - 4) + 2 = 0$

$\Rightarrow \quad r_1 = \dfrac{15}{\cos\theta + 3\sin\theta}$

$\Rightarrow \quad \dfrac{15}{r_1} = (\cos\theta + 3\sin\theta)$

$\Rightarrow \quad \dfrac{15}{AB} = (\cos\theta + 3\sin\theta)$

Similarly, $\dfrac{10}{AC} = 2\cos\theta + \sin\theta$

and $\quad \dfrac{6}{AD} = \cos\theta - \sin\theta$

It is given that

$$\left(\frac{15}{AB}\right)^2 + \left(\frac{10}{AC}\right)^2 = \left(\frac{6}{AD}\right)^2$$

$\Rightarrow \quad (\cos\theta + 3\sin\theta)^2 + (2\cos\theta + \sin\theta)^2 = \cos\theta - \sin\theta)^2$

$\Rightarrow \quad 4\cos^2\theta + 9\sin^2\theta + 12\sin\theta\cos\theta = 0$

$\Rightarrow \quad (2\cos\theta + 3\sin\theta)^2 = 0$

$\Rightarrow \quad (2\cos\theta + 3\sin\theta) = 0$

$\Rightarrow \quad \tan\theta = -\dfrac{2}{3}$

Hence, the equation of the required line is

$$y + 4 = -\frac{2}{3}(x + 5)$$

$\Rightarrow \quad 3y + 12 = -2x - 10$

$\Rightarrow \quad 2x + 3y + 22 = 0$

41. As we know that the point of intersection of any two perpendicular sides of a triangle is also called ortho-centre.

Given lines are $xy = 0$ and $x + y = 1$

$\Rightarrow \quad x = 0, y = 0$ and $x + y = 1$

42. Ans. $(m^2 - 1)x - my + b(m^2 + 1) + am = 0$

43. The given lines are $x + 3y = 4$ and $6x - 2y = 7$ which are mutually perpendicular to each other. Thus, $PQRS$ must be a rhombus.

44. Equation of PS is $2x + 9y = 22$.

Equation of any line parallel to PS is

$2x + 9y + k = 0$

Which is passing through $(1, -1)$

So, $\quad k = 7$

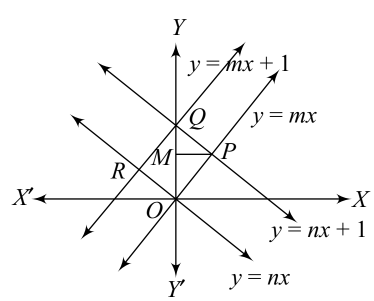
Hence, the equation of the required line

is $2x + 9y + 7 = 0$.

45.

46.

47.



Area of a parallelogram $OPQR$

$= 2$ (ar of ΔOPQ)

$$= 2 \times \frac{1}{2} \times OQ \times PM$$
$$= OQ \times PM$$
$$= \frac{1}{|m-n|}$$

48. We have $3x + 4mx + 4 = 9$

$\Rightarrow \quad x(3 + 4m) = 5$

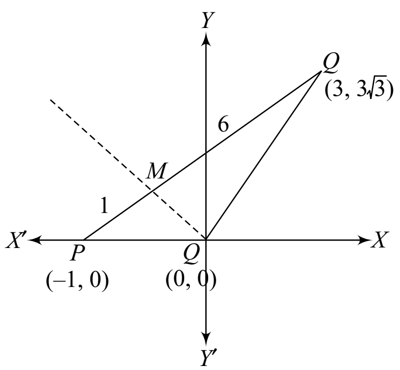
$\Rightarrow \quad x = \frac{5}{(3+4m)}$

When $m = -1$, then $x = -5$

When $m = -2$, then $x = -1$

Thus, the number of possible integral values of m is 2.

49.



Since OM is the internal bisector of $\angle PQR$.

$$\frac{QM}{PM} = \frac{OQ}{OP} = \frac{6}{1}$$

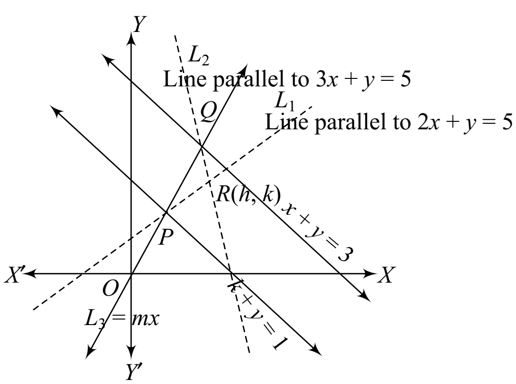
Thus, the co-ordinates of $M = \left(-\frac{3}{7}, \frac{3\sqrt{3}}{7}\right)$

Therefore, the equation of OM is

$$y = -\sqrt{3}\,x$$

$\Rightarrow \quad y + \sqrt{3}x = 0$

50.



Let L: $y = mx$

Then the co-ordinates of P and Q are

$$P = \left(\frac{1}{m+1}, \frac{m}{m+1}\right) \text{ and}$$
$$Q = \left(\frac{3}{m+1}, \frac{3m}{m+1}\right)$$

Equation of the lines L_1 and L_2 are

$$L_1: 2x - y = \frac{2-m}{m+1}$$
$$L_2: 3x + y = \frac{9+3m}{m+1}$$

Let the co-ordinates of R be (h, k)

Thus, $h = \frac{11+2m}{5(m+1)}, k = \frac{12+9m}{5(m+1)}$

Eliminating m from the above relations, we get

$$5h - 15k + 25 = 0$$

$\Rightarrow \quad h - 3k + 5 = 0$

Hence, the locus of (h, k) is

$$x - 3y + 5 = 0$$

51. Equation of any line passing through (8, 2) is

$$y - 2 = m(x - 8)$$

Put $\quad y = 0, m(x - 8) = -2.$

$\Rightarrow \quad (x-8) = -\frac{2}{m}$

$\Rightarrow \quad x = 8 - \frac{2}{m}$

Thus, $OP = 8 - \frac{2}{m}$

Put $x = 0$, then $y = 2 - 8m$

Thus, $OQ = 2 - 8m$

Let $\quad S = OP + OQ = \left(8 - \frac{2}{m}\right) + (2 - 8m)$

$\Rightarrow \quad \frac{dS}{dm} = \frac{2}{m^2} - 8$

For maximum or minimum,

$$\frac{dS}{dm} = 0$$

$\Rightarrow \quad \frac{2}{m^2} - 8 = 0$

$\Rightarrow \quad m^2 = \frac{1}{4}$

$\Rightarrow \quad m = \pm\frac{1}{2}$

Now, $\frac{d^2S}{dm^2} = -\frac{2}{m^3}$

$\Rightarrow \quad \left(\frac{d^2S}{dm^2}\right)_{m=-\frac{1}{2}} = 16 > 0$

Thus, the sum will provide us the least value.
The least value is = 8 + 4 + 2 + 4 = 18.

52. Equation of any line passing through (2, 2) is

$$y - 2 = m(x - 2)$$

Let the points A and B are

$$A = \left(\frac{2m-2}{m+\sqrt{3}}, -\sqrt{3}\left(\frac{2m-2}{m+\sqrt{3}}\right)\right)$$

and $$B = \left(\frac{2m-2}{m-\sqrt{3}}, \sqrt{3}\left(\frac{2m-2}{m-\sqrt{3}}\right)\right)$$

Since the triangle OAB is equilateral, so

$$OA = OB = AB$$

$$\Rightarrow \left(\frac{2m-2}{(m+\sqrt{3})}\right)^2 + 3\left(\frac{2m-2}{(m+\sqrt{3})}\right)^2$$

$$= \left(\frac{2m-2}{(m-\sqrt{3})}\right)^2 + 3\left(\frac{2m-2}{(m-\sqrt{3})}\right)^2$$

$$\Rightarrow \frac{4}{(m+\sqrt{3})^2} = \frac{4}{(m-\sqrt{3})^2}$$

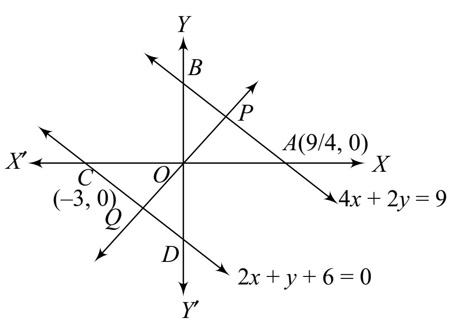
$$\Rightarrow (m+\sqrt{3})^2 = (m-\sqrt{3})^2$$

$$\Rightarrow m = 0$$

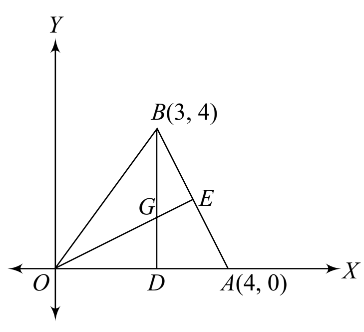
Hence the required equation of the line is $y = 2$.

53. Triangles OPA and OQC are similar.

Thus, $\dfrac{OP}{OQ} = \dfrac{OA}{OC} = \dfrac{9/4}{3} = \dfrac{3}{4}$



54. Equation of the altitude BD is $x = 3$.



Slope of AB is $\dfrac{4-0}{3-4} = -4$

and Slope of OE is $\dfrac{1}{4}$.

Equation of OE is $y = \dfrac{1}{4}x$

Thus the line BD and OE meet in $\left(3, \dfrac{3}{4}\right)$

Hence, the centroid is $\left(3, \dfrac{3}{4}\right)$.

55. Let the point P be (x, y)

Given

$$\sqrt{(x-a_1)^2 + (y-b_1)^2} = \sqrt{(x-a_2)^2 + (y-b_2)^2}$$

$$(x-a_1)^2 + (y-b_1)^2 = (x-a_2)^2 + (y-b_2)^2$$

$$a_1^2 - 2a_1x + b_1^2 - 2b_1y = a_2^2 - 2a_2x + b_2^2 - 2b_2y$$

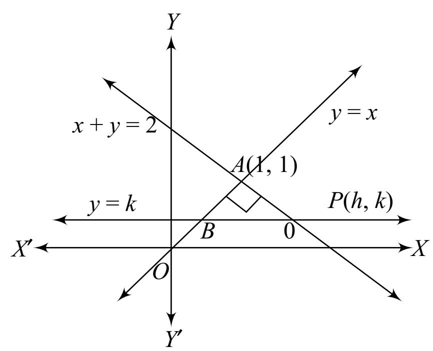
$$a_1^2 - a_2^2 + b_1^2 - b_2^2 = 2(a_1 - a_2)x + 2(b_1 - b_2)y$$

$$(a_1 - a_2)x + (b_1 - b_2)y = \frac{1}{2}(a_1^2 - a_2^2 + b_1^2 - b_2^2)$$

Thus, the value of c is $\dfrac{1}{2}(a_2^2 - a_1^2 + b_2^2 - b_1^2)$.

56. Ans. (a)

57.



The co-ordinates of A, B and C are (1, 1), (k, k) and $(2 - k, k)$.

Given $$\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ k & k & 1 \\ 2-k & k & 1 \end{vmatrix} = 4h^2$$

$$\Rightarrow \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ k & 0 & 1-k \\ 2-k & 2k-2 & k-1 \end{vmatrix} = 4h^2 \begin{pmatrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{pmatrix}$$

$$\Rightarrow \frac{1}{2}\begin{vmatrix} 0 & 1-k \\ 2k-2 & k-1 \end{vmatrix} = 4h^2$$

$$\Rightarrow \begin{vmatrix} 0 & 1-k \\ k-1 & k-1 \end{vmatrix} = 4h^2$$

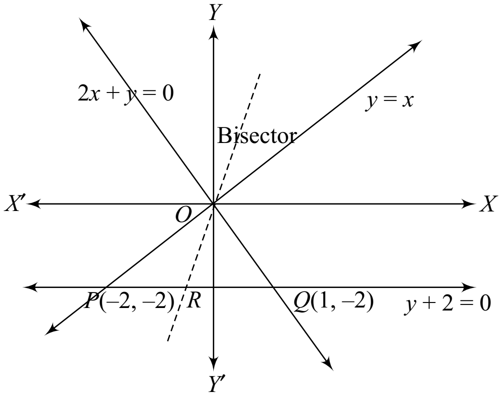
$$\Rightarrow (k-1)^2 = 4h^2$$

$$\Rightarrow (k-1) = \pm 2h$$

Hence, the locus of P is $2x - y + 1 = 0$.

58. Ans. $a_0 = 1$

59.



In ΔOPQ, $\dfrac{PR}{RQ} = \dfrac{OP}{OQ} = \dfrac{2\sqrt{2}}{\sqrt{5}}$

But statement II is false.

60. (a) The point of intersection L_1: $x + 3y - 5 = 0$ and L_3: $5x + 2y - 12 = 0$ is $P(2, 1)$.

If L_1, L_2 and L_3 are concurrent, the point P must satisfy $L_2 = 0$

Thus, $6 - k - 1 = 0$

$k = 5$.

(b) Let L_1 and L_2 are parallel.

Then $\dfrac{1}{3} = \dfrac{3}{-\text{k}} \Rightarrow k = -9$

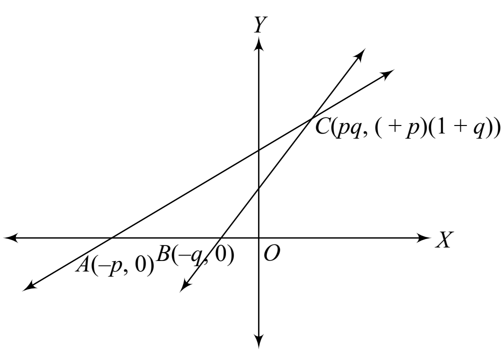
Again, let L_2 and L_3 are parallel.

Then $\dfrac{3}{5} = \dfrac{-\text{k}}{2} \Rightarrow k = -\dfrac{6}{5}$.

(c) For $k \neq 5, -9, -\dfrac{6}{5}$ they will form a triangle.

(d) For $k = 5, -9, -\dfrac{6}{5}$ they will not form a triangle.

64.



Intersection point of $y = 0$ with first line is $A(-p, 0)$

Intersection point of $y = 0$ with second line is $B(-q, 0)$

Intersection point of the two lines is

$C(pq, (p + 1)(q + 1))$

Altitude from C to AB is $x = pq$

Altitude from B to AC is

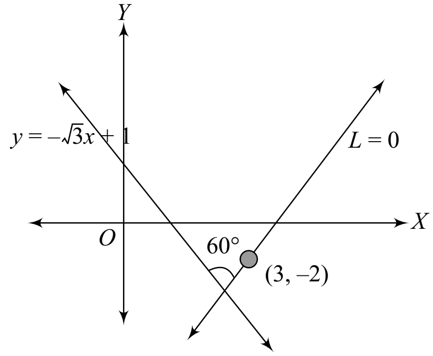
$$y = -\frac{q}{1+q}(x + p)$$

On solving, we get,

$x = pq$ and $y = -pq$

Thus, the locus of the orthocentre is $x + y = 0$ which is a straight line.

62.



Equation of a line passing through $(3, -2)$ is

$(y + 2) = m(x - 3)$

It is given that,

$$\tan(60°) = \left|\frac{m + \sqrt{3}}{1 - m\sqrt{3}}\right|$$

$$\Rightarrow \quad \left|\frac{m + \sqrt{3}}{1 - m\sqrt{3}}\right| = \sqrt{3}$$

$$\Rightarrow \quad \frac{m + \sqrt{3}}{1 - m\sqrt{3}} = \pm\sqrt{3}$$

$$\Rightarrow \quad (m + \sqrt{3}) = \pm\sqrt{3}(1 - m\sqrt{3})$$

$$\Rightarrow \quad m = 0, m = \sqrt{3}$$

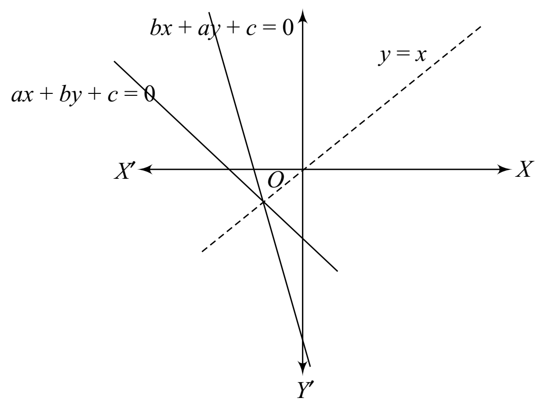
Hence, the equation is

$$(y + 2) = \sqrt{3}(x - 3)$$

$$\Rightarrow \quad \sqrt{3}x - y - (2 + 3\sqrt{3}) = 0$$

$$\Rightarrow \quad y - \sqrt{3}x + (2 + 3\sqrt{3}) = 0$$

63.



Clearly, the point of intersection of the lines $ax + by + c = 0$ and $bx + ay + c = 0$ lie on the line $y = x$.

Let the point be (r, r).

Given $\sqrt{(r-1)^2 + (r-1)^2} < 2\sqrt{2}$

$$\Rightarrow \quad \sqrt{2(r-1)^2} < 2\sqrt{2}$$

$$\Rightarrow \quad \sqrt{2}\,|(r-1)| < 2\sqrt{2}$$

$\Rightarrow \quad |(r-1)| < 2$

$\Rightarrow \quad -2 < (r-1) < 2$

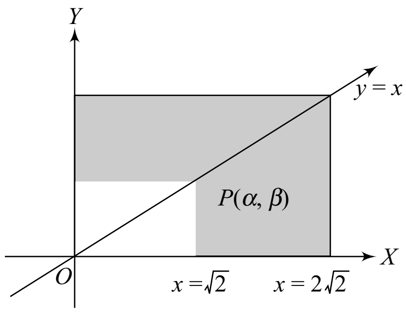
$\Rightarrow \quad -1 < r < 3$

Thus, $(-1, -1)$ lies on the opposite side of origin for both lines.

Therefore, $-a - b + c < 0$

$\Rightarrow \quad a + b - c > 0$

64.



Given $2 \le d_1(P) + d_2(P) \le 4$

$$\Rightarrow \quad 2 \le \left|\frac{\alpha-\beta}{\sqrt{2}}\right| + \left|\frac{\alpha+\beta}{\sqrt{2}}\right| \le 4$$

$$\Rightarrow \quad 2\sqrt{2} \le |\alpha-\beta| + |\alpha+\beta| \le 4\sqrt{2}$$

$\Rightarrow \quad 2\sqrt{2} \le 2\alpha \le 4\sqrt{2}$, when $\alpha > \beta$, for $P(\alpha, \beta)$

$$\Rightarrow \quad \sqrt{2} \le \alpha \le 2\sqrt{2}.$$

Area of the region $= (2\sqrt{2})^2 - (\sqrt{2})^2$

$= 8 - 2$

$= 6$

CHAPTER

2

Pair of Straight Lines

Concept Booster

1. Introduction

A pair of lines is the locus of a point moving on two lines. Let the two lines be

$$a_1x + b_1y + c_1 = 0$$

and $$a_2x + b_2y + c_2 = 0$$

The joint equation of the given equations is

$$(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0$$

and conversely if joint equation of two lines be

$$(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0,$$

then their separate equations will be

$$a_1x + b_1y + c_1 = 0 \text{ and } a_2x + b_2y + c_2 = 0$$

Notes:

1. In order to find the joint equation of two lines make RHS of equation of the straight lines to zero and then multiply the equations.
2. In order to find the separate equations of two lines, when their joint equation is given. Simply factorize the joint equation and reduces into two linear factors.

2. Homogeneous Equation

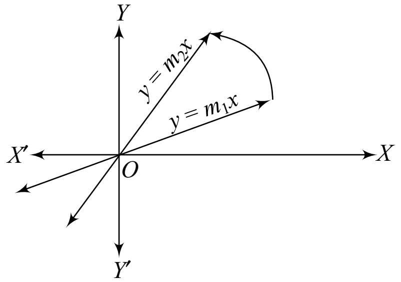
In an equation, if the degree of each term throughout the equation is same, it is known as homogeneous equation.

Thus $ax^2 + 2hxy + by^2 = 0$ is a homogeneous equation of 2nd degree and $ax^3 + 2hx^2y + by^3 = 0$ is a homogeneous equation of degree 3.

3. Pair of Straight Lines through the Origin

Theorem

Any homogeneous equation of 2nd degree in x and y represents two straight lines which are passing through the origin.



A homogeneous equation of 2nd degree in x and y is $ax^2 + 2hxy + by^2 = 0$

$$\Rightarrow \quad y^2 + \left(\frac{2h}{b}\right)x + \frac{a}{b}x^2 = 0$$

$$\Rightarrow \quad y = \frac{-\left(\frac{2h}{b}\right)x \pm x\sqrt{\frac{4h^2}{b^2} - \frac{4a}{b}}}{2}$$

$$= \left(-\left(\frac{h}{b}\right) \pm \sqrt{\frac{h^2}{b^2} - \frac{a}{b}}\right)x$$

$$= \left(-\left(\frac{h}{b}\right) \pm \sqrt{\frac{h^2 - ab}{b^2}}\right)x$$

$$= \left(\frac{-h \pm \sqrt{h^2 - ab}}{b}\right)x$$

$$= \left(\frac{-h + \sqrt{h^2 - ab}}{b}\right)x \text{ or } \left(\frac{-h - \sqrt{h^2 - ab}}{b}\right)x$$

$$\Rightarrow \quad (y - m_1x)(y - m_2x) = 0,$$

where $m_1 + m_2 = -\frac{2h}{b}$ and $m_1 \cdot m_2 = \frac{a}{b}$

$$\Rightarrow \quad y - m_1x = 0 \text{ or } y - m_2x = 0$$

both of which are pass through the origin.

4. Angle Between the Lines Represented by $ax^2 + 2hxy + by^2 = 0$

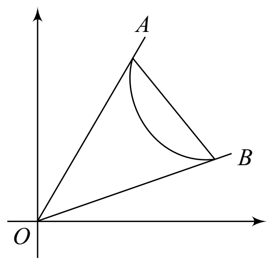
Let the lines are represented by

$$ax^2 + 2hxy + by^2 = 0$$

are $\quad y - m_1x = 0$ and $y - m_2x = 0$

where $\quad m_1 + m_2 = -\dfrac{2h}{b}$

and $\quad m_1 \cdot m_2 = \dfrac{2a}{b}$



Let θ be the angle between them.

Then $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1m_2}\right|$

$$= \left|\frac{\sqrt{(m_1+m_2)^2 - 4m_1m_2}}{1 + m_1m_2}\right|$$

$$= \left|\frac{\sqrt{\dfrac{4h^2}{b^2} - 4\dfrac{a}{b}}}{1 + \dfrac{a}{b}}\right|$$

$$= \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$$

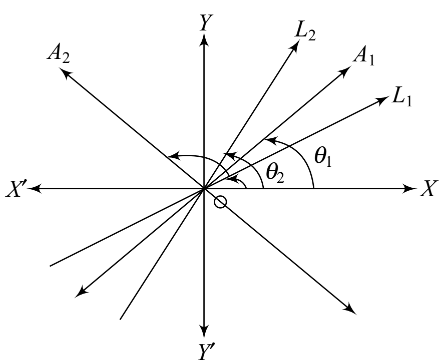
Thus $\theta = \tan^{-1}\left(\left|\dfrac{2\sqrt{h^2 - ab}}{a + b}\right|\right)$.

(i) *Condition of parallelism*
In this case, $\theta = 0$
$\Rightarrow \quad h^2 = ab$

(ii) *Condition of perpendicularity*
In this case, $\theta = \dfrac{\pi}{2}$
$\Rightarrow \quad a + b = 0$

(iii) *Condition of coincidency*
In this case also, $\theta = 0$
$\Rightarrow \quad h^2 = ab$.

5. Bisectors of Angles between the Lines Represented by $ax^2 + 2hxy + by^2 = 0$



Let the given equation represents the straight lines

$$y - m_1x = 0 \text{ and } y - m_2x = 0$$

Then $m_1 + m_2 = -\dfrac{2h}{b},\ m_1m_2 = \dfrac{a}{b}$.

The equation of the bisectors of the angles between the given straight lines are

$$\frac{y - m_1x}{\sqrt{1 + m_1^2}} = \frac{y - m_2x}{\sqrt{1 + m_2^2}}$$

and $$\frac{y - m_1x}{\sqrt{1 + m_1^2}} = -\frac{y - m_2x}{\sqrt{1 + m_2^2}}$$

The joint equation of the bisectors is

$$\Rightarrow \left(\frac{y - m_1x}{\sqrt{1 + m_1^2}} - \frac{y - m_2x}{\sqrt{1 + m_2^2}}\right)\left(\frac{y - m_1x}{\sqrt{1 + m_1^2}} + \frac{y - m_2x}{\sqrt{1 + m_2^2}}\right) = 0$$

$$\Rightarrow \left(\frac{y - m_1x}{\sqrt{1 + m_1^2}}\right)^2 = \left(\frac{y - m_2x}{\sqrt{1 + m_2^2}}\right)^2$$

$$\Rightarrow (y - m_1x)^2(1 + m_2^2) = (y - m_2x)^2(1 + m_1^2)$$

$$\Rightarrow (1 + m_2^2)(y^2 - 2m_1xy + m_1^2x^2) = (1 + m_1^2)(y^2 - 2m_2xy + m_2^2x^2)$$

$$\Rightarrow (m_1{}^2 - m_2^2)(x^2 - y^2) + 2(m_1m_2 - 1)(m_1 - m_2)xy = 0$$

$$\Rightarrow (m_1 + m_2)(x^2 - y^2) + 2(m_1m_2 - 1)xy = 0$$

$$\Rightarrow \left(-\frac{2h}{b}\right)(x^2 - y^2) + 2\left(\frac{a}{b} - 1\right)xy = 0$$

$$\Rightarrow \left(-\frac{2h}{b}\right)(x^2 - y^2) = -2\left(\frac{a}{b} - 1\right)xy$$

$$\Rightarrow \frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

which is the required equation of bisectors of the lines represented by

$$ax^2 + 2hxy + by^2 = 0$$

6. General Equation of 2nd Degree is $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Theorem

The general equation of 2nd degree is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

which represents a pair of straight lines if

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$$

Proof: The general equation of 2nd degree is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \ldots\text{(i)}$$

$$\Rightarrow by^2 + 2(hx + f)y + (ax^2 + 2gx + c) = 0$$

$$\Rightarrow y = \frac{-2(hx + f) \pm \sqrt{4(hx + f)^2 - 4b(ax^2 + 2gx + c)}}{2b}$$

$$= \frac{-(hx + f) \pm \sqrt{(h^2 - ab)x^2 + 2(gh - af)x + (g^2 - ac)}}{b}$$

Equation (i) represents two straight lines if LHS of Eq (i) can be resolved into two linear factors, therefore the expression under the square root should be a perfect square.

Hence, $4(gh - af)^2 - 4(h^2 - ab)(g^2 - ac) = 0$

$\Rightarrow \quad g^2h^2 + a^2f^2 - 2afgh - h^2g^2 + abg^2 + ach^2 - a^2bc = 0$

$\Rightarrow \quad a(af^2 + bg^2 + ch^2 - 2fgh - abc) = 0$

$\Rightarrow \quad abc + 2fgh - af - bg - ch^2 = 0$

which is the required condition.

Result 1

The lines represented by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

are parallel if

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

and $\quad h^2 - ab = 0$

Result 2

The lines represented by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

are perpendicular if

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

and $\quad a + b = 0.$

Result 3

The point of intersection of the lines represented by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

is obtained by

$$\frac{\delta f}{\delta x} = 0 \quad \text{and} \quad \frac{\delta f}{\delta y} = 0$$

i.e $\quad 2ax + 2hy + 2g = 0$

$\Rightarrow \quad ax + hy + g = 0$

and $\quad 2hx + 2by + 2f = 0$

$\Rightarrow \quad hx + by + f = 0$

Result 4

The point of intersection of the lines represented by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

is $\left(\sqrt{\dfrac{f^2 - bc}{h^2 - ab}}, \sqrt{\dfrac{g^2 - ac}{h^2 - ab}}\right)$

or $\left(\dfrac{bg - hf}{h^2 - ab}, \dfrac{af - gh}{h^2 - ab}\right)$

Result 5

The angle between the lines represented by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \text{ is}$$

$$\theta = \tan^{-1}\left|\frac{2\sqrt{h^2 - ab}}{(a + b)}\right|$$

Result 6

The lines represented by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

will be coincident if

$$h^2 - ab = 0,\ g^2 - ac = 0 \text{ and } f^2 - bc = 0$$

Result 7

The pair of bisectors of the lines represented by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \ldots\text{(i)}$$

is

$$\frac{(x - \alpha)^2 - (y - \beta)^2}{(a - b)} = \frac{(x - \alpha)(x - \beta)}{h},$$

where (α, β) be the point of intersection of the pair of straight lines represented by Eq. (i).

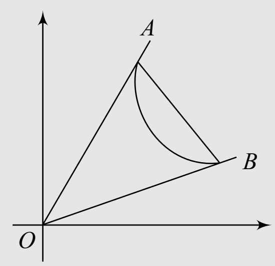
Result 8

The combined equation of the straight lines joining the origin to the points of intersection of a second degree curve

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

and a straight line $lx + my + n = 0$ is

$$ax^2 + 2hxy + by^2 + 2gx + 2gx\left(\frac{lx + my}{-n}\right) + 2fy\left(\frac{lx + my}{-n}\right) + c\left(\frac{lx + my}{-n}\right)^2 = 0$$



After simplification, the above equation will be reduced to $Ax^2 + 2Hxy + By^2 = 0$ and it will represent two straight lines, which are passing through the origin.

Let θ be the angle between them, then

$$\theta = \tan^{-1}\left|\frac{2\sqrt{H^2 - AB}}{A + B}\right|$$

Exercises

Level I

(Problems Based on Fundamentals)

ABC OF PAIR OF STRAIGHT LINES

1. Find the joint equation of the lines $x - y + 2 = 0$ and $2x + y + 4 = 0$.
2. Find the joint equation of the lines $x = y$ and $x + y = 1$.
3. Find the separate equations of the lines represented by $x^2 - 3xy + 2y^2 = 0$.
4. Find the separate equations of the lines represented by $x^2 - y^2 + 2y - 1 = 0$.
5. Find the straight lines which are represented by $x^2 + 5xy + 6y^2 = 0$.

6. Find the area of the triangle formed by the lines $y^2 - 5xy + 6y^2 = 0$ and $y = 6$.
7. Find the orthocentre of the triangle formed by the lines $xy = 0$ and $x + y = 2$.
8. Find the circumcentre of the triangle formed by the lines $xy - x - y + 1 = 0$ and $x + y = 4$.
9. Find the angle between the lines represented by $2013x^2 + 2014xy - 2013y^2 = 0$.
10. Find the angles between the lines represented by $2x^2 - 4\sqrt{3}xy + 6y^2 = 0$.
11. Find the angle between the lines represented by $x^2 + 4xy + y^2 = 0$.

BISECTORS OF THE LINES

12. Find the equation of the bisectors of the angle between the lines represented by $3x^2 + 5xy + 4y^2 = 0$.
13. If $y = mx$ is one of the bisectors of the lines $x^2 + 4xy + y^2 = 0$, find the value of m.
14. Prove that the lines $9x^2 + 14xy + 16y^2 = 0$ are equally inclined to the lines $3x^2 + 2xy + 4y^2 = 0$.
15. Prove that the bisectors of the angle between the lines $ax^2 + acxy + cy^2 = 0$ and
$$\left(2013 + \frac{1}{c}\right)x^2 + xy + \left(2013 + \frac{1}{a}\right)y^2 = 0$$
are always same.
16. If pairs of straight lines $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ be such that each pair bisects the angle between the other pair, prove that $pq = -1$.

GENERAL EQUATION OF A STRAIGHT LINE

17. For what values of m, does the equation $mx^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0$ represents two straight lines?
18. For what value of λ, does the equation $12x^2 - 10xy + 2y^2 + 11x - 5y + \lambda = 0$ represents a pair of straight lines?
19. If $\lambda x^2 + 10xy + 3y^2 - 15x - 21y + 18 = 0$ represents a pair of straight lines, find the value of λ.
20. Find the separate equation of lines represented by $x^2 - 5xy + 4y^2 + x + 2y - 2 = 0$
21. Prove that the equation
$$x^2 - 2\sqrt{3}xy + 3y^2 - 3x + 3\sqrt{3}y - 4 = 0$$
represents two parallel straight lines and also find the distance between them.
22. Find the equation of the straight lines passing through the point (1, 1) and parallel to the lines represented by the equation
$$x^2 - 5xy + 4y^2 + x + 2y - 2 = 0.$$
23. If the equation
$$3x^2 + 5xy - py^2 + 2x + 3y = 0$$
represents two perpendicular straight lines,then find the value of $p + 2010$.
24. Prove that the equation
$$16x^2 + 24xy + 9y^2 + 40x + 30y - 75 = 0$$
represents two parallel straight lines.
25. Find the point of intersection of the lines represented by $x^2 - 5xy + 4y^2 + x + 2y - 2 = 0$.
26. Find the point of intersection of the straight lines represented by the equation
$$3x^2 - 2xy - 8y^2 - 4x + 18y - 7 = 0$$
27. Find the point of intersection of the straight lines is given by the equation
$$3y^2 - 8xy - 3x^2 - 29x + 3y - 18 = 0$$
28. If the equation
$$12x^2 + 7xy - py^2 - 18x + qy + 6 = 0$$
represents two perpendicular lines, then find the values of p and q.
29. Prove that the four lines given by
$$3x^2 + 8xy - 3y^2 = 0$$
and $3x^2 + 8xy - 3y^2 + 2x - 4y - 1 = 0$
form a square.
30. Find the angle between the lines represented by
$$3x^2 - 2xy - 8y^2 - 4x + 18y - 7 = 0$$
31. Find the angle between the lines represented by
$$x^2 - 5xy + 4y^2 + x + 2y - 2 = 0$$
32. If the angle between the lines represented by
$$2x^2 + 5xy + 3y^2 + 7y + 4 = 0$$
is $\tan^{-1}(m)$, then find m.
33. Prove that the lines represented by
$$x^2 - 8xy + 16y^2 + 2x - 8y + 1 = 0$$
is coincident
34. Find the equation of bisectors of the lines represented by $x^2 - 5xy + 2y^2 + x + 2y - 2 = 0$.
35. Find the angle between the lines joining the origin to the points of intersection of the straight line $y = 3x + 2$ with the curve $x^2 + 2xy + 3y^2 + 4x + 8y - 11 = 0$.
36. Find the equation of the straight lines joining the origin to the points of intersection of the line $y = mx + c$ and the curve $x^2 + y^2 = b^2$.
37. Find the equation of the straight lines joining the origin to the points of intersection of the line $lx + my = 1$ and the curve $y^2 = 4bx$.
38. Find the equation of bisectors of the angles represented by the lines $12x^2 - 10xy + 2y^2 + 9x + 2y - 12 = 0$.
39. Prove that the lines joining the origin to the points of intersection of the line $3x - 2y = 1$ and the curve $3x^2 + 5xy - 3y^2 + 2x + 3y = 0$ are perpendicular to each other.
40. If the equation $ax^2 - 2hxy + by^2 + 2gx + 2fy + c = 0$.
represents two straight lines, prove that the product of the perpendiculars drawn from the origin to the lines is $\dfrac{c}{\sqrt{(a-b)^2 + 4h^2}}$.
41. Find the value of m, if the lines joining the origin and the point of intersection of $y = mx + 1$ and $x^2 + 3y^2 = 1$ perpendicular to one another.

Level II

(Mixed Problems)

1. The orthocentre of the triangle formed by the lines $xy = 0$ and $2x + 3y = 6$ is
(a) (1, 1) (b) (0, 0) (c) (1, 2) (d) (2, 1)

2. The orthocentre of the triangle formed by the lines $xy - x - y + 1 = 0$ and $3x + 4y = 12$ is
(a) (1,1) (b) (0, 0) (c) (2,2) (d) (3,3)

3. The image of the pair of lines represented by $ax^2 + 2hxy + by^2 = 0$ by the line mirror $y = 0$ is
(a) $ax^2 + 2hxy + by^2 = 0$ (b) $ax^2 + 2hxy - by^2 = 0$
(c) $ax^2 - 2hxy + by^2 = 0$ (d) $ax^2 - 2hxy - by^2 = 0$

4. The image of the pair of lines represented by $ax^2 + 2hxy + by^2 = 0$ by the line mirror $x = 0$ is
(a) $ax^2 + 2hxy + by^2 = 0$ (b) $ax^2 + 2hxy - by^2 = 0$
(c) $ax^2 - 2hxy + by^2 = 0$ (d) $ax^2 - 2hxy - by^2 = 0$

5. The point of intersection of the two lines given by $2x^2 - 5xy + 2y^2 + 3x + 3y + 1 = 0$ is
(a) $\left(-\frac{1}{2}, \frac{1}{2}\right)$ (b) $\left(\frac{1}{2}, -\frac{1}{2}\right)$
(c) $\left(-\frac{1}{3}, \frac{1}{3}\right)$ (d) $\left(\frac{1}{3}, -\frac{1}{3}\right)$

6. If the equation $12x^2 + 7xy - py^2 - 18x + qy + 6 = 0$ represents two perpendicular lines, the value of $p + q + 7$ is
(a) 15 (b) 20 (c) 25 (d) 30

7. The angle between the lines $x^2 + 4xy + y^2 = 0$ is
(a) $\pi/2$ (b) $\pi/6$ (c) $\pi/3$ (d) $\pi/4$

8. The angle between the lines given by $x^2 + 2013xy - y^2 = 0$ is
(a) $\pi/2$ (b) $\pi/6$ (c) $\pi/3$ (d) $\pi/4$

9. The area of the triangle formed by the lines $y^2 - 9xy + 18x^2 = 0$ and $y = 9$ is
(a) 27/4 (b) 31/4 (c) 18/5 (d) 27

10. If $x^2 - 3xy + \lambda y^2 + 3x - 5y + 2 = 0$ represents a pair of straight lines, the value of λ is
(a) 1 (b) 4 (c) 3 (d) 2

11. If the equation $2x^2 - 3xy - ay^2 + x + by - 1 = 0$ represents two perpendicular lines, then $a + b + 2$ is
(a) 2 (b) 5 (c) 6 (d) 3

12. If $xy + x + y + 1 = 0$ and $x + ay - 3 = 0$ are concurrent, the value of $a + 10$ is
(a) 5 (b) 7 (c) 6 (d) 4

13. The circumcentre of the triangle formed by the lines $xy + 2x + 2y + 4 = 0$ and $x + y + 2 = 0$ is
(a) (0, 0) (b) (–1, –1) (c) (–1, –2) (d) (–2, –2)

14. The distance between the pair of parallel lines $9x^2 - 24xy + 16y^2 - 12x + 16y - 12 = 0$ is
(a) 5 (b) 8 (c) 8/5 (d) 5/8

15. If one of the lines given by $6x^2 - xy + 4cy^2 = 0$ is $3x + 4y = 0$, then the value of c is
(a) 1 (b) –1 (c) –3 (d) 5

Level III

(Problems for JEE Advanced)

1. Find the internal angles of the triangle formed by the pair of the straight lines $x^2 - 4xy + y^2 = 0$ and the straight line $x + y + 4\sqrt{6} = 0$. Give the co-ordinates of the vertices of the triangle so formed and also the area of the triangle. **[Roorkee, 1983]**

2. From a point $A(1, 1)$, straight lines AL and AM are drawn at right angles to the pair of straight lines $3x^2 + 7xy + 2y^2 = 0$. Find the equations of the pair of straight lines AL and AM. Also find the area of the quadrilateral $ALOM$, where O is the origin of co-ordinates. **[Roorkee, 1984]**

3. The base of a triangle passes through a fixed point (f, g) and its sides are, respectively, bisected at right angles by the lines $y^2 - 8xy - 9x^2 = 0$. Determine the locus of its vertex. **[Roorkee, 1985]**

4. Show that the four straight lines given by $12x^2 + 7xy + 12y^2 = 0$ and $12x^2 + 7xy - 12y^2 - x + 7y - 1 = 0$ lie along the sides of a square. **[Roorkee, 1986]**

5. The distance between the two parallel lines given by the equation $x^2 + 2\sqrt{3}xy + 3y^2 - 3x - 3\sqrt{3}y - 4 = 0$ is
(a) 3/2 units (b) 2 units
(c) 5/2 units (d) 3 units
[Roorkee, 1989]

6. Pair of straight lines perpendicular to each other are represented by
(a) $2x^2 = 2y(x + y)$ (b) $x^2 + y^2 + 3 = 0$
(c) $2x^2 = y(2x + y)$ (d) $x^2 = 2(x - y)$
[Roorkee, 1990]

7. If one of the lines represented by $ax^2 + 2hxy + by^2 = 0$ bisects the angle between positive directions of the axes, then a, b, h satisfy the relation
(a) $a + b = 2|h|$ (b) $(a + b)^2 = 4h^2$
(c) $a - b = 2|h|$ (d) $(a - b)^2 = 4h^2$
[Roorkee, 1992]

8. Show that all the chords of the curve $3x^2 - y^2 - 2x + 4y = 0$ which subtend a right angle at the origin are concurrent. Does this result also hold for the curve $3x^2 + 3y^2 - 2x + 4y = 0$? **[Roorkee Main, 1992]**

9. A pair of straight lines drawn through the origin form with the line $2x + 3y = 6$ an isosceles triangle right angled at the origin. Find the equation of the pair of straight lines and the area of the triangle correct to two places of decimals. **[Roorkee Main,1993]**

10. The distance between the two parallel lines given by the equation is
(a) 3/2 units (b) 2 units
(c) 5/2 units (d) 3 units
[Roorkee, 1994]

11. Mixed term xy is to be removed from the general equation of the second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$. One should rotate the axes through an angle given by $\tan(2\theta)$ equal to

(a) $\frac{a-b}{2h}$ (b) $\frac{2h}{a+b}$ (c) $\frac{a+b}{2h}$ (d) $\frac{2h}{a-b}$

[Roorkee, 1996]

12. What is the conditions for the second degree polynomials in x and y to represent a pair of straight lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$? **[Roorkee, 1997]**
13. A variable line L passing through the point $B(2, 5)$ intersects the lines $2x^2 - 5xy + 2y^2 = 0$ at P and Q. Find the locus of the point R on L such that distances BP, BR and BQ are in harmonic progression.

[Roorkee Main, 1998]

Level IV

(Tougher Problems for JEE Advanced)

1. Find the area of the triangle formed by the pair of lines $ax^2 + 2hxy + by^2 = 0$ and the lines $lx + my = 1$.
2. If (α, β) be the centroid of the triangle whose sides are the lines $ax^2 + 2hxy + by^2 = 0$ and $lx + my = 1$, prove that
$$\frac{\alpha}{bl-hm} = \frac{\beta}{am-hl} = \frac{2}{3} \cdot \frac{1}{bl^2 - 2hlm + am^2}$$
3. A triangle has the lines $ax^2 + 2hxy + by^2 = 0$ for two of its sides and the point (l, m) for its orthocentre. Prove that the third side has the equation
$$(a + b)(lx + my) = am^2 - 2hlm + bl^2$$
4. If the equation
$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$
represents a pair of straight lines, prove that the distance between the lines $= 2\sqrt{\frac{g^2 - ac}{a(a+b)}}$.
5. Prove that two of the four lines represented by the joint equation $ax^4 + bx^3y + cx^2y^2 + dxy^3 + ay^4 = 0$ will bisect the angles between the other two if $c + 6a = 0$ and $b + d = 0$.
6. Find the new equation of the curve
$$4(x - 2y + 1)^2 + 9(2x + y + 2)^2 = 25,$$
if the lines $2x + y + 2 = 0$ and $x - 2y + 1 = 0$ are taken as the new x and y axes, respectively.
7. The lines $x^2 - 3xy + 2y^2 = 0$ are shifted parallel to themselves so that their point of intersection comes to $(1, 1)$. Find the combined equation of the lines in the new position.
8. If the straight lines represented by
$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$
are equidistant from the origin, prove that
$$f^4 - g^4 = c(bf^2 - ag^2)$$
9. If the lines joining the origin and the point of intersection of the curves
$$a_1x^2 + 2h_1xy + b_1y^2 + 2g_1x = 0$$
and $a_2x^2 + 2h_2xy + b_2y^2 + 2g_2x = 0$
are mutually perpendicular, prove that
$$g_1(a_1 + b_1) = g_2(a_2 + b_2).$$
10. If one of the lines denoted by the line pair $ax^2 + 2hxy + by^2 = 0$ bisects the angle between the co-ordinate axes, prove that $(a + b)^2 = 4h^2$

Integer Type Questions

1. If $ax^3 - 9x^2y - xy^2 + 4y^2 = 0$ represents three straight lines such that two of them are perpendicular, then find the sum of the values of a.
2. If the curve
$$(\tan^2\theta + \cos^2\theta)x^2 - 2xy\tan\theta + (\sin^2\theta)y^2 = 0$$
represents two straight lines, which makes angles θ_1 and θ_2 with the x-axis, find $(\tan\theta_1 - \tan\theta_2)$.
3. If pairs of straight lines $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ be such that each pair bisects the angle between the other pair, find the value of $(pq + 4)$.
4. If two of the lines represented by
$$ax^4 - 10x^3y + 12x^2y^2 - 20xy^3 + ay^4 = 0$$
bisects the angle between the other two, find the value of a.
5. If the equation $4x^2 + 10xy + my^2 + 5x + 10y = 0$ represents a pair of straight lines, find m.

Comprehensive Link Passages

Passage I

Let the lines represented by $2x^2 - 5xy + 2y^2 = 0$ be the two sides of a parallelogram and the line $5x + 2y = 1$ be one of its diagonals. Then

1. the one of the sides of the parallelogram passing through the origin is
(a) $x - 2y = 0$ (b) $x + 2y = 0$
(c) $x - 3y = 0$ (d) $x + 3y = 0$
2. the equation of the other diagonal is
(a) $x - 2y = 0$ (b) $11x - 10y = 0$
(c) $10x - 11y = 0$ (d) $10x + 11y = 0$
3. the area of the parallelogram is
(a) $\frac{1}{36}$ sq. u. (b) $\frac{1}{54}$ sq. u.
(c) $\frac{1}{72}$ sq. u. (d) $\frac{1}{96}$ sq. u.

Passage II

A general equation of 2nd degree is
$$8x^2 + 8xy + 2y^2 + 26x + 13y + 15 = 0 \quad \ldots\text{(i)}$$
Then

1. the given Eq. (i) represents two straight lines, which are
(a) $2x + y + 5 = 0$, $4x + 2y + 3 = 0$
(b) $x + 2y + 5 = 0$, $2x + 4y + 3 = 0$
(c) $2x - y + 5 = 0$, $4x - 2y + 3 = 0$
(d) $x - 2y + 5 = 0$, $2x - 4y + 3 = 0$
2. the given Eq. (i) represents
(a) two perpendicular lines
(b) two parallel lines
(c) two intersecting lines
(d) none of these

3. the distance between the lines is

(a) $\frac{7}{2\sqrt{5}}$ (b) $\frac{5}{2\sqrt{7}}$ (c) $\frac{2}{5\sqrt{7}}$ (d) $\frac{5}{7\sqrt{2}}$

Matrix Match (For JEE-Advanced Examination Only)

1. Match the following columns:

Column I		Column II	
(A)	The separate equations of $x^2 - y^2 + 2y - 1 = 0$ are	(P)	$x - 2 = 0$, $y - 2 = 0$
(B)	The separate equations of $x^2 - y^2 + 4x + 4 = 0$ are	(Q)	$x - y = 0$, $x + y + 1 = 0$
(C)	The separate equations of $x^2 - x - y^2 - y = 0$ are	(R)	$x - y + 1 = 0$ $x + y - 1 = 0$
(D)	The separate equations of $xy - 2x - 2y + 4 = 0$ are	(S)	$x + y + 2 = 0$ $x - y + 2 = 0$

2. Match the following columns:

Column I		Column II	
(A)	The joint equation of the lines $x = y$ and $x = -y$ is	(P)	$xy - 3y - 2x + 6 = 0$
(B)	The joint equation of the lines $x = 2$ and $y = 3$ is	(Q)	$x^2 + y^2 = 0$
(C)	The joint equation of the lines $x = y$ and $x = 1 - y$ is	(R)	$x^2 - y^2 + 4y - 4 = 0$
(D)	The joint equation of the lines $x + y - 2 = 0$ and $x - y + 2 = 0$ is	(S)	$x^2 - y^2 - x + y = 0$

3. Match the following columns:

Column I		Column II	
(A)	The orthocentre of the triangle formed by the lines $xy = 0$ and $x + y = 2013$ is	(P)	(11/2, 9/2)
(B)	The orthocentre of the triangle formed by the lines $x^2 - y^2 + 4y - 4 = 0$ and $y + 2014 = 0$ is	(Q)	(–2, –2)
(C)	The circumcentre of the triangle formed by the lines $xy - y = 0$ and $x + y = 10$ is	(R)	(0, 2)
(D)	The circumcentre of the triangle formed by the lines $xy - x - y + 1 = 0$ and $x + y + 4 = 0$ is	(S)	(0, 0)

4. Match the following columns:

Column I		Column II	
(A)	The image of the pair of lines represented by $2x^2 + 3xy + 3y^2 = 0$ with respect to the line mirror $y = 0$ is	(P)	$24x^2 + 10xy + y^2 = 0$
(B)	The image of the pair of lines represented by $x^2 + 5xy + 6y^2 = 0$ with respect to the line mirror $x = 0$ is	(Q)	$2x^2 - 3xy + 3y^2 = 0$
(C)	The image of the pair of lines represented by $x^2 + 10xy + 24y^2 = 0$ with respect to the line $y = x$ is	(R)	$x^2 - 5xy + 6y^2 = 0$

Questions asked in Previous Years' JEE-Advanced Examinations

1. Show that all chords of the curve $3x^2 - y^2 - 2x + 4y = 0$, which subtend a right angle at the origin, passes through a fixed point. Find the co-ordinates of the fixed point. **[IIT-JEE, 1991]**

No questions asked in between 1992 to 1993.

2. The equations to a pair of of opposite sides of a parallelogram are $x^2 - 5x + 6 = 0$ and $y^2 - 6y + 5 = 0$. The equations to its diagonals are
   (a) $x + 4y = 12$ and $y = 4x - 7$
   (b) $4x + y = 13$ and $4y = x - 7$
   (c) $4x + y = 13$ and $y = 4x - 7$
   (d) $y - 4x = 13$ and $y + 4x = 7$ **[IIT-JEE, 1994]**

No questions asked in between 1995 to 1998.

3. Let PQR be a right angled isosceles triangle, right angled at $P(2, 1)$. If the equation of the line QR is $2x + y = 3$, the equations representing the pair of lines PQ and PR is
   (a) $3x^2 - 3y^2 + 8xy + 20x + 10y + 25 = 0$
   (b) $3x^2 - 3y^2 + 8xy - 20x - 10y + 25 = 0$
   (c) $3x^2 - 3y^2 + 8xy + 10x + 15y + 20 = 0$
   (d) $3x^2 - 3y^2 - 8xy - 10x - 15y - 20 = 0$
   [IIT-JEE, 1999]

No questions asked in 2000.

4. Let $2x^2 + y^2 - 3xy = 0$ be the equations of a pair of tangents drawn from the origin O to a circle of radius 3 with centre in the first quadrant. If A is one of the points of contact, find the length of OA. **[IIT-JEE, 2001]**

No questions asked in between 2002 to 2003.

5. The area of the triangle formed by the angle bisectors of the pair of lines $x^2 - y^2 + 2y - 1 = 0$ and the line $x + y = 3$ (in sq units) is
(a) 1 (b) 2 (c) 3 (d) 4
[IIT-JEE, 2004]

No questions asked in between 2005 to 2007.

6. Let a and b be non-zero real numbers. Then the equation $(ax^2 + by^2 + c)(x^2 - 5xy + 6y^2) = 0$ represents
(a) four straight lines, when $c = 0$ and a, b are of the same signs.
(b) two straight lines and a circle, when $a = b$, and c is of sign opposite to a.
(c) two straight lines and a hyperbola, when a and b are of the same sign and c is of sign opposite to a.
(d) a circle and an ellipse, when a and b are of the same sign and c is of sign opposite to that of a.
[IIT-JEE, 2008]

No questions asked in between 2009 to 2014.

ANSWERS

LEVEL I

1. $2x^2 - y^2 - xy + 8x - 2y + 8 = 0$
2. $x^2 - y^2 - x + y = 0$
3. $x - y = 0$ and $x - 2y = 0$
4. $x + y - 1 = 0$ and $x - y + 1 = 0$
5. $(x + 2y) = 0$ and $(x + 3y) = 0$
6. 3
7. (0, 0)
8. (2, 2)
9. 90°
10. 0°
11. 60°
12. $5x^2 - 2xy - 5y^2 = 0$
13. $(\sqrt{3} - 2)$ and $(-\sqrt{3} - 2)$
15. $ac(x^2 - y^2) - 2(a - c)xy = 0$
16. $pq = -1$
17. $m = 6$
18. $x - y - 1 = 0$, $x - 4y + 2 = 0$
19. 5/2
20. $x - y = 0$ and $x - 4y + 3 = 0$
21. 2013
23. (2, 1)
24. (1, 1)
26. $q = 1, -23/2$
28. $\theta = \tan^{-1}(2)$
29. $\theta = \tan^{-1}\left(\dfrac{3}{5}\right)$
32. $m = \dfrac{1}{5}$
34. $5x^2 - 2xy - 5y^2 - 18x + 24y + 11 = 0$
35. $7x^2 - 2xy - y^2 = 0$
36. $(c^2 - b^2m^2)x^2 + 2b^2mxy + (c^2 - b^2)y^2 = 0$
37. $4blx^2 + 4bmxy - y^2 = 0$
38. $\dfrac{(x-14)^2 - (y-69/2)^2}{\left(14 - \dfrac{69}{2}\right)} = \dfrac{(x-14)(y-69/2)}{-5}$
41. $\pm\sqrt{3}$

LEVEL II

1. (b)	2. (a)	3. (a)	4. (c)	5. (d)
6. (b)	7. (a)	8. (a)	9. (a)	10. (d)
11. (b)	12. (c)	13. (b)	14. (c)	15. (c)

LEVEL III

1. $16\sqrt{3}$ sq. units.; triangle is equilateral
2. 7/10
5. (c)
6. (a)
7. (d)
8. 2nd part: Not Concurrent
9. $5(x^2 - y^2) - 24xy = 0$; 2. 77
10. (c)
11. (d)
12. $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$
13. $17x - 10y = 0$

LEVEL IV

1. $\dfrac{\sqrt{h^2 - ab}}{(am^2 - 2hlm + bl^2)}$
6. $4x^2 = 9y^2 = 5$
7. $x^2 - 3xy + 2y^2 + x - y = 0$

INTEGER TYPE QUESTIONS

1. 1 2. 2 3. 3 4. 3
5. 4

COMPREHENSIVE LINK PASSAGE

Passage I: 1. (a) 2. (b) 3. (c)
Passage II: 1. (a) 2. (b) 3. (a)

MATRIX MATCH

1. (A) → (R); (B) → (S);
   (C) → (Q); (D) → (P)
2. (A) → (Q); (B) → (P); (C) → (S); (D) → (R)
3. (A) → (S); (B) → (R); (C) → (P); (D) → (Q);
4. (A) → (Q); (B) → (R); (C) → (P);

QUESTIONS ASKED IN IIT-JEE EXAMINATIONS

1. (1, –2)
2. (c)
3. (b)
4. $(9+3\sqrt{10})$ units.
5. (b)
6. (b)

HINTS AND SOLUTIONS

LEVEL I

1. Hence, the joint equation of the given lines is
   $(x-y+2)(2x+y+4)=0.$
   $\Rightarrow \quad 2x^2-y^2-xy+8x-2y+8=0$
2. Hence, the joint equation of the lines is
   $(x-y)(x+y-1)=0.$
   $\Rightarrow \quad x^2-y^2-x+y=0$
3. The given equation is
   $x^2-3xy+2y^2=0$
   $\Rightarrow \quad x^2-xy-2xy+2y^2=0$
   $\Rightarrow \quad x(x-y)-2y(x-y)=0$
   $\Rightarrow \quad (x-y)(x-2y)=0$
   Hence, the separate equations of the given lines are $x-y=0$ and $x-2y=0$.
4. The given equation is
   $x^2-y^2+2y-1=0$
   $\Rightarrow \quad x^2-(y^2-2y+1)=0$
   $\Rightarrow \quad x^2-(y^2-1)^2=0$
   $\Rightarrow \quad (x+y-1)(x-y+1)=0.$
   Hence, the separate equations of the given lines are $x+y-1=0$ and $x-y+1=0$.
5. The given equation $x^2+5xy+6y^2=0$ can be written as
   $(x+2y)(x+3y)=0$
   $\Rightarrow \quad (x+2y)=0$ and $(x+3y)=0$
   Hence, the required lines are
   $(x+2y)=0$ and $(x+3y)=0$
6. The given equation of the lines
   $y^2-5xy+6y^2=0$
   can be reduced to $y=2x$ and $y=3x$.
   Then, $A=(2,6)$ and $B=(3,6)$
   Hence, the area of the triangle OAB is
   $$=\frac{1}{2}\begin{vmatrix}0 & 0\\ 3 & 6\\ 2 & 6\\ 0 & 0\end{vmatrix}=\frac{1}{2}(18-12)=3 \text{ sq. u.}$$
7. The equation $xy=0$ gives to $x=0$ and $y=0$.
   Clearly, OAB is a right-angled triangle and right angle at O.
   As we know that the point of intersection of two perpendicular sides of a triangle is known as the othrocentre of the triangle.
   Hence, the orthocentre is (0, 0).
8. The given equation $xy-x-y+1=0$ can be reduced to $(x-1)(y-1)=0$
   $\Rightarrow \quad x-1=0$ and $y-1=0$
   $\Rightarrow \quad x=1$ and $y=1$.
   Here, ABC be a right-angled triangle and right angled at A.
   As we know that, the mid-point of the hypotenuse is the circumcentre of the right-angle triangle.
   Now, the mid-point of BC is (2, 2).
   Hence, the circumcentre is (2, 2).
9. Here, $a=2013$, $b=-2013$ and $h=-1007$. Since, $a+b=2013-2013=0$, so the angle between the lines is 90°.
10. Here, $a=2$, $b=6$ and $h=-2\sqrt{3}$.
    Now, $h^2-ab=12-12=0$.
    Hence, the angle between them is 0°.
11. Here, $a=1$, $b=1$ and $h=2$.
    Let θ be the angle between them, then
    $$\tan(\theta)=\left(\frac{2\sqrt{h^2-ab}}{a+b}\right)$$
    $$=\left(\frac{2\sqrt{4-1}}{1+1}\right)=\sqrt{3}=\tan\frac{\pi}{3}$$
    $$\Rightarrow \quad \theta=\frac{\pi}{3}$$
    Hence, the angle between them is 60°.
12. The given equation is $3x^2-5xy+4y^2=0$
    Here, $a=3$, $b=4$ and $h=-5/2$
    Hence, the equation of the bisector is
    $$\frac{x^2-y^2}{a-b}=\frac{xy}{h}$$
    $$\Rightarrow \quad \frac{x^2-y^2}{3-4}=\frac{xy}{-(5/2)}$$
    $$\Rightarrow \quad -(x^2-y^2)=-\frac{2}{5}xy$$
    $$\Rightarrow \quad 5x^2-2xy-5y^2=0$$

13. Since $y = mx$ is one of the bisector of the given lines $x^2 + 4xy + y^2 = 0$, then $y = mx$ will satisfy the given lines.

Therefore, $x^2 + 4mx^2 + m^2x^2 = 0$

$\Rightarrow \quad 1 + 4m + m^2 = 0$

$\Rightarrow \quad m^2 + 4m + 1 = 0$

$\Rightarrow \quad (m+2)^2 = (\sqrt{3})^2$

$\Rightarrow \quad m = \pm\sqrt{3} - 2$

Hence, the values of m are

$(\sqrt{3} - 2)$ and $(-\sqrt{3} - 2)$

14. The given pair of lines are

$9x^2 + 14xy + 16y^2 = 0 \quad \ldots(i)$

and $3x^2 + 2xy + 4y^2 = 0 \quad \ldots(ii)$

The two pairs will be equally inclined if the two pairs of the straight lines have the same bisectors.

The equation of the bisectors of the angle between the lines represented by $9x^2 + 14xy + 16y^2 = 0$ is

$$\frac{x^2 - y^2}{9 - 16} = \frac{xy}{7}$$

$\Rightarrow \quad x^2 + xy - y^2 = 0$

The equation of the bisector of the angle between the lines represented by $3x^2 + 2xy + 4y^2 = 0$ is

$$\frac{x^2 - y^2}{3 - 4} = \frac{xy}{1}$$

$\Rightarrow \quad x^2 + xy - y^2 = 0$

Hence, the pair of lines of Eq. (i) are equally inclined to the pair of lines of Eq. (ii).

15. The given equations of the lines are

$ax^2 + acxy + cy^2 = 0 \quad \ldots(i)$

and

$$\left(2013 + \frac{1}{c}\right)x^2 + xy + \left(2013 + \frac{1}{a}\right)y^2 = 0 \quad \ldots(ii)$$

The equation of bisectors of Eq. (i) is

$$\frac{x^2 - y^2}{a - c} = \frac{xy}{\sqrt{3} - 2/2}$$

$\Rightarrow \quad ac(x^2 - y^2) = 2(a - c)xy$

$\Rightarrow \quad ac(x^2 - y^2) - 2(a - c)xy = 0$

Also, the equation of bisectors of Eq. (ii) is

$$\frac{x^2 - y^2}{\left(2013 + \frac{1}{c}\right) - \left(2013 + \frac{1}{a}\right)} = \frac{xy}{1/2}$$

$$\Rightarrow \quad \frac{ac(x^2 - y^2)}{(a - c)} = 2xy$$

$\Rightarrow \quad ac(x^2 - y^2) = 2(a - c)xy$

$\Rightarrow \quad ac(x^2 - y^2) - 2(a - c)xy = 0$

Hence, the result.

16. From the problem, it is clear that, the bisectors of the angles between the lines given by

$x^2 - 2pxy - y^2 = 0 \quad \ldots(i)$

is $x^2 - 2qxy - y^2 = 0 \quad \ldots(ii)$

The equation of the bisectors of Eq. (i) is

$$\frac{x^2 - y^2}{1 - (-1)} = \frac{xy}{-q}$$

$\Rightarrow \quad qx^2 + 2xy + qy^2 = 0 \quad \ldots(iii)$

Here, Eqs (ii) and (iii) are identical.

Thus, comparing the co-efficients, we get

$$\frac{1}{-q} = \frac{-2p}{-2} = \frac{-1}{q}$$

$\Rightarrow \quad pq = -1$

Hence, the result.

17. The given equation is

$mx^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0 \quad \ldots(i)$

Here, $a = m$, $h = -5/2$, $b = -6$, $g = 7$, $f = 5/2$ and $c = 4$.

The Eq. (i) represents a pair of straight lines if

$abc - 2fgh - af^2 - bg^2 - ch^2 = 0$

$$\Rightarrow \quad -24m - \frac{175}{2} - \frac{25}{4}m + 294 - 25 = 0$$

$$\Rightarrow \quad -\left(24m + \frac{25}{4}\mathrm{m}\right) = \frac{175}{2} - 269$$

$$\Rightarrow \quad -\frac{121}{4}m = -\frac{363}{2}$$

$\Rightarrow \quad m = 6$

Hence, the value of m is 6.

18. Given equation is

$x^2 - 5xy + 4y^2 + x + 2y - 2 = 0$

$(x - y)(x - 4y) + x + 2y - 2 = 0 \quad \ldots(i)$

The joint equation can be written as

$(x - y + c_1)(x - 4y + c_2) = 0$

$(x - y)(x - 4y) + (c_1 + c_2)x - (4c_1 + c_2)y + c_1c_2 = 0 \quad \ldots(ii)$

Comparing Eqs (i) and (ii), we get

$(c_1 + c_2) = 1$, $(4c_1 + c_2) = -2$, $c_1c_2 = -2$

Solving, we get

$c_1 = -1$, $c_2 = 2$

Hence, the separate equations of the lines are

$x - y - 1 = 0$ and $x - 4y + 2 = 0$.

19. The given equation is

$$x^2 - 2\sqrt{3}xy + 3y^2 - 3x + 3\sqrt{3}y - 4 = 0 \quad \ldots(i)$$

Here, $a = 1$, $b = 3$ and $h = -\sqrt{3}$

Now, $h^2 - ab = 3 - 3 = 0$

Thus, the given equation represents two parallel straight lines.

Equation (i) can be reduces to

$$(x - \sqrt{3}y)^2 - 3(x - \sqrt{3}y) - 4 = 0$$

$\Rightarrow \quad m^2 - 3m - 4 = 0, \quad$ where $m = (x - \sqrt{3}y)$

$\Rightarrow \quad (m-4)(m+1) = 0$

$\Rightarrow \quad m = 4$ and $m = -1$

$\Rightarrow \quad x - \sqrt{3}y - 4 = 0$

and $\quad x - \sqrt{3}y + 1 = 0$

Hence, the required distance $= \left|\dfrac{1+4}{\sqrt{1+3}}\right| = \dfrac{5}{2}$.

20. The given equation is

$$x^2 - 5xy + 4y^2 + x + 2y - 2 = 0$$

$\Rightarrow \quad (x-y)(x-4y) + x + 2y - 2 = 0 \qquad \ldots$(i)

Thus, the Eq. (i) represents two parallel straight lines, whose joint equation is

$$(x - y + \lambda)(x - 4y - \mu) = 0$$

which is passing through (1, 1).

Therefore, $\lambda = 0$ and $\mu = 3$

Hence, the parallel lines are

$$x - y = 0 \text{ and } x - 4y + 3 = 0.$$

21. The given equation is

$$3x^2 + 5xy - py^2 + 2x + 3y = 0 \qquad \ldots\text{(i)}$$

Since, the Eq. (i) represents two perpendicular straight lines,

so, $\quad a + b = 0$

co-efficients of x^2 + co-efficients of $y^2 = 0$

$\Rightarrow \quad 3 - p = 0$

$\Rightarrow \quad p = 3$

Hence, the value of $p + 2010 = 2013$.

22. Here, $a = 16$, $h = 12$ and $b = 9$

Now, $h^2 - ab = 144 - 144 = 0$

Hence, it represents two parallel straight lines.

23. The given equation is

$$x^2 - 5xy + 4y^2 + x + 2y - 2 = 0.$$

Now, $\dfrac{\delta f}{\delta x} = 0 \quad \Rightarrow \quad 2x - 5y + 1 = 0$

and $\dfrac{\delta f}{\delta y} = 0 \quad \Rightarrow \quad 8y - 5x + 2 = 0$

Solving, we get, $x = 2$ and $y = 1$

Hence. the required point of intersection is (2, 1).

24. The given equation is

$$3x^2 - 2xy - 8y^2 - 4x + 18y - 7 = 0$$

Here, $a = 3$, $h = -1$, $b = -8$, $g = -2$, $f = 9$ and $c = -7$.

Therefore, $h^2 - ab = 1 + 24 = 25$, $bg - hf = 16 + 9 = 25$, $af - gh = 27 - 2 = 25$

Hence, the point of intersection is

$$= \left(\frac{bg - hf}{h^2 - ab}, \frac{af - gh}{h^2 - ab}\right)$$

$$= \left(\frac{25}{25}, \frac{25}{25}\right) = (1, 1)$$

25. Do yourself

26. Clearly $12 - p = 0$

$\Rightarrow \quad p = 12$

Also, the given represents a pair of straight lines if

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} 12 & \frac{7}{2} & -9 \\ \frac{7}{2} & -12 & \frac{q}{2} \\ -9 & \frac{q}{2} & 6 \end{vmatrix} = 0$$

$\Rightarrow \quad q = 1, -23/2$

27. We have,

$$3x^2 + 8xy - 3y^2 = 0$$

$\Rightarrow \quad 3x^2 + 9xy - xy - 3y^2 = 0$

$\Rightarrow \quad 3x(x + 3y) - y(x + 3y) = 0$

$\Rightarrow \quad (x + 3y)(3x - y) = 0$

$\Rightarrow \quad (x + 3y) = 0, (3x - y) = 0$

Also, $3x^2 + 8xy - 3y^2 + 2x - 4y - 1 = 0$

$\Rightarrow \quad (x + 3y)(3x - y) + 2x - 4y - 1 = 0 \qquad \ldots$(i)

The joint equation of the above equation can be written as

$$(x + 3y + c_1)(3x - y + c_2)$$

$$\Rightarrow \quad (x + 3y)(3x - y) + (3c_1 + c_2)x - (c_1 - 3c_2)y + c_1c_2 = 0 \qquad \ldots\text{(ii)}$$

Comparing Eqs (i) and (ii), we get

$$(3c_1 + c_2) = 2, (c_1 - 3c_2) = 4, c_1c_2 = -1$$

Solving, we get

$$c_1 = 1, c_2 = -1$$

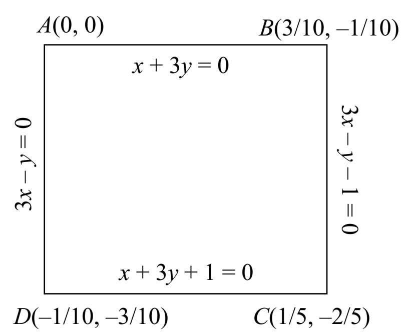
Thus, the separate equations are

$$x + 3y + 1 = 0 \text{ and } 3x - y - 1 = 0$$

Therefore, the four lines are

$$x + 3y = 0, 3x - y = 0$$

$$x + 3y + 1 = 0 \text{ and } 3x - y - 1 = 0$$



Clearly, $\angle BAD = 90°$

and $\quad AB = AD = \dfrac{1}{\sqrt{10}}$

Thus, $ABCD$ is a square

28. We have,

$$\tan\theta = \left|\left(\frac{2\sqrt{h^2-ab}}{a+b}\right)\right|$$

$$= \left|\frac{2\sqrt{1+24}}{3-8}\right| = \left|\frac{2\times 5}{-5}\right| = 2$$

$\theta = \tan^{-1}(2)$

29. The given equation is

$x^2 - 5xy + 4y^2 + x + 2y - 2 = 0$

Here, $a = 1$, $h = -5/2$, $b = 4$, $g = 1/2$, $f = 1$, $c = -2$.

Let θ be the angle between them.

Then, $\tan\theta = \left(\frac{2\sqrt{h^2-ab}}{a+b}\right)$

$$= \left(\frac{2\sqrt{\frac{25}{4}-4}}{1+4}\right) = \frac{3}{5}$$

$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{3}{5}\right)$

Hence, the angle between them is $\tan^{-1}\left(\frac{3}{5}\right)$.

32. We have

$$\tan\theta = \left|\left(\frac{2\sqrt{h^2-ab}}{a+b}\right)\right|$$

$\Rightarrow \quad = \left|\left(\frac{2\sqrt{\frac{25}{4}-6}}{2+3}\right)\right| = \frac{2\cdot\frac{1}{2}}{5} = \frac{1}{5}$

$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{1}{5}\right)$

$\Rightarrow \quad \tan^{-1}(m) = \tan^{-1}\left(\frac{1}{5}\right)$

$\Rightarrow \quad m = \frac{1}{5}$

33. The given equation is

$x^2 - 8xy + 16y^2 + 2x - 8y + 1 = 0$ …(i)

Here, $a = 1$, $h = -4$, $b = 16$, $g = 1$, $f = -4$ and $c = 1$.

Now, $h^2 - ab = 16 - 16 = 0$,

$g^2 - ac = 1 - 1 = 0$ and

$f^2 - bc = 16 - 16 = 0$

Hence, the Eq. (i) represents two coincident straight lines.

34. The given equation is

$x^2 - 5xy + 2y^2 + x + 2y - 2 = 0$ …(i)

Now, $\frac{\delta f}{\delta x} = 0 \Rightarrow 2x - 5y + 1 = 0$

and $\frac{\delta f}{\delta y} = 0 \Rightarrow -5x + 4y + 2 = 0$

Solving, we get

$x = 2$ and $y = 1$.

Therefore, the point of intersection is (2, 1).

Hence, the equation of bisector is

$$\frac{(x-\alpha)^2-(y-\beta)^2}{(a-b)} = \frac{(x-\alpha)(x-\beta)}{h}$$

$\Rightarrow \quad \frac{(x-2)^2-(y-1)^2}{(1-2)} = \frac{(x-2)(y-1)}{(-5/2)}$

$\Rightarrow \quad 5x^2 - 2xy - 5y^2 - 18x + 24y + 11 = 0$

which is a homogeneous equation of 2nd degree.

35. The equation of the straight line is

$y = 3x + 2$

$\Rightarrow \quad \frac{y-3x}{2} = 1$ …(i)

The equation of the given curve is

$x^2 + 2xy + 3y^2 + 4x + 8y - 11 = 0$ …(ii)

Making the Eq. (ii) homogeneous in the second degree in x and y by means of Eq. (i), we get

$$x^2 + 2xy + 3y^2 + 4\left(\frac{y-3x}{2}\right)x$$
$$+ 8\left(\frac{y-3x}{2}\right)y - 11\left(\frac{y-3x}{2}\right)^2 = 0$$

On simplification, we get

$7x^2 - 2xy - y^2 = 0$

which is the required equation of the line joining the origin to the points of intersection of lines (i) and (ii).

Here, $a = 7$, $h = -1$ and $b = -1$

Let θ be the angle between them.

Then, $\tan\theta = \left(\frac{2\sqrt{h^2-ab}}{a+b}\right)$

$$= \left(\frac{2\sqrt{1+7}}{7-1}\right) = \frac{2\sqrt{8}}{6} = \frac{2\sqrt{2}}{3}$$

$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{2\sqrt{2}}{3}\right)$

36. The equation of the straight line is

$y = mx + c$

$\Rightarrow \quad \left(\frac{y-mx}{c}\right) = 1$ …(i)

The equation of the given curve is

$x^2 + y^2 = b^2$ …(ii)

Making the Eq. (ii) homogeneous of the second degree in x and y by means of Eq. (i), we get

$$x^2 + y^2 = b^2\left(\frac{y - mx}{c}\right)^2$$

$\Rightarrow$ $c^2(x^2 + y^2) = b^2(y - mx)^2$

$\Rightarrow$ $(c^2 - b^2m^2)x^2 + 2b^2mxy + (c^2 - b^2)y^2 = 0$

which is the required equation of the straight lines joining the origin to the point of intersection of the lines (i) and (ii).

37. The given line is

$lx + my = 1$

$\Rightarrow$ $\left(\frac{lx + my}{1}\right) = 1$

The given curve is

$y^2 = 4bx$ …(ii)

Making the Eq. (ii) homogeneous of the second degree in x and y by means of Eq. (i), we get

$y^2 = 4bx(lx + my)$

$\Rightarrow$ $4blx^2 + 4bmxy - y^2 = 0$

38. The given curve is

$$12x^2 - 10xy + 2y^2 + 9x + 2y - 12 = 0$$

Now, $\frac{\delta f}{\delta x} = 0$ gives $24x - 10y + 9 = 0$

and $\frac{\delta f}{\delta y} = 0$ gives $5x - 2y - 1 = 0$

Solving, we get

$$x = 14,\ y = \frac{69}{2}$$

Hence, the equation of the bisectors is

$$\frac{(x-a)^2 - (y-b)^2}{(a-b)} = \frac{(x-a)(y-b)}{h}$$

$$\frac{(x-14)^2 - (y-69/2)^2}{\left(14 - \frac{69}{2}\right)} = \frac{(x-14)(y-69/2)}{-5}$$

39. The given curve is

$3x^2 + 5xy - 3y^2 + 2x + 3y = 0$ …(i)

The equation of the straight line is

$3x - 2y = 1$ …(ii)

Making the Eq. (i) homogeneous of the second degree in x and y by means of Eq. (ii), we get

$3x^2 + 5xy - 3y^2 + (2x + 3y)(3x - 2y) = 0$

$\Rightarrow$ $9x^2 + 10xy - 9y^2 = 0$

clearly, they are mutually perpendicular to each other.

40. Let $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

$= (lx + my + n)(l_1x + m_1y + n_1)$

$= ll_1x^2 + mm_1y^2 + (lm_1 + l_1m)xy$

$\quad + (ln_1 + l_1n)x + (mn_1 + nm_1)y + nn_1$

Comparing the co-efficients, we get

$ll_1 = a,\ mm_1 = b,\ nn_1 = c,\ (lm_1 + ml_1) = 2h,$

$(ln_1 + nl_1) = 2g,\ (mn_1 + nm_1) = 2f$

Let p_1 and p_2 be the length of perpendicular from the origin.

Thus, $p_1p_2 = \frac{n}{\sqrt{l^2 + m^2}} \times \frac{n_1}{\sqrt{l_1^2 + m_1^2}}$

$$= \frac{nn_1}{\sqrt{(l^2 + m^2)(l_1^2 + m_1^2)}}$$

$$= \frac{nn_1}{\sqrt{l^2l_1^2 + m^2m_1^2 + l^2m_1^2 + l_1^2m^2}}$$

$$= \frac{nn_1}{\sqrt{l^2l_1^2 + m^2m_1^2 + (lm_1 + l_1m)^2 - 2ll_1mm_1}}$$

$$= \frac{c}{\sqrt{a^2 + b^2 + 4h^2 - 2ab}}$$

$$= \frac{c}{\sqrt{(a-b)^2 + 4h^2}}$$

41. Given curve is $x^2 + 3y^2 = 1$ …(i)

and given line is $y = mx + 1$ …(ii)

Making the Eq. (i) homogeneous of the second degree in x and y by means of Eq. (ii), we get

$x^2 + 3y^2 = (y - mx)^2$

$\Rightarrow$ $x^2 + 3y^2 = y^2 + m^2x^2 - 2mxy$

$\Rightarrow$ $(1 - m^2)x^2 + 2mxy + 2y^2 = 0$

Clearly, $(1 - m^2) + 2 = 0$

$\Rightarrow$ $m^2 = 3$

$\Rightarrow$ $m = \pm\sqrt{3}$

Hence, the value of m is $\pm\sqrt{3}$

Level III

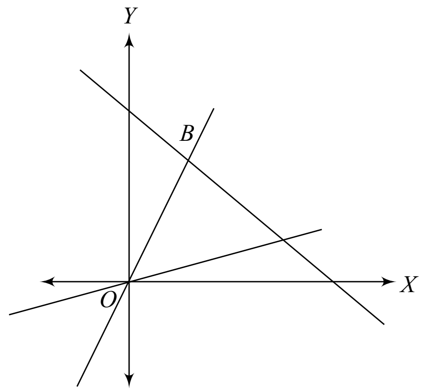
1. We have $x^2 - 4xy + y^2 = 0$

$\Rightarrow$ $y^2 - 4x.y + x^2 = 0$

$\Rightarrow$ $y = \frac{4x \pm \sqrt{16x^2 - 4x^2}}{2}$

$$= \frac{4x \pm 2\sqrt{3}\,x}{2} = (2 \pm 2\sqrt{3})\,x$$

$$= (2 + 2\sqrt{3})x,\ y = (2 - 2\sqrt{3})x$$



Thus

$$OB: y = (2 + 2\sqrt{3})x \text{ and } OA: y = (2 - 2\sqrt{3})x$$

and $\quad AB: x + y + 4\sqrt{6} = 0$

Clearly, the angle between OA and OB is $\dfrac{\pi}{3}$.

Solving, we get

$$O = (0, 0),\ A = \left(\frac{4\sqrt{2}}{2-\sqrt{3}}, \frac{8\sqrt{2}(1-\sqrt{3})}{(2-\sqrt{3})}\right)$$

and $\quad B = \left(-\dfrac{4\sqrt{2}}{2+\sqrt{3}}, \dfrac{8\sqrt{2}(1+\sqrt{3})}{(2+\sqrt{3})}\right)$

Hence, the area of the

$$\Delta OAB = \frac{1}{2}[64(1+\sqrt{3}) + 64(1-\sqrt{3})]$$

$$= \frac{1}{2}(64 + 64)$$

$$= 64 \text{ sq. u.}$$

Also, the angle between OA and AB is $\dfrac{\pi}{3}$.

and also the angle between OB and AB is $\dfrac{\pi}{3}$.

Hence, the triangle is equilateral.

2. Given lines are $3x^2 + 7xy + 2y^2 = 0$

$\Rightarrow \quad 3x^2 + 6xy + xy + 2y^2 = 0$

$\Rightarrow \quad 3x(x + 2y) + (x + 2y) = 0$

$\Rightarrow \quad (x + 2y)(3x + y) = 0$

$\Rightarrow \quad (x + 2y) = 0, (3x + y) = 0$

Thus, the equation of AL and AM are

$$2x - y + \lambda = 0,\ 3x - y + \mu = 0$$

which is passing through $A(1, 1)$

$$\lambda = -1, \mu = -2$$

Hence, the equations of AL and AM are

$$2x - y - 1 = 0,\ 3x - y - 2 = 0$$

Solving, we get

$$L = \left(-\frac{1}{5}, \frac{3}{5}\right) \text{ and } M = \left(\frac{2}{5}, -\frac{1}{5}\right)$$

Hence, the area of the quadrilateral $ALMO$

$$= \frac{1}{2}\begin{vmatrix} 1 & 1 \\ -\frac{1}{5} & \frac{3}{5} \\ 0 & 0 \\ \frac{2}{5} & -\frac{1}{5} \\ 1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left(\frac{3+1+2+1}{5}\right) = \frac{7}{10} \text{ sq. u.}$$

3. Given lines are

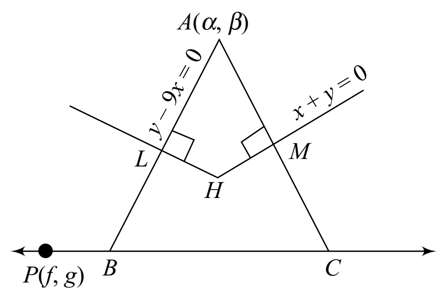
$\quad y^2 - 8xy - 9x^2 = 0$

$\Rightarrow \quad y^2 - 9xy + xy - 9x^2 = 0$

$\Rightarrow \quad y(y - 9x) + x(y - 9x) = 0$

$\Rightarrow \quad (y + x)(y - 9x) = 0$

$\Rightarrow \quad (y + x) = 0, (y - 9x) = 0$



Equations of AB and AC are

$$x + 9y = (\alpha + 9\beta) \text{ and } x - y = (\alpha + \beta)$$

Here, B and C are the images of A w.r.t. the lines $y = 9x$ and $y = -x$

So, $C = (-\beta, -\alpha)$ and $B = \left(\dfrac{9\beta - 40\alpha}{41}, \dfrac{40\beta + 9\alpha}{41}\right)$

Now, P, B and C are collinear, so,

$$\begin{vmatrix} f & g & 1 \\ \frac{9\beta - 40\alpha}{41} & \frac{40\beta + 9\alpha}{41} & 1 \\ -\beta & -\alpha & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} f & g & 1 \\ (9\beta - 40\alpha) & (40\beta + 9\alpha) & 41 \\ \beta & \alpha & -1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} f & g & 1 \\ (9\beta - 40\alpha) & (40\beta + 9\alpha) & 41 \\ (f + \beta) & (g + \alpha) & 0 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} (9\beta - 40\alpha) & (40\beta + 9\alpha) \\ (f + \beta) & (g + \alpha) \end{vmatrix} - 41\begin{vmatrix} f & g \\ (f + \beta) & (g + \alpha) \end{vmatrix} = 0$$

$\Rightarrow \quad (g + \alpha)(9\beta - 40\alpha) - (f + \beta)(40\beta + 9\alpha)$
$\quad -41(f\alpha - g\beta) = 0$

$\Rightarrow \quad 40(\alpha^2 + \beta^2) + 10(4g + 5f)\alpha + 10(4f - 5g)\beta = 0$

$\Rightarrow \quad 4(\alpha^2 + \beta^2) + (4g + 5f)\alpha + (4f - 5g)\beta = 0$

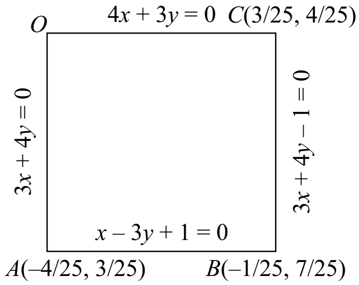
Hence, the locus of $A(\alpha, \beta)$ is

$$4(x^2 + y^2) + (4g + 5f)x + (4f - 5g)y = 0$$

4. The equations of the four sides are

$(3x + 4y) = 0$, $4x - 3y = 0$,

$(3x + 4y - 1) = 0$ and $4x - 3y + 1 = 0$



Clearly, the angle between OA and AB is 90°.

and $OA = AB = \frac{1}{5}$

So, $ABCD$ is a square.

5. We have

$$x^2 + 2\sqrt{3}xy + 3y^2 - 3x - 3\sqrt{3}y - 4 = 0$$

$\Rightarrow \quad (x + \sqrt{3}y)^2 - 3(x + \sqrt{3}y) - 4 = 0$

$\Rightarrow \quad a^2 - 3a - 4 = 0, a = (x + \sqrt{3}y)$

$\Rightarrow \quad a^2 - 3a - 4 = 0$

$\Rightarrow \quad (a - 4)(a + 1) = 0$

$\Rightarrow \quad a = -1, 4$

Thus, the parallel lines are

$$x + \sqrt{3}y - 4 = 0, x + \sqrt{3}y + 1 = 0$$

Hence, the distance between them is

$$= \left|\frac{1-(-4)}{\sqrt{1+3}}\right| = \frac{5}{2}$$

6. We have $2x^2 = 2y(x + y)$

$2x^2 - 2xy - 2y^2 = 0$

Now, $a + b$

Co-eff of x^2 + Co-eff of y^2

$= 2 - 2$

$= 0$

Hence, the result.

7. Given lines are $ax^2 + 2hxy + by^2 = 0$

The line bisects the axes is $y = x$.

So, $\quad ax^2 + 2hx \cdot x + bx^2 = 0$

$\Rightarrow \quad ax^2 + 2hx^2 + bx^2 = 0$

$\Rightarrow \quad a + 3h + b = 0$

$\Rightarrow \quad a + b = -2h$

$\Rightarrow \quad (a + b)^2 = (-2h)^2$

$\Rightarrow \quad (a + b)^2 = 4h^2$

8. Let $lx + my =1$ be any chord of the curve $3x^2 - y^2 - 2x + 4y = 0$.

Let the equation of the pair of straight lines and passing through the point of intersections of the chord and the curve is given by

$(3x^2 - y^2) - (2x + 4y)(lx + my) = 0$

$(3 - 2l)x^2 + (4m - 1)y^2 - (2m + 4l) xy = 0$

Since the chord subtends a right angle at the origin, so

$(3 - 2l) + (4m - 1) = 0$

$\Rightarrow \quad 2 - 2l + 4m = 0$

$\Rightarrow \quad 1 - l + 2m = 0$

$\Rightarrow \quad l - 2m = 0$

which shows that all such chords pass through a fixed point (1, –2) and hence are concurrent.

If we repeat the process for the curve $3x^2 + 3y^2 - 2x + 4y = 0$, we get

$(3 - 2l) + (3 + 4m) = 0$

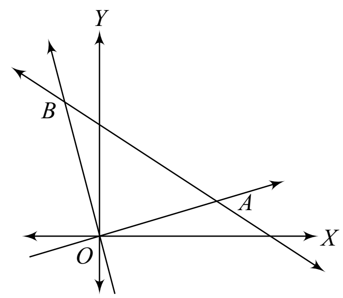
$\Rightarrow \quad (6 - 2l + 4m) = 0$

$\Rightarrow \quad (3 - l + 2m) = 0$

$\Rightarrow \quad l - 2m = 3$

which shows that such chords $lx + my = 1$ are not concurrent.

9.



Suppose the lines OA and OB are

$y = m_1x$, $y = m_2x$ respectively.

Here, $OA = OB$ and $\angle AOB = 90°$

So, $\quad \angle OAB = 45° = \angle OBA$

Thus,

$$\tan(45°) = \frac{m_1 + \frac{2}{3}}{1 - \frac{2}{3}m_1}$$

$\Rightarrow \quad \frac{3m_1 + 2}{3 - 2m_1} = 1$

$\Rightarrow \quad (3m_1 + 2) = (3 - 2m_1)$

$\Rightarrow \quad 5m_1 = 1$

$\Rightarrow \quad m_1 = \frac{1}{5}.$

Also, $\tan(135°) = \frac{m_2 + \frac{2}{3}}{1 - \frac{2}{3}m_2}$

$\Rightarrow \quad \frac{3m_2 + 2}{3 - 2m_2} = -1$

$\Rightarrow \quad 3m_2 + 2 = -3, 2m_2$

$\Rightarrow \quad m_2 = -5$

Hence, the equations of the pair of straight lines

$$(y + 5x)\left(y - \frac{1}{5}x\right) = 0$$

$\Rightarrow \quad (y + 5x)(5y - x) = 0$

$\Rightarrow \quad 5(x^2 - y^2) - 24xy = 0$

Solving, we get

the co-ordinates of A and B as

$$A=\left(\frac{30}{13},\frac{6}{13}\right);\ B=\left(-\frac{6}{13},\frac{30}{13}\right)$$

Thus, the area of the triangle OAB is

$$=\frac{1}{2}\begin{vmatrix} 0 & 0 \\ \frac{30}{13} & \frac{6}{13} \\ -\frac{6}{13} & \frac{30}{13} \\ 0 & 0 \end{vmatrix}$$

$$=\frac{1}{2}\left(\frac{900+36}{169}\right)$$

$$=\frac{1}{2}\times\frac{936}{169}=\frac{469}{169}=2.76 \text{ sq. u.}$$

10. See Q. 15
11. Let the axes be rotated through an angle θ so that $x \to x\cos\theta - y\sin\theta$
and $y \to x\sin\theta + y\sin\theta$

Given curve is

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

$$\Rightarrow a(x\cos\theta-y\sin\theta)^2+a(x\sin\theta+y\cos\theta)^2 + 2h(x\cos\theta-y\sin\theta)(x\sin\theta+y\cos\theta) + 2g(x\cos\theta-y\sin\theta)+2f(x\sin^2\theta+y\cos^2\theta)=0$$

$$\Rightarrow (b-a)\sin 2\theta+2h\cos 2\theta=0$$

Remove the term xy, we get

$$(-a\sin 2\theta+b\sin 2\theta)+2h(\cos^2\theta-\sin^2\theta)=0$$

$$\Rightarrow (b-a)\sin 2\theta+2h\cos 2\theta=0$$

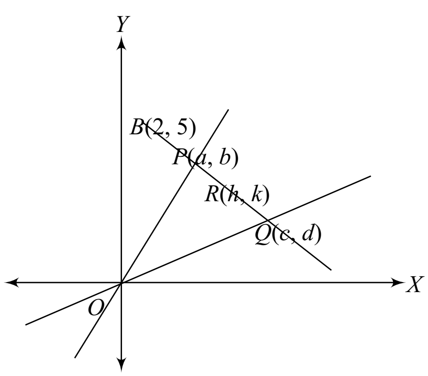
$$\Rightarrow \tan(2\theta)=\frac{2h}{a-b}$$

12. A general equation of 2nd degree represents a pair of straight lines if

$$abc+2fgh-af^2-bg^2-ch^2=0$$

13. We have,

$$2x^2-5xy+2y^2=0$$
$$\Rightarrow 2x^2-4xy-xy+2y^2=0$$
$$\Rightarrow 2x(x-2y)-y(x-2y)=0$$
$$\Rightarrow (x-2y)(2x-y)=0$$
$$\Rightarrow (x-2y)=0,\ (2x-y)=0$$
$$\Rightarrow x=2y \text{ and } y=2x$$



Let $P=(a,b)$, $Q=(c,d)$ and $\underline{R}=(h,k)$

and $RB=r$, $PB=r_1$ and $QB=r_2$

So, $P=(2+r_1\cos\theta, 5+r_1\cos\theta)$

and $Q=(2+r_2\cos\theta, 5+r_2\sin\theta)$

and $R=(2+r\cos\theta, 5+r\sin\theta)$

Since P lies on $y=2x$,

so, $\frac{1}{r_1}=2\cos\theta-\sin\theta$

Also, Q lies on $x=2y$

So, $\frac{1}{r_2}=\frac{\cos\theta-2\sin\theta}{8}$

It is given that $\frac{2}{r}=\frac{1}{r_1}+\frac{1}{r_2}$

$$\Rightarrow \frac{2}{r}=\frac{17\cos\theta-10\sin\theta}{8}$$

$$\Rightarrow \frac{2}{r}=\frac{17}{8}\left(\frac{h-2}{r}\right)-\frac{10}{8}\left(\frac{k-5}{r}\right)$$

$$\Rightarrow 16=17h-34-10k+50$$

$$\Rightarrow 17h-10k=0$$

Hence, the locus of R is $17x-10y=0$.

Level IV

1. Let three lines be

$$y=m_1x,\ y=m_2x,\ lx+my=1$$

where $m_1+m_2=-\frac{2h}{b},\ m_1m_2=\frac{a}{b}$

Thus area of a triangle OAB, where

$$O=(0,0),\ A=\left(\frac{1}{l+mm_1},\frac{m_1}{l+mm_1}\right)$$

and $B=\left(\frac{1}{l+mm_2},\frac{m_2}{l+mm_2}\right)$

is

$$=\frac{1}{2}\begin{vmatrix} 0 & 0 \\ \frac{1}{l+mm_1} & \frac{m_1}{l+mm_1} \\ \frac{1}{l+mm_2} & \frac{m_2}{l+mm_2} \\ 0 & 0 \end{vmatrix}$$

$$=\frac{1}{2}\left(\frac{(m_2-m_1)}{(l+mm_1)(l+mm_2)}\right)$$

$$=\frac{1}{2}\left(\frac{\sqrt{(m_1+m_2)^2-4m_1m_2}}{l^2+(m_1+m_2)lm+(m_1m_2)m^2}\right)$$

$$= \frac{1}{2}\left(\frac{\sqrt{\frac{4h^2}{b^2} - \frac{4a}{b}}}{l^2 - \frac{2h}{b}lm + \frac{a}{b}m^2}\right)$$

$$= \frac{1}{2}\left(\frac{2\sqrt{h^2 - ab}}{bl^2 - 2hlm + am^2}\right)$$

$$= \left(\frac{\sqrt{h^2 - ab}}{bl^2 - 2hlm + am^2}\right) \text{sq. u}$$

2. Let three lines be

$y = m_1x$, $y = m_2x$, and $lx + my = 1$

where $m_1 + m_2 = -\frac{2h}{b}, m_1m_2 = \frac{a}{b}$

Thus, the points O, A and B are

$$O = (0, 0),\ A = \left(\frac{1}{l + mm_1}, \frac{m_1}{l + mm_1}\right)$$

and $$B = \left(\frac{1}{l + mm_2}, \frac{m_2}{l + mm_2}\right)$$

Therefore, $$\alpha = \frac{1}{3}\left(\frac{1}{l + mm_1} + \frac{1}{l + mm_2}\right)$$

and $$\beta = \frac{1}{3}\left(\frac{m_1}{l + mm_1} + \frac{m_2}{l + mm_2}\right)$$

$$\Rightarrow \quad \alpha = \frac{1}{3}\left(\frac{2l + (m_1 + m_2)m}{(l + mm_1)(l + mm_2)}\right)$$

and $$\beta = \frac{1}{3}\left(\frac{(m_1 + m_2)l + 2(m_1m_2)m}{(l + mm_1)(l + mm_2)}\right)$$

$$\Rightarrow \quad \alpha = \frac{2}{3}\left(\frac{bl - hm}{bl^2 - 2hlm + am^2}\right)$$

and $$\beta = \frac{2}{3}\left(\frac{am - hl}{bl^2 - 2hlm + am^2}\right)$$

Thus,

$$\frac{\alpha}{(bl - hm)} = \frac{\beta}{(am - hl)} = \frac{2}{3}\left(\frac{1}{bl^2 - 2hlm + am^2}\right)$$

Hence, the result.

3. Let the lines be $y = m_1x$ and $y = m_2x$

where $m_1 + m_2 = -\frac{2h}{b}, m_1m_2 = \frac{a}{b}$

Equation of OA is $y = m_1x$

BE is perpendicular to OA

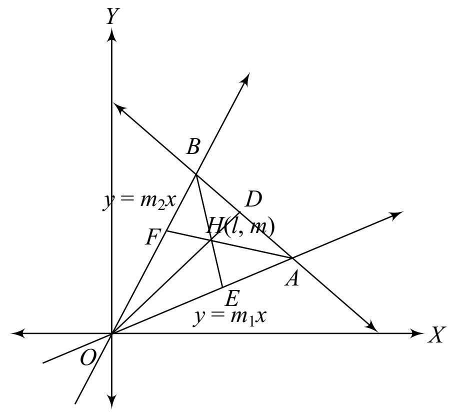
So, BE: $x + m_1y + \lambda_1 = 0$ which is passing through $H(l, m)$, we get

$l + mm_1 + \lambda_1 = 0$

$\Rightarrow \quad \lambda_1 = -(l + mm_1)$

Thus, BE: $x + m_1y - (l + mm_1) = 0$

Similarly, AF: $x + m_2y - (l + mm_2) = 0$



Solving OA and AF, we get

$$A = \left(\frac{l + mm_2}{1 + m_1m_2}, \frac{m_1(l + mm_2)}{1 + m_1m_2}\right)$$

Solving OB and BE, we get

$$B = \left(\frac{l + mm_1}{1 + m_1m_2}, \frac{m_2(l + mm_1)}{1 + m_1\text{m}_2}\right)$$

Equation of AB is $lx + my + \lambda_3 = 0$ which is passing through A.

So, $$l\left(\frac{l + mm_2}{1 + m_1m_2}\right) + m\left(\frac{m_1(l + mm_2)}{1 + m_1m_2}\right) + \lambda_3 = 0$$

$$\Rightarrow \quad \left(\frac{l^2 + lm(m_2 + m_1) + m^2m_1m_2}{1 + m_1m_2}\right) + \lambda_3 = 0$$

$$\Rightarrow \quad \lambda_3 = -\left(\frac{l^2 + lm(m_2 + m_1) + m^2m_1m_2}{1 + m_1m_2}\right)$$

$$\Rightarrow \quad \lambda_3 = -\left(\frac{l^2 - \left(\frac{2h}{b}\right)lm + \frac{a}{b}m^2}{1 + \frac{a}{b}}\right)$$

$$\Rightarrow \quad \lambda_3 = -\left(\frac{bl^2 - 2hlm + am^2}{a + b}\right)$$

Hence, the equation of the third side AB is

$$lx + my - \left(\frac{bl^2 - 2hlm + am^2}{a + b}\right) = 0$$

$\Rightarrow \quad (lx + my)(a + b) = (bl^2 - 2hlm + am^2)$

Hence, the result.

4. Given equation is

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ …(i)

Equation (i) represents a pair of parallel straight lines

$lx + my + n_1 = 0$ and $lx + my + n_2 = 0$

Thus, $(lx + my + n_1)(lx + my + n_2)$

$= ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

Comparing, the co-efficients, we get

$a = l^2$, $b = m^2$, $n_1n_2 = c$

and $h = lm$, $l(n_1 + n_2) = 2g$, $m(n_1 + n_2) = 2f$

Hence, the distance between the lines

$$= \left|\frac{n_1 - n_2}{\sqrt{l^2 + m^2}}\right|$$

$$= \left|\frac{\sqrt{(n_1 + n_2)^2 - 4n_1n_2}}{\sqrt{l^2 + m^2}}\right|$$

$$= 2\sqrt{\frac{g^2 - ac}{a(a + b)}}$$

6. We have

$$4(x - 2y + 1)^2 + 9(2x + y + 2)^2 = 25$$

$\Rightarrow \quad 4X^2 + 9Y^2 = 25,$

where $X = x - 2y + 1$,

$Y = 2x + y + 2$

which represents an ellipse.

7. Given lines are $x^2 - 3xy + 2y^2 = 0$

$\Rightarrow \quad (x - y)(x - 2y) = 0$

Let the equation of the lines be

$(x - y + \lambda)(x - 2y + \mu) = 0$

which is passing through (1, 1), i.e.

$(1 - 1 + \lambda)(1 - 2 + \mu) = 0$

$\Rightarrow \quad (1 - 1 + \lambda) = 0, (1 - 2 + \mu) = 0$

$\Rightarrow \quad \lambda = 0, \mu = -1$

Hence, the combine equation is

$(x - y)(x - 2y + 1) = 0$

$\Rightarrow \quad (x - y)(x - 2y) + (x - y) = 0$

$\Rightarrow \quad x^2 - 3xy + 2y^2 + x - y = 0$

8. Given equation is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \ldots\text{(i)}$$

Let the lines represented by Eq. (i) be

$l_1x + m_1y + n_1 = 0$ and $l_1x + m_2y + n_2 = 0$

We have

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

$= (l_1x + m_1y + n_1)(l^2x + m_2y + n_2)$

Comparing the co-efficients, we get

$l_1l_2 = a$, $m_1m_2 = b$, $n_1n_2 = c$

$l_1n_2 + l_2n_1 = 2g$, $l_1m_2 + l_2m_1 = 2h$

and $m_1n_2 + m_2n_1 = 2f$,

Since they are equidistant from the origin, so

$$\frac{n_1}{\sqrt{l_1^2 + m_1^2}} = \frac{n_2}{\sqrt{l_2^2 + m_2^2}}$$

$\Rightarrow \quad n_1^2(l_2^2 + m_2^2) = n_2^2(l_1^2 + m_1^2)$

$\Rightarrow \quad n_1^2l_2^2 - n_2^2l_1^2 = n_2^2m_1^2 - n_1^2m_2^2$

$\Rightarrow \quad n_1^2l_2^2 - n_2^2l_1^2 = n_2^2m_1^2 - n_1^2m_2^2$

$\Rightarrow \quad (n_1l_2 - n_2l_1)(n_1l_2 + n_2l_1)$

$= (n_2m_1 - n_1m_2)(n_2m_1 + n_1m_2)$

$\Rightarrow \quad (n_1l_2 - n_2l_1)(2g) = (n_2m_1 - n_1m_2)(2f)$

$\Rightarrow \quad (n_1l_2 - n_2l_1)^2(g^2) = (n_2m_1 - n_1m_2)^2(f^2)$

$\Rightarrow \quad [(n_1l_2 + n_2l_1)^2 - 4n_1n_2l_1l_2(g^2)]$

$= (n_2m_1 + n_1m_2)^2 - 4m_1m_2n_1n_2(f^2)$

$\Rightarrow \quad (4g^2 - 4ac)g^2 = (4f^2 - 4bc)f^2$

$\Rightarrow \quad (g^2 - ac)g^2 = (f^2 - bc)f^2$

$\Rightarrow \quad (g^4 - g^2ac) = (f^4 - f^2bc)$

$\Rightarrow \quad f^4 - g^4 = c(f^2b - g^2a)$

Hence, the result.

9. Given curves are

$$a_1x^2 + 2h_1xy + b_1y^2 + 2g_1x = 0 \quad \ldots\text{(i)}$$

and $$a_2x^2 + 2h_2xy + b_2y^2 + 2g_2x = 0 \quad \ldots\text{(ii)}$$

Multiplying Eq. (i) by g_1 and Eq. (ii) by g_2, we get

$$a_1g_1x^2 + 2g_1h_1xy + b_1g_1y^2 + 2g_1^2x = 0 \quad \ldots\text{(iii)}$$

and $$a_2g_2x^2 + 2h_2g_2xy + b_2g_2y^2 + 2g_2^2x = 0 \quad \ldots\text{(iv)}$$

Subtracting Eq. (iii) and Eq. (iv), we get

$(a_1g_1 - a_2g_2)x^2 + 2(g_1h_1 - g_2h_2)xy$

$+ (b_1g_1 - b_2g_2)y^2 = 0$

These lines will be right angles to each other if

$(a_1g_1 - a_2g_2) + (b_1g_1 - b_2g_2) = 0$

$\Rightarrow \quad (a_1g_1 + b_1g_1) + (a_2g_2 + b_2g_2)$

$\Rightarrow \quad g_1(a_1 + b_1) = g_2(a_2 + b_2)$

10. Given lines are $ax^2 + 2hxy + by^2 = 0$.

The line bisects the axes is $y = x$.

So, $\quad ax^2 + 2hx{\cdot}x + bx^2 = 0$

$\Rightarrow \quad ax^2 + 2hx^2 + bx^2 = 0$

$\Rightarrow \quad a + 2h + b = 0$

$\Rightarrow \quad a + b = -2h$

$\Rightarrow \quad (a + b)^2 = (-2h)^2$

$\Rightarrow \quad (a + b)^2 = 4h^2$

Integer Type Questions

1. As we know that if $ax^3 + bx^2y + cxy^2 + dy^3 = 0$ represents two perpendicular lines then

$a^2 + ac + bd + d^2 = 0$

$a^2 - a - 36 + 16 = 0$

$a^2 - a - 20 = 0$

$(a - 5)(a + 4) = 0$

$a = 5, -4$

Hence, the sum of the values of a is 1.

2. Let $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$

Now, $$m_1 + m_2 = -\frac{2h}{b} = \frac{2\tan\theta}{\sin^2\theta}$$

and $$m_1m_2 = \frac{\tan^2\theta + \cos^2\theta}{\sin^2\theta}$$

Thus,

$$\tan\theta_1 - \tan\theta_2$$

$$= \sqrt{\frac{4\tan^2\theta}{\sin^4\theta} - \frac{4(\tan^2\theta + \sin^2\theta)}{\sin^2\theta}}$$

$$= \frac{2}{\sin^2\theta}\sqrt{\tan^2\theta - \sin^2\theta(\tan^2\theta + \cos^2\theta)}$$

$$= \frac{2\sin\theta}{\sin^2\theta}\sqrt{\sec^2\theta - (\tan^2\theta + \cos^2\theta)}$$

$$= \frac{2}{\sin\theta}\sqrt{1 - \cos^2\theta}$$

$$= \frac{2}{\sin\theta} \times \sin\theta$$

$$= 2.$$

3. From the problem, it is clear that, the bisectors of the angles between the lines given by

$$x^2 - 2pxy - y^2 = 0 \quad \ldots\text{(i)}$$

and $x^2 - 2qxy - y^2 = 0$ …(ii)

The equation of the bisectors of line (i) is

$$\frac{x^2 - y^2}{1-(-1)} = \frac{xy}{-q}$$

$\Rightarrow \quad -qx^2 + 2xy + qy^2 = 0$ …(iii)

Here, Eqs (ii) and (iii) are identical.

Thus, comparing the co-efficients, we get

$$\frac{1}{-q} = \frac{-2p}{-2} = \frac{-1}{q}$$

$\Rightarrow \quad pq = -1$

Hence, the value of $pq + 4$ is 3.

4. As we know that

$$ax^4 + bx^3y + cx^2y^2 + dx^3 + ay^4 = 0$$

bisects the angle between the other two

then $6a + c + b + d = 0$

$\Rightarrow \quad 6a - 10 + 12 - 20 = 0$

$\Rightarrow \quad 6a = 18$

$\Rightarrow \quad a = 3.$

5. We have

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 4 & 5 & 5/2 \\ 5 & m & 5 \\ 5/2 & 5 & 0 \end{vmatrix} = 0$$

$$\Rightarrow \frac{5}{2}\left(25 - \frac{5m}{2}\right) - 5\left(20 - \frac{25}{2}\right) = 0$$

$$\Rightarrow \frac{125}{2} - \frac{25m}{4} - 100 + \frac{125}{2} = 0$$

$$\Rightarrow 125 - \frac{25m}{4} - 100 = 0$$

$$\Rightarrow 25 - \frac{25m}{4} = 0$$

$$\Rightarrow 25 = \frac{25m}{4}$$

$$\Rightarrow m = 4$$

Previous Years' JEE-Advanced Examinations

1. Let the equation of the chord be $lx + my = 1$

The joint equation of the curve

$3x^2 - y^2 - 2x + 4y = 0$ and the chord

be $lx + my = 1$ to the origin is

$$3x^2 - y^2 - (2x - 4y)(lx + my) = 0$$

$$(3 - 2l)x^2 + (4l - 2m)xy + (4m - 1)y^2 = 0$$

which subtends a right angle at the origin, if co-efficient of x^2 + x co-efficient of $y^2 = 0$

$\Rightarrow \quad 3 - 2l + 4m - 1 = 0$

$\Rightarrow \quad l - 2m = 1$

Now, $lx + my = 1$

$\Rightarrow \quad lx + my = l - 2m$

$\Rightarrow \quad l(x - 1) + m(y + 2) = 0$

$$\Rightarrow (x-1) + \frac{m}{l}(y+2) = 0$$

$\Rightarrow \quad (x - 1) + \lambda(y + 2) = 0$

$\Rightarrow \quad (x - 1) = 0, (y + 2) = 0$

$\Rightarrow \quad x = -1, y = -2$

Hence, the chord passes through the fixed point is $(1, -2)$.

No questions asked in between 1992 to 1993.

2. The sides of the parallelogram are

$$x^2 - 5x + 6 = 0 \text{ and } y^2 - 6y + 5 = 0$$

$$(x - 2)(x - 3) = 0 \text{ and } (y - 1)(y - 5) = 0$$

$$x = 2, x = 3 \text{ and } y = 1, y = 5$$

Thus, the angular points are

$$A = (2, 1), B = (,2\ 5), C = (3, 5), D = (3, 1)$$

Equation of the diagonal AC is

$$(y - 2) = \left(\frac{5-1}{3-2}\right)(x - 1)$$

$\Rightarrow \quad (y - 2) = 4(x - 1)$

$\Rightarrow \quad y = 4x - 7$

Equation of the diagonal BD is

$$(y - 5) = \left(\frac{1-5}{3-2}\right)(x - 2)$$

$\Rightarrow \quad (y - 5) = -4(x - 2)$

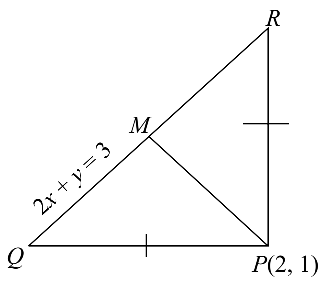
$\Rightarrow \quad 4x + y = 13.$

3. Ans. (b)

Clearly, the slope of PM is $-1/2$

Here, PM is equally inclined with PQ and PR.

Let the slope of PQ be m, then the slope of PR will be $-1/m$.



Thus, $\dfrac{m-\left(-\frac{1}{2}\right)}{1+m\left(-\frac{1}{2}\right)}=\dfrac{-\frac{1}{2}-\left(-\frac{1}{m}\right)}{1+\left(-\frac{1}{2}\right)\left(-\frac{1}{m}\right)}$

$\Rightarrow \quad \dfrac{2m+1}{2-m}=\dfrac{2-m}{2m+1}$

$\Rightarrow \quad (2m+1)^2=(2-m)^2$

$\Rightarrow \quad 4m^2+4m+1=4-4m+m^2$

$\Rightarrow \quad 3m^2+8m-3=0$

$\Rightarrow \quad 3m^2+9m-m-3=0$

$\Rightarrow \quad 3m(m+3)-1(m+3)=0$

$\Rightarrow \quad (m+3)(3m-1)=0$

$\Rightarrow \quad m=-3, \dfrac{1}{3}$

Thus, the equations of PQ and PR are

$$y-1=-3(x-2),\ y-1=\frac{1}{3}(x-2)$$

$\Rightarrow \quad (y-1)=-3(x-2),\ 3(y-1)=(x-2)$

Hence, the joint equation of PQ and PR is

$\{3(x-2)+(y-1)\}\{(x-2)-3(y-1)\}=0$

$\Rightarrow \quad 3(x-2)^2-3(y-1)^2+8(x-2)(y-1)=0$

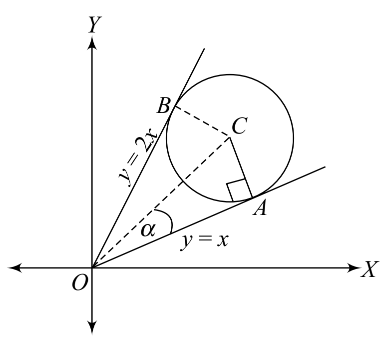
$\Rightarrow \quad 3x^2-3y^2+8xy-20x-10y+25=0$

4. Given equations of tangents are

$2x^2+y^2-3xy=0$

$(2x-y)(x-y)=0$

$y=x,\ y=2x$



Let 2α be the angle between these pair of tangents.

Then $\tan(2\alpha)=\left|\dfrac{m_2-m_1}{1+m_1m_2}\right|=\dfrac{2-1}{1+2}=\dfrac{1}{3}$

$\Rightarrow \quad \tan(2\alpha)=\dfrac{1}{3}$

$\Rightarrow \quad \dfrac{2\tan\alpha}{1-\tan^2\alpha}=\dfrac{1}{3}$

$\Rightarrow \quad 6\tan\alpha=1-\tan^2\alpha$

$\Rightarrow \quad \tan^2\alpha+6\tan\alpha-1=0$

$\Rightarrow \quad \tan(\alpha)=\dfrac{-6\pm\sqrt{36+4}}{2}$

$\Rightarrow \quad \tan(\alpha)=-3\pm\sqrt{10}$

$\Rightarrow \quad \tan(\alpha)=\sqrt{10}-3$

Since $0<\alpha<\dfrac{\pi}{4}$.

Thus,

$\tan\alpha=\dfrac{CA}{OA}=\dfrac{3}{OA}$

$\Rightarrow \quad OA=\dfrac{3}{\tan\alpha}$

$\Rightarrow \quad OA=\dfrac{3}{\sqrt{10}-3}$

$\Rightarrow \quad OA=3(\sqrt{10}+3)$

$\Rightarrow \quad OA=9+3\sqrt{10}$

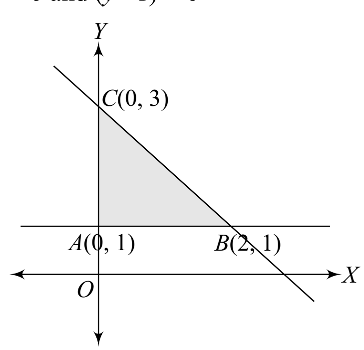
5. Equation of the angle bisectors of

$x^2-y^2+2y-1=0$ is

$\Rightarrow \quad \dfrac{x^2-(y-1)^2}{1+1}=\dfrac{x(y-1)}{0}$

$\Rightarrow \quad x(y-1)=0$

$\Rightarrow \quad x=0$ and $(y-1)=0$



Hence, the area of the

$\Delta ABC=\dfrac{1}{2}\times AB\times AC$

$=\dfrac{1}{2}\times 2\times 2=2$ sq. units

6. Given curve is

$(ax^2+bx^2+c)(x^2-5xy+6y^2)=0$

$\Rightarrow \quad (ax^2+bx^2+c)=0,\ (x^2-5xy+6y^2)=0$

$\Rightarrow \quad (ax^2+by^2+c)=0,\ (x-2y)(x-3y)=0$

$\Rightarrow \quad (ax^2+by^2+c)=0,\ (x-2y)=0,\ (x-3y)=0$

$\Rightarrow \quad x^2+y^2=\dfrac{c}{a},\ (x-2y)=0,\ (x-3y)=0$

which represents two straight lines and a circle.

CHAPTER 3

Circle

CONCEPT BOOSTER

1. INTRODUCTION

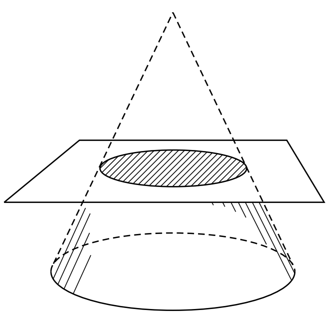
The word 'circle' is derived from the Greek word *kirkos* which means a circle, from the base *ker* which means to turn or bend. The origins of the words 'circus' and 'circuit' are closely related.

The circle has been known since before the beginning of recorded history. Natural circles would have been observed, such as the Moon, Sun, and a short plant stalk blowing in the wind on sand, which forms a circle-like shape in the sand. The circle is the basis for the wheel, which, with related inventions such as gears, made much of modern civilisations possible. In mathematics, the study of the circle has helped the development of geometry, astronomy, and calculus.

2. MATHEMATICAL DEFINITIONS

Definition 1

The intersection of a right circular cone and a plane is a circle, in which the plane is perpendicular to the axis or parallel to the base of the cone.



Definition 2

A circle is a conic section whose eccentricity is zero.

Definition 3

A conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a circle if

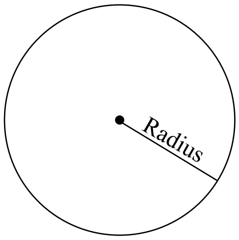
(i) $a = b$,

(ii) $h = 0$, and

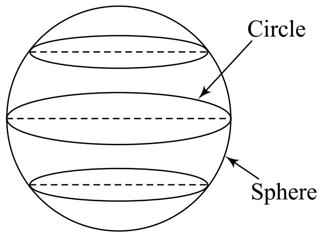
(iii) $\Delta \neq 0$, where $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

Definition 4

It is the locus of a point which moves in a plane in such a way that its distance from a fixed point is always the same. The fixed point is called the *centre* and the distance between the point and the centre is known as the *radius* of the circle.



Definition 5



In 3D geometry, the section of a sphere by a plane is a circle.

Definition 6

Let z be a complex number and a be a positive real number. If $|z| = a$, the locus of z is a circle.

Definition 7

Let z be a complex number and $a \in R^+$, $b \in R$. If $|z - b| = a$, the locus of z will be a circle, with the centre at b and the radius a.

Definition 8

Let z_1, z_2 and z_3 be three non-zero complex numbers such that $\left|\frac{z - z_1}{z - z_2}\right| = k$, where $k \neq 1$. The locus of z is a circle.

Definition 9

Let z, z_1 and z_2 be three non-zero complex numbers such that $|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$. The locus of z is a circle with the centre at $\left(\frac{z_1 + z_2}{2}\right)$ and the radius $\frac{1}{2}|z_1 - z_2|$.

Definition 10

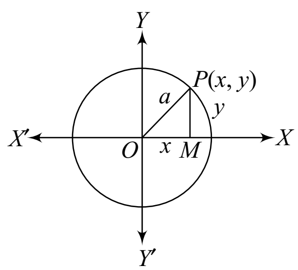
Let z, z_1 and z_2 be three non-zero complex numbers such that $\arg\left(\dfrac{z - z_2}{z - z_1}\right) = \dfrac{\pi}{2}$, the locus of z is a circle.

Definition 11

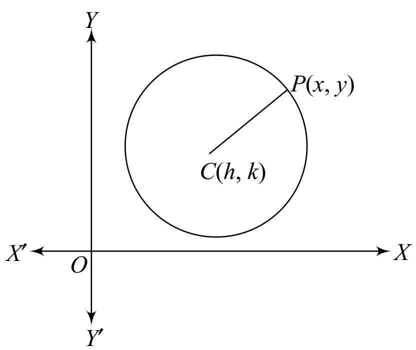
Let z, z_1 and z_2 be three non-zero complex numbers such that $\arg\left(\dfrac{z - z_2}{z - z_1}\right) = \alpha$, where $\alpha \neq 0, \pi$. The locus of z is a circle.

3. Standard Equation of a Circle

(i) $x^2 + y^2 = a^2$, where the centre is (0, 0) and the radius is a.



(ii) $(x - h)^2 + (y - k)^2 = a^2$, where the centre is (h, k) and the radius is 'a'.



4. General Equation of a Circle

The general equation of a circle is $x^2 + y^2 + 2gx + 2fy + c = 0$, where the centre is $(-g, -f)$ and the radius is $\sqrt{g^2 + f^2 - c}$.

As we know that the equation any of circle, when centre of a circle is not the origin, is

$$(x - h)^2 + (y - k)^2 = a^2$$

$$\Rightarrow \quad x^2 - 2hx + h^2 + y^2 - 2ky + k^2 = a^2$$

$$\Rightarrow \quad x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - a^2 = 0$$

$$\Rightarrow \quad x^2 + y^2 + 2gx + 2fy + c = 0,$$

where $h = -g$, $k = -f$ and $c = h^2 + k^2 - a^2$

$$\Rightarrow \quad x^2 + y^2 + 2gx + 2fy + c = 0,$$

where $h = -g$, $k = -f$ and

$$a = \sqrt{h^2 + k^2 - c} = \sqrt{g^2 + f^2 - c}$$

5. Nature of the Circle

The nature of the circle depends on its radius.

Let the equation of a circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ and its radius is $\sqrt{g^2 + f^2 - c}$.

(i) If $(g^2 + f^2 - c) > 0$, it represents a real circle.

(ii) If $(g^2 + f^2 - c) < 0$, it represents a virtual circle or an imaginary circle.

(iii) If $(g^2 + f^2 - c) = 0$, it represents a point circle.

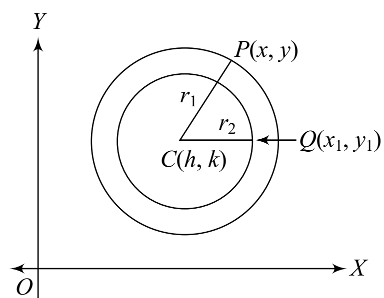
6. Concentric Circles

Two circles having the same centre, say (h, k) and different radii, say r_1 and r_2 respectively, are called concentric circles.

Thus, $\quad (x - h)^2 + (y - k)^2 = r_1^2$

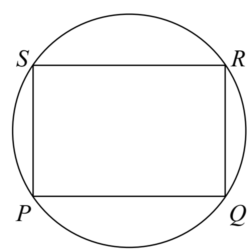
and $\quad (x_1 - h)^2 + (y_1 - k)^2 = r_2^2$

are two concentric circles.



7. Concyclic Points

If P, Q, R and S lie on the same circle, the points P, Q, R and S are known as concyclic points.



8. Parametric Equation of a Circle

If the radius of a circle, whose centre is the origin, makes an angle θ with the positive direction of x-axis, then θ is called a parameter and $0 \leq \theta < 2\pi$.

(i) If the equation of a circle be $x^2 + y^2 = a^2$, its parametric equations are

$$x = a\cos\theta, \; y = a\sin\theta,$$

where θ is a parameter.

(ii) If the equation of a circle be $(x - h)^2 + (y - k)^2 = a^2$, its parametric equations are

$$x = h + a\cos\theta, \; y = k + a\sin\theta,$$

where θ is a parameter.

9. Diametric form of a Circle

The equation of a circle, when the end-points (x_1, y_1) and (x_2, y_2) of a diameter are given is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.$$

10. Equation of a Circle Passing through Three Non-collinear Points

Let the equation of a circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

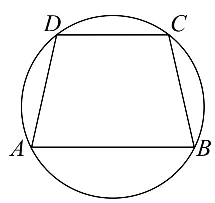
If three points (x_1, y_1), (x_2, y_2), (x_3, y_3) lie on a circle, then

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0,$$
$$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0 \text{ and}$$
$$x_3^2 + y_3^2 + 2gx_3 + 2fy_3 + c = 0.$$

Eliminating g, f and c, we get

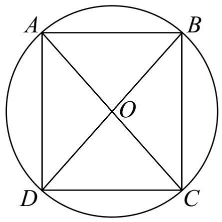
$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

11. Cyclic Quadrilateral



If all the vertices of a quadrilateral lie on a circle, the quadrilateral is called a cyclic quadrilateral and the four vertices are known as concyclic points.

12. Condition for Concyclic Points



If A, B, C and D are concyclic points, then

$$OA.\ OD = OB.\ OC,$$

where O is the point of intersection of the chords AD and BC, and O is not the centre of a circle.

If the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ cut the x-axis and y-axis in four concyclic points, then

$$a_1a_2 = b_1b_2.$$

Two conics

$$a_1x^2 + 2h_1xy + b_1y^2 + 2g_1x + 2f_1y + c_1 = 0$$

and $$a_2x^2 + 2h_2xy + b_2y^2 + 2g_2x + 2f_2y + c_2 = 0$$

will intersect each other in four concyclic points, if $\dfrac{a_1 - b_1}{a_2 - b_2} = \dfrac{h_1}{h_2}$.

13. Intercepts Made on the Axes by a Circle

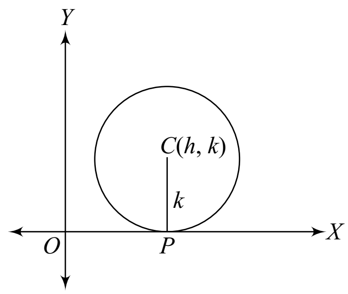
Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$. Then the

(i) length of x-intercept = $2\sqrt{g^2 - c}$

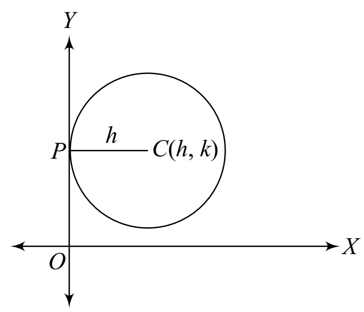
(ii) length of y-intercept = $2\sqrt{f^2 - c}$.

14. Different Forms of a Circle

(i) **When the circle touches the x-axis**
The equation of a circle is $(x - h)^2 + (y - k)^2 = k^2$.



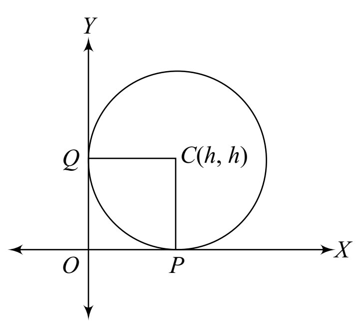
(ii) **When the circle touches the y-axis**
The equation of a circle is $(x - h)^2 + (y - k)^2 = h^2$.



(iii) **When the circle touches both the axes**
The equation of a circle is

$$(x - h)^2 + (y - h)^2 = h^2$$

or $$(x - k)^2 + (y - k)^2 = k^2$$

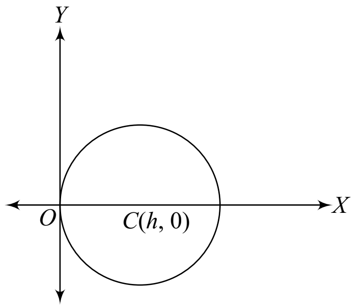


(iv) **When the circle passes through the origin and the centre lies on x-axis**

The equation of a circle is

$$(x - h)^2 + y^2 = h^2$$

or $$x^2 + y^2 - 2hx = 0$$

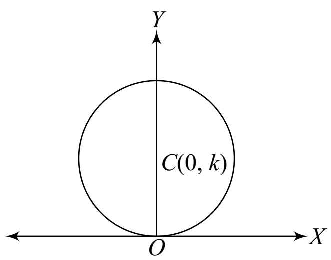


(v) **When the circle passes through the origin and the centre lies on y-axis**

The equation of a circle is

$$x^2 + (y - k)^2 = k^2$$

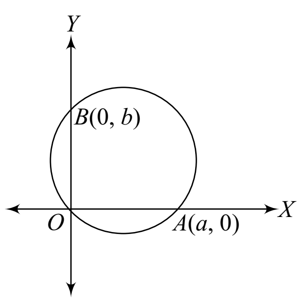
or $$x^2 + y^2 - 2ky = 0$$



(vi) **When the circle passes through the origin and has intercepts a and b on the x and y axes respectively**

The equation of the circle is

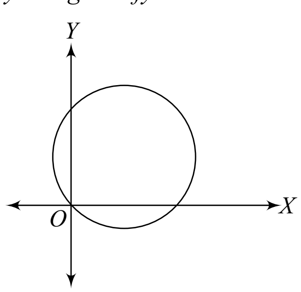
$$x^2 + y^2 - ax - by = 0$$



(vii) **When the circle passes through the origin**

The equation of the circle is

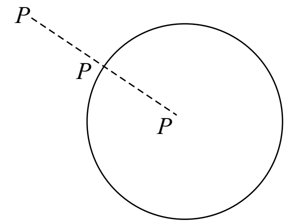
$$x^2 + y^2 + 2gx + 2fy = 0$$



15. Position of a Point with Respect to a Circle

(i) A point (x_1, y_1) lies outside, on and inside of a circle $x^2 + y^2 = a^2$, if

$$x_1^2 + y_1^2 - a^2 > 0, = 0, < 0$$



(ii) A point (x_1, y_1) lies outside, on and inside of a circle $x^2 + y^2 + 2gx + 2fy + c = 0$, if $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c > 0, = 0, < 0$.

16. Shortest and Largest Distance of a Circle from a Point

Let $P(x_1, y_1)$ be any point and the circle be

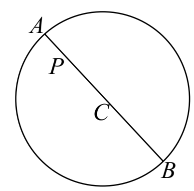
$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

Case I: When P lies inside the circle.

Draw a line through P passing through the centre and intersects the circle at A and B, respectively.

Shortest distance, $PA = CA - CP$

Longest distance, $PB = CP + CB$.

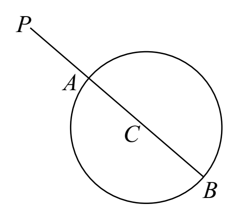


Case II: When P lies outside on the circle.

Draw a line through P passing through the centre and intersects the circle at A and B, respectively.

Shortest distance = $PA = CP - CA$

Longest distance = $PB = CP + CB$

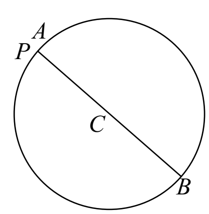


Case III: When P lies on the circle

Draw a line through P passing through the centre and intersects the circle at A and B, respectively.

Shortest distance = 0

Longest distance, PB = 2 radius.



17. Intersection of a Line and a Circle

Let the circle be $x^2 + y^2 = a^2$ ...(i)

and the line be $y = mx + c$ ...(ii)

From Eqs (i) and (ii), we get

$$x^2 + (mx + c)^2 = a^2$$

$$\Rightarrow \quad (1 + m^2)x^2 + 2mcx + (c^2 - a^2) = 0$$

(i) The line $y = mx + c$ will intersect the circle in two distinct points, if

$$D > 0$$

$\Rightarrow \quad 4m^2c^2 - 4(1 + m^2)(c^2 - a^2) > 0$

$\Rightarrow \quad m^2c^2 - [c^2 + m^2c^2 - a^2(1 + m^2)] > 0$

$\Rightarrow \quad c^2 - a^2(1 + m^2) > 0$

$\Rightarrow \quad a^2(1 + m^2) > c^2$

$$\Rightarrow \quad a^2 > \frac{c^2}{(1+m^2)}$$

$$\Rightarrow \quad a > \frac{|c|}{\sqrt{(1+m^2)}}$$

$\Rightarrow$ radius > the length of the perpendicular from the origin to the line $y = mx + c$.

(ii) The line $y = mx + c$ will intersect the circle in two coincident points if

$D = 0$.

$\Rightarrow \quad 4m^2c^2 - 4(1 + m^2)(c^2 - a^2) = 0$

$\Rightarrow \quad m^2c^2 - (1 + m^2)(c^2 - a^2) = 0$

$\Rightarrow \quad c^2 = a^2(1 + m^2)$

$$\Rightarrow \quad a^2 = \frac{c^2}{(1+m^2)}$$

$$\Rightarrow \quad a = \frac{|c|}{\sqrt{(1+m^2)}}.$$

$\Rightarrow$ radius = the length of the perpendicular from the centre of a circle to the line $y = mx + c$.

(iii) The line $y = mx + c$ will not intersect the circle if

$D < 0$

$\Rightarrow \quad 4m^2c^2 - 4(1 + m^2)(c^2 - a^2) < 0$

$\Rightarrow \quad m^2c^2 - [c^2 + m^2c^2 - a^2(1 + m^2)] < 0$

$\Rightarrow \quad c^2 - a^2(1 + m^2) < 0$

$\Rightarrow \quad a^2(1 + m^2) < c^2$

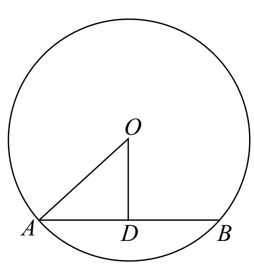
$$\Rightarrow \quad a^2 < \frac{c^2}{(1+m^2)}$$

$$\Rightarrow \quad a < \frac{|c|}{\sqrt{(1+m^2)}}$$

$\Rightarrow$ radius < the length of the perpendicular from the origin to the line $y = mx + c$.

18. Length of Intercept Cut off from a Line by a Circle

The length of the intercept cut off from the line $y = mx + c$ by the circle $x^2 + y^2 = a^2$ is $2 \times \sqrt{\left(\frac{a^2(1+m^2) - c^2}{(1+m^2)}\right)}$.



Proof: We have,

$$OD = \left|\frac{c}{\sqrt{1+m^2}}\right|$$

In ΔAOD, $AD^2 = OA^2 - OD^2$

$$= a^2 - \frac{c^2}{(1+m^2)}$$

$$= \frac{a^2(m^2+1) - c^2}{(1+m^2)}$$

$$\Rightarrow \quad AD = \sqrt{\frac{a^2(m^2+1) - c^2}{(1+m^2)}}$$

Thus, the length of the intercept = AB

$= 2AD$

$$= 2 \times \sqrt{\frac{a^2(m^2+1) - c^2}{(1+m^2)}}$$

19. Tangent and Secant

If a line intersects the curve in two coincident points, it is called a tangent and if a line intersects the curve in two distinct points, it is called a *secant*.

Here, PT is a tangent and MN a secant.

Different forms of the Equations of Tangents

1. Point form

(i) The equation of tangent to the circle $x^2 + y^2 = a^2$ at (x_1, y_1) is $xx_1 + yy_1 = a^2$.

Proof: Let C be the centre of the circle.

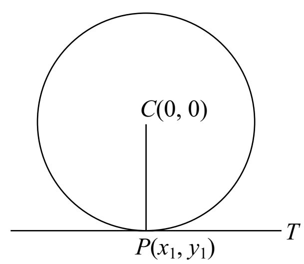
Now slope of CP

$$= \frac{y_1 - 0}{x_1 - 0} = \frac{y_1}{x_1}.$$

Here, PT is the perpendicular to CP.

Thus, slope of $PT = -\frac{x_1}{y_1}$.

Hence, the equation of the tangent is



$$\Rightarrow \quad y - y_1 = -\frac{x_1}{y_1}(x - x_1)$$

$$\Rightarrow \quad yy_1 - y_1^2 = -xx_1 + x_1^2$$

$$\Rightarrow \quad xx_1 + yy_1 = x_1^2 + y_1^2 = a^2$$

(ii) The equation of tangent to the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ at (x_1, y_1) is

$x^2 + y^2 + xx_1 + yy_1 + c = 0.$

Proof: Let $C(-g, -f)$ be the centre of the circle.

Now, slope of $CP = \frac{y_1 + f}{x_1 + g}$.

Here, PT is the perpendicular to CP.

Thus, slope of $PT = -\frac{x_1 + g}{y_1 + f}$.

Hence, the equation of the tangent at (x_1, y_1) is

$$(y - y_1) = -\frac{x_1 + g}{y_1 + f}(x - x_1)$$

$\Rightarrow \quad xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$

> ***Note:*** The equation of the tangent to the 2nd degree curve $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is $axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0$

In order to find out the equation of tangent to any 2nd degree curve, the following points must be kept in your mind

x^2 is replaced by xx_1,

y^2 is replaced by yy_1,

xy is replaced by $\frac{xy_1 + x_1y}{2}$,

x is replaced by $\frac{x + x_1}{2}$,

y is replaced by $\frac{y + y_1}{2}$,

and c will remain c.

This method is applicable only for a 2nd degree conic.

2. Parametric form

The equation of a tangent to the circle $x^2 + y^2 = a^2$ at $(a\cos\theta, a\sin\theta)$ is $x\cos\theta + y\sin\theta = a$

Proof: The equation of the chord joining the points $(a\cos\theta, a\sin\theta)$ and $(a\cos\varphi, a\sin\varphi)$ is

$$x\cos\left(\frac{\theta + \varphi}{2}\right) + y\sin\left(\frac{\theta + \varphi}{2}\right) = a\cos\left(\frac{\theta - \varphi}{2}\right).$$

The equation of the tangent to the circle

$(x - h)^2 + (y - k)^2 = a^2$ at $(h + a\cos\theta, k + a\sin\theta)$ is

$(x - h)\cos\theta + (y - k)\sin\theta = a$

Condition of tangency

The line $y = mx + c$ will be a tangent to the circle $x^2 + y^2 = a^2$ if $c^2 = a^2(1 + m^2)$.

Proof: The line $y = mx + c$ will be a tangent to the circle $x^2 + y^2 = a^2$ if radius = the length of perpendicular from the centre of a circle to the line $y = mx + c$.

$$\Rightarrow \quad a = \frac{|c|}{\sqrt{(1 + m^2)}}$$

$$\Rightarrow \quad a^2 = \frac{c^2}{(1 + m^2)}$$

$$\Rightarrow \quad c^2 = a^2(1 + m^2)$$

which is the required condition.

3. Slope form

The equation of tangent with slope m to a circle $x^2 + y^2 = a^2$ is $y = mx \pm a\sqrt{(1 + m^2)}$ and the co-ordinates of the point of contact are

$$\left(\pm\frac{am}{\sqrt{(1 + m^2)}}, \mp\frac{a}{\sqrt{(1 + m^2)}}\right)$$

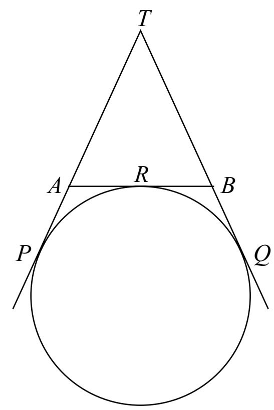
> ***Note:*** Equation of any tangent to the circle can be considered as $y = mx + a\sqrt{1 + m^2}$.

Number of tangents

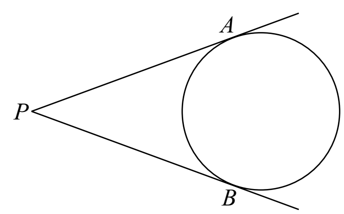
(i) If a point lies outside of a circle, the two tangents can be drawn. Here, TP and TQ are two tangents.

(ii) If a point lies on the circle, then one tangent can be drawn. Here, ARB be a tangent.

(iii) If a point lies inside the circle, then no tangent can be drawn.



Tangents from a point to a circle



Let the point be (x_1, y_1) and the circle be $x^2 + y^2 = a^2$.

The equation of any tangent from a point (x_1, y_1) to a circle $x^2 + y^2 = a^2$ is

$$(y - y_1) = m(x - x_1) \quad \ldots\text{(i)}$$

The line (i) will be the tangent to the given circle if the length of the perpendicular from the centre of a circle = radius of a circle

$$\left|\frac{(mx_1 - y_1)}{\sqrt{1 + m^2}}\right| = a$$

$$\Rightarrow \quad (mx_1 - y_1)^2 = a^2(1 + m^2)$$

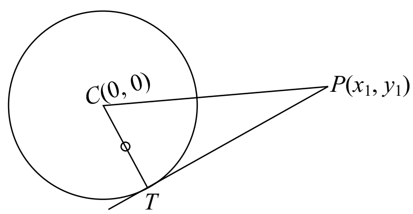
$$\Rightarrow \quad m^2x_1^2 - 2mx_1y_1 + y_1^2 = a^2 + a^2m^2$$

which is a quadratic in m, gives two values of m.

Put those values of m in Eq (i), we get the required equations of tangents.

Length of tangent from a point to a circle

(i) The length of the tangent from a point (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is $\sqrt{x_1^2 + y_1^2 - a^2}$.



(ii) The length of the tangent from a point (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$.

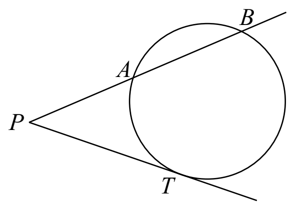
Power of a point with respect to a circle

The power of a point P with respect to any circle is

$$PA \cdot PB.$$

From the geometry, we can write

$$PA \cdot PB = PT^2$$



Thus, the power of a point is the square of the length of the tangent to a circle from that point.

(i) The power of a point (x_1, y_1) with respect to a circle $x^2 + y^2 = a^2$ is $(x_1^2 + y_1^2 - a^2)$.

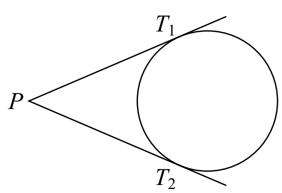
(ii) The power of a point (x_1, y_1) with respect to a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$(x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c).$$

Pair of tangents

(i) The equation of a pair of tangents to a circle $x^2 + y^2 = a^2$ from a point (x_1, y_1) is

$$SS_1 = T^2,$$



where $S \equiv x^2 + y^2 - a^2$;

$$S_1 \equiv x_1^2 + y_1^2 - a^2; T \equiv xx_1 + yy_1 - a^2.$$

(ii) The equation of a pair of tangents to a circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ is}$$

$$SS_1 = T^2,$$

where $S \equiv x^2 + y^2 + 2gx + 2fy + c$; S_1

$$\equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

$$T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$$

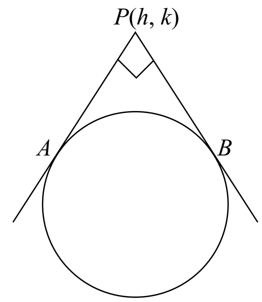
(iii) The angle between the pair of tangents from (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is

$$2\tan^{-1}\left(\frac{a}{\sqrt{S_1}}\right), \text{ where } S_1 = x_1^2 + y_1^2 - a^2.$$

20. Director Circle

The locus of the point of intersection of two perpendicular tangents to a circle is known as the director circle. It is a concentric circle having radius $\sqrt{2}$ times the radius of the original circle.

The equation of the director circle to the circle $x^2 + y^2 = a^2$ is $x^2 + y^2 = 2a^2$.

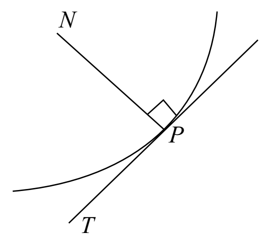


21. Normal

If a line is perpendicular to the point of contact to the tangent is called a normal.

The relation between tangent and normal is

$$m(T) \times m(N) = -1.$$

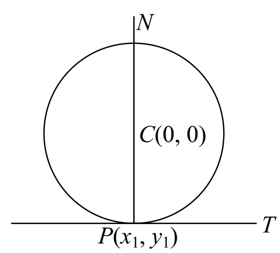


Different Forms of the Equation of Normals

1. Point form

(i) The equation of a normal to the circle $x^2 + y^2 = a^2$ at (x_1, y_1) is $\frac{x}{x_1} = \frac{y}{y_1}$.

Proof: Let PT be a tangent and PN be a normal. Clearly PN must be a perpendicular to PT.



Equation of tangent at P is

$$xx_1 + yy_1 = a^2 \quad \text{...(i)}$$

Equation of normal at P is

$$xy_1 - yx_1 + k = 0 \quad \text{...(ii)}$$

which is passing through (0, 0)

Therefore, $k = 0$.

Hence, the equation of the normal is $xy_1 - yx_1 + k = 0$

$$\Rightarrow \quad \frac{x}{x_1} = \frac{y}{y_1}$$

(ii) The equation of a normal to the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ at (x_1, y_1) is

$$\frac{x - x_1}{x_1 + g} = \frac{y - y_1}{y_1 + f}$$

Proof: As we know that the normal always passes through the centre of a circle.

Thus, the equation of the normal at P to the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$y - y_1 = \frac{y_1 + f}{x_1 + g}(x - x_1)$$

$$\Rightarrow \quad \frac{x - x_1}{x_1 + g} = \frac{y - y_1}{y_1 + f}$$

(iii) The equation of a normal to a conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

at (x_1, y_1) is

$$\frac{x - x_1}{ax_1 + hy_1 + g} = \frac{y - y_1}{hx_1 + by_1 + f}$$

2. Parametric form

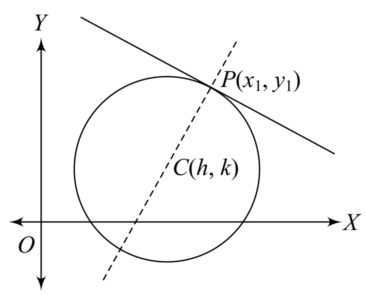
The equation of the normal to the circle $x^2 + y^2 = a^2$ at

$(a \cos \theta, a \sin \theta)$ is $\dfrac{x}{\cos \theta} = \dfrac{y}{\sin \theta}$

3. Slope form

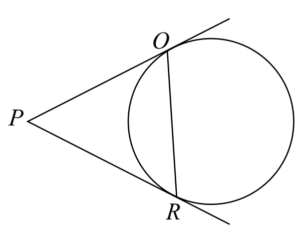
The equation of a normal to the circle $x^2 + y^2 = a^2$ is $y = mx$.

4. Normal always passes through the centre of the circle



5. Line $y = mx + c$ will be a normal to the circle $x^2 + y^2 = a^2$ if $c = 0$.

22. Chord of Contact



From any external point, two tangents can be drawn to a given circle. The chord joining the points of contact of the two tangents is called the chord of contact of tangents.

Here, QR is the chord of contact of tangents.

(i) The equation of the chord of contact of tangents drawn from a point (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is

$$xx_1 + yy_1 = a^2.$$

(ii) The equation of the chord of contact of tangents drawn from a point (x_1, y_1) to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

is $\quad xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

(iii) The chord of contact exists only when if the point P not lies inside of the circle.

(iv) The length of the chord of contact, $T_1T_2 = \dfrac{2LR}{\sqrt{R^2 + L^2}}$, where R = radius and L = length of tangent.

(v) Area of the triangle formed by the pair of tangents and its chord of contact = $\dfrac{RL^2}{R^2 + L^2}$.

(vi) Tangents of the angle between the pair of tangents from the point $(x_1, y_1) = \dfrac{2RL}{L^2 - R^2}$.

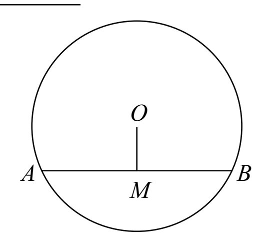
(vii) Equation of the circle circumscribing the triangle ΔPT_1T_2 is

$$(x - x_1)(x + g) + (y - y_1)(y + f) = 0$$

(viii) The distance between the chord of contact of tan gents to $x^2 + y^2 + 2gx + 2fy + c = 0$ from the origin and the point (g, f) is

$$\left| \frac{g^2 + f^2 - c}{2\sqrt{(g^2 + f^2)}} \right|$$

23. Chord Bisected at a Given Point

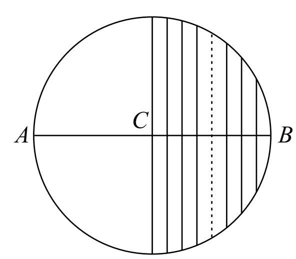


(i) The equation of the chord of the circle $x^2 + y^2 = a^2$ bisected at a point (x_1, y_1) is given by $T = S_1$, where $S_1 \equiv x_1^2 + y_1^2 - a^2$ and $T \equiv xx_1 + yy_1 - a^2$.

(ii) The equation of the chord of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ bisected at a point (x_1, y_1) is given by

$T = S_1$, where $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ and $T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$

24. Diameter of a Circle



The locus of the mid-points of a system of parallel chords of a circle is known as diameter of the circle.

The diameter of a circle always passes through the centre of a circle and perpendicular to the parallel chords.

Let the circle be $x^2 + y^2 = a^2$ and parallel chord be $y = mx + c$.

Equation of any diameter to the given circle is perpendicular to the given parallel chord is $my + x + \lambda = 0$ which passes through the centre of a circle.

Thus $\quad 0 + 0 + \lambda = 0$

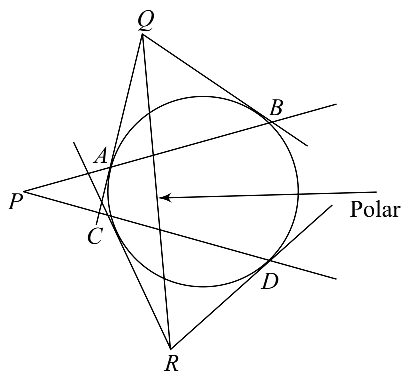
$\Rightarrow \quad \lambda = 0$

Hence, the required equation of the diameter is $x + my = 0$.

25. Pole and Polar

If from a point P, any straight line is drawn to meet the circle in Q and R and if tangents to the circle at Q and R meet in T, the locus of T is called the polar of P with respect to the circle.

The point P is known as the pole of its polar.



Polar of a circle exists only when the point P lies either outside or inside of the given circle.

(i) The equation of a polar of the point (x_1, y_1) with respect to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$

(ii) The equation of polar of the point (x_1, y_1) with respect to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

(iii) If a point P lies outside of a circle, then the polar and the chord of contact of this point P are the same straight line.

(iv) If a point P lies on the circle, then the polar and the tangent to the circle at P are the same straight line.

(v) The pole of a line $lx + my + n = 0$ with respect to the circle $x^2 + y^2 = a^2$ is $\left(-\frac{a^2 l}{n}, -\frac{a^2 m}{n}\right)$.

(vi) If the polar of a point P with respect to a circle passes through Q, then the polar of Q with respect to the circle will pass through P.
Here, the points P and Q are called the conjugate points.

(vii) **Conjugate Points:** Two points are said to be conjugate points with respect to a circle, if the polar of either passes through the other.

(viii) If the pole of a line L_1 with respect to a circle lies on another line L_2, then the pole of the other line L_2 with respect to the same circle will lie on the first line L_1.
Here, the lines L_1 and L_2 are conjugate lines.

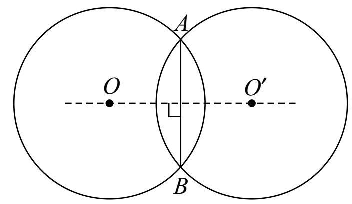
(ix) **Conjugate lines:** Two straight lines are said to be conjugate lines if the pole of either lies on the other.

(x) If the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are conjugate to each other with respect to the circle $x^2 + y^2 = c^2$, then $a_1a_2 + b_1b_2 = \frac{c_1c_2}{r^2}$.

(xi) If O be the centre of a circle and P be any point, OP is perpendicular to the polar of P.

(xii) If O be the centre of a circle and P be any point, then if OP (produced, if it is necessary) meet the polar of P in Q, then $OP \cdot OQ = (\text{radius})^2$.

26. Common Chord of Two Circles



The chord joining the points of intersection of two given circles is called their common chord.

The equation of the common chord between two circles is $S_1 - S_2 = 0$

$$\Rightarrow \quad 2(g_1 - g_2)x + 2(f_1 - f_2)y = c_1 - c_2,$$

where $\quad S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$

and $\quad S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

(i) The length of the common chord $= PQ = 2 \cdot PM = 2\sqrt{(C_1P)^2 - (C_1M)^2}$,
where
C_1P = radius of the circle $S_1 = 0$ and
C_1M = Length of the perpendicular from C_1 on the common chord PQ.

(ii) The common chord PQ of two circles becomes of the maximum length, when it is a diameter of the smaller one between them.

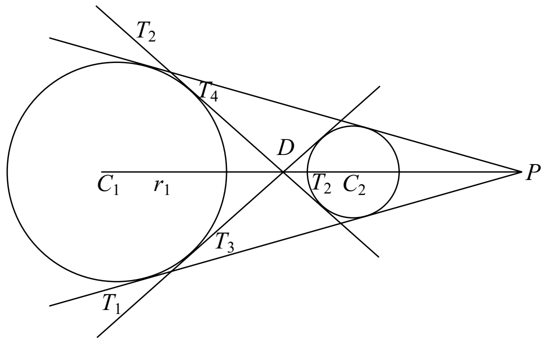
(iii) If the circle on the common chord be a diameter, then the centre of the circle passing through P and Q lie on the common chord of the two circles.

(iv) If the circle $S_1 = 0$, bisects the circumference of the circle $S_2 = 0$, then their common chord will be the diameter of the circle $S_2 = 0$.

27. Intersection between Two Circles

Let the two circles be $(x - \alpha)^2 + (y - \beta)^2 = r_1^2$ and $(x - \gamma)^2 + (y - \delta)^2 = r_2^2$, where centres are $C_1(\alpha, \beta)$ and $C_2(\gamma, \delta)$ and radii are r_1 and r_2, respectively.

(i) **When two circles do not intersect**



Then $C_1C_2 > r_1 + r_2$

Thus four common tangents can be drawn.

Let P be the point of intersection of two transverse tangents and D be the point of intersection of two direct common tangents.

Then the co-ordinates of P and D are

$$\left(\frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \frac{r_1y_2 + r_2y_1}{r_1 + r_2}\right) = (h, k)$$

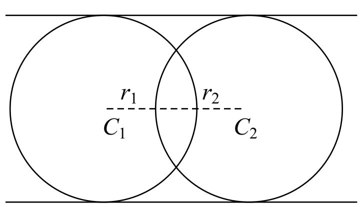
and $$\left(\frac{r_1x_2 - r_2x_1}{r_1 - r_2}, \frac{r_1y_2 - r_2y_1}{r_1 - r_2}\right) = (\lambda, \mu)$$

respectively

The equation of the transverse common tangents is $(y - k) = m_1(x - h)$ and the equation of direct common tangent is $(y - \mu) = m_2(x - \lambda)$.

Now values of m_1 and m_2 can be obtained from the length of the perpendicular from the centre C_1 or C_2 on the tangent is equal to r_1 or r_2. Put two values of m_1 and m_2 on the common tangent equations, then we get the required results.

(ii) **When two circles intersect**



Then $|r_1 - r_2| < C_1C_2 < r_1 + r_2$

Thus two common tangents can be drawn.

Let two direct common tangents intersect at D externally in the ratio $r_1 : r_2$.

Then the co-ordinates of D are

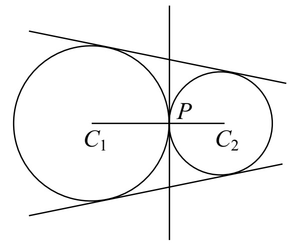
$$\left(\frac{r_1x_2 - r_2x_1}{r_1 - r_2}, \frac{r_1y_2 - r_2y_1}{r_1 - r_2}\right) = (h, k)$$

Hence the equation of the direct common tangent is $y - k = m(x - h)$.

Now values of m can be obtained from the length of the perpendicular from the centre C_1 or C_2 on the tangent is equal to r_1 or r_2.

Put two values of m on the common tangent equation, then we get the required equation of direct common tangents.

(iii) **When two circles touch each other externally**



Then $C_1C_2 = r_1 + r_2$

Thus three common tangents can be drawn.

Let the point of contact be P.

Then the co-ordinates of P are

$$\left(\frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \frac{r_1y_2 + r_2y_1}{r_1 + r_2}\right)$$

Hence the equation of the common transverse tangent is $S_1 - S_2 = 0$ which is the same as the equation of the common chord.

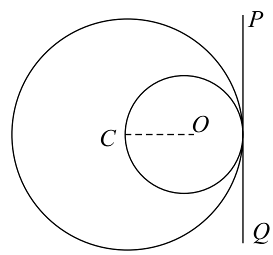
Let two direct common tangents intersect at D externally in the ratio $r_1 : r_2$.

Then the co-ordinates of D are $\left(\frac{r_1x_2 - r_2x_1}{r_1 - r_2}, \frac{r_1y_2 - r_2y_1}{r_1 - r_2}\right)$ $= (h, k)$

Hence the equation of the direct common tangent is $(y - k) = m(x - h)$.

Now values of m can be obtained from the length of perpendicular from the centre C_1 or C_2 on the tangent is equal to r_1 or r_2. Put two values of m on the common tangent equation, then we get the required equation of direct common tangents.

(iv) **When two circles touch each other internally**



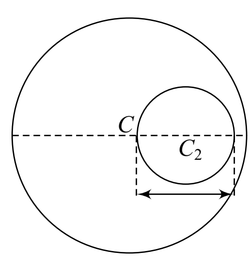
Then $C_1C_2 = |r_1 - r_2|$

Thus, only one tangent can be drawn.

Equation of the common transverse tangent is

$$S_1 - S_2 = 0.$$

(v) **When one circle lies inside the other one**



Then $C_1C_2 > |r_1 - r_2|$

Thus, no common tangent can be drawn.

(vi) The direct common tangents meet at a point which divides the line joining the centres of the circles externally in the ratio of their radii.

(vii) Transverse common tangents meet at a point which divides the line joining the centres of the circles internally in the ratio of their radii.

(viii) The length of an external common tangent and internal common tangent to the two circles is given by the length of external common tangent

$$L_{ex} = \sqrt{d^2 - (r_2 - r_1)^2}$$

and the length of internal common tangent

$$L_{in} = \sqrt{d^2 - (r_2 + r_1)^2},$$

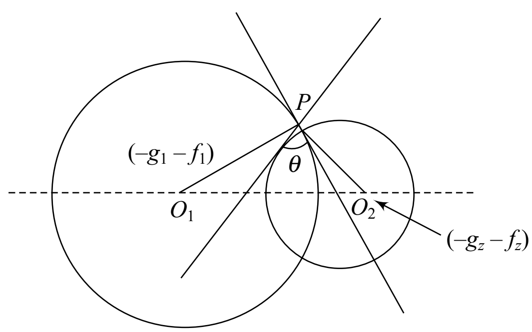
(it is applicable only when $d > r_1 + r_2$)

Where d is the distance between the centres of two circles and r_1 and r_2 are the radii of two circles where $C_1C_2 = d$.

29. Angle of Intersection of Two Circles

Let the two circles be

$$S_1: x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$
$$S_2: x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$



Let C_1 and C_2 are the centres of the given circles and r_1 and r_2 are the radii of the circles.

Thus $C_1 = (-g_1, -f_1)$ and $C_2 = (-g_2, -f_2)$

$$r_1 = \sqrt{g_1^2 + f_1^2 - c_1},$$

and $$r_2 = \sqrt{g_2^2 + f_2^2 - c_2}$$

Let $$d = C_1C_2 = \sqrt{(g_1 - g_2)^2 + (f_1 - f_2)^2}$$
$$= \sqrt{g_1^2 + g_2^2 + f_1^2 + f_2^2 - 2(g_1g_2 + f_1f_2)}$$

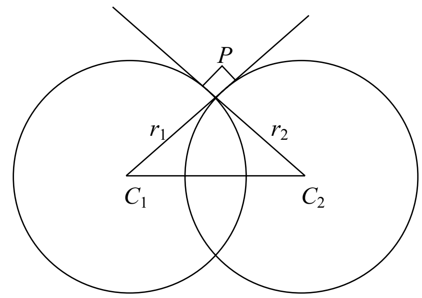
In C_2PC_2, $$\cos\alpha = \left(\frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right)$$

$$\Rightarrow \quad \cos(180° - \theta) = \left(\frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right)$$

Orthogonal Circles

If the angle between two circles is 90°, then the circles are said to be orthogonal circles.

Condition of Orthogonality

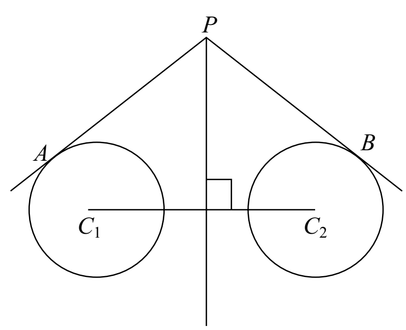


Here, $\theta = 90°$, then

$$\Rightarrow \quad \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2} = 0$$
$$r_1^2 + r_2^2 - d^2 = 0$$
$$\Rightarrow \quad r_1^2 + r_2^2 = d^2$$
$$\Rightarrow \quad g_1^2 + f_1^2 - c_1 + g_2^2 + f_2^2 - c_2$$
$$= g_1^2 + f_1^2 + g_2^2 + f_2^2 - 2(g_1g_2 + f_1f_2)$$
$$\Rightarrow \quad 2(g_1g_2 + f_1f_2) = c_1 + c_2$$

which is the required condition.

30. Radical Axis



The radical axis of two circles is the locus of a point which moves in a plane in such a way that the lengths of the tangents drawn from it to the two circles are same.

Consider

$$S_1: x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$

and $$S_2: x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

Let $P(x_1, y_1)$ be a point such that $PA = PB$

Thus, $PA^2 = PB^2$

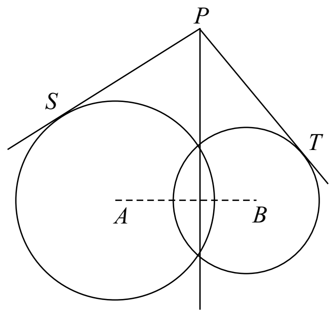
$$\Rightarrow \quad x_1^2 + y_1^2 + 2g_1x_1 + 2f_1y_1 + c_1$$
$$= x_2^2 + y_2^2 + 2g_2x_2 + 2f_2y_2 + c_2$$
$$\Rightarrow \quad 2(g_1 - g_2)x + 2(f_1 - f_2)y = c_2 - c_1$$

31. Properties of the Radical Axis

(i) **The radical axis and common chord are identical.**
Since the radical axis and common chord of two circles $S_1 = 0$ and $S_2 = 0$ are the same straight line $S_1 - S_2 = 0$, they are identical.

The only difference is that, the common chord exists only if the circles intersect in two real points, while the radical axis exists for all pair of circles irrespective of their position.

(ii) **The radical axis is perpendicular to the straight line which joins the centres of the circles.**



Consider

$$S_1 : x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$

and $S_2 : x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

Here, $A(-g_1, -f_1)$ and $B(-g_2, -f_2)$ are the centres of the circles.

Now the slope of $AB = \dfrac{f_1 - f_2}{g_1 - g_2}$.

Equation of the radical axis is

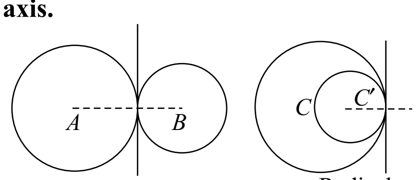
$$2(g_1 - g_2)x + 2(f_1 - f_2)y = c_2 - c_1.$$

$\therefore$ Slope of the radical axis is $-\dfrac{g_1 - g_2}{f_1 - f_2}$.

Clearly the product of their slopes is –1.

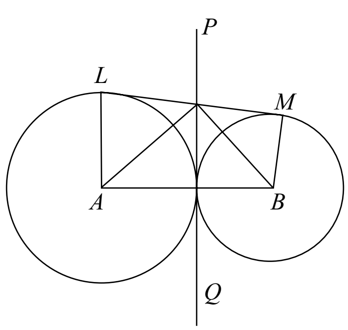
Hence AB and radical axis are perpendicular to each other.

(iii) If two circles touch each other externally or internally, the common tangents itself becomes the radical axis.



Radical axis   Radical axis

(iv) **The radical axis bisects common tangents of two circles.**



Radical axis

In this case the radical axis bisects the common tangent

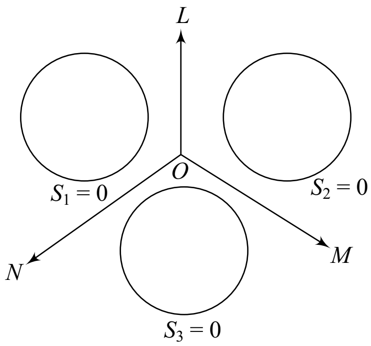
(v) **The radical axis of three circles taken in pairs are concurrent.**

Let

$$S_1 : x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0,$$
$$S_2 : x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

and

$$S_3 : x^2 + y^2 + 2g_3x + 2f_3y + c_3 = 0$$



Now $S_1 - S_2 : 2(g_1 - g_2)x + 2(f_1 - f_2)y = c_2 - c_1$

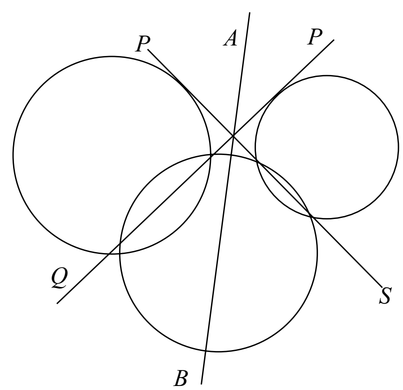
$S_2 - S_3 : 2(g_2 - g_3)x + 2(f_2 - f_3)y = c_3 - c_2$

$S_3 - S_1 : 2(g_3 - g_1)x + 2(f_3 - f_1)y = c_1 - c_3$

Adding we get, both the sides are identical.

Thus three radical axes are concurrent.

(vi) **If two circles cut a third circle othogonally, the radical axis of the two circles will pass through the centre of the third circle.**



Let $S_1 : x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0,$ …(i)

$S_2 : x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ …(ii)

and $S_3 : x^2 + y^2 + 2g_3x + 2f_3y + c_3 = 0$ …(iii)

Since (i) and (ii) both cut (iii) orthogonally, then

$$2(g_1g_3 + f_1f_3) = c_1 + c_3$$
$$2(g_2g_3 + f_2f_3) = c_2 + c_3$$

Subtracting, we get

$$2g_3(g_1 - g_2) + 2f_3(f_1 - f_2) = c_1 - c_2 \quad \text{…(iv)}$$

Also radical axis of (i) and (ii) is

$$S_1 - S_2 : 2(g_1 - g_2)x + 2(f_1 - f_2)y = c_2 - c_1$$

Since it will pass through the centre of third circle, so we get

$$-2g_3(g_1 - g_2) - 2f_3(f_1 - f_2) = -c_1 + c_2$$

$\Rightarrow \quad 2g_3(g_1 - g_2) + 2f_3(f_1 - f_2) = c_1 - c_2$

which is identical with (iv).

(vii) **Radical axis need not always pass through the mid-point of the line joining the centres of the two circles**.
It will pass through the mid-point of the line joining the centres of the two circles only if they have equal radii.

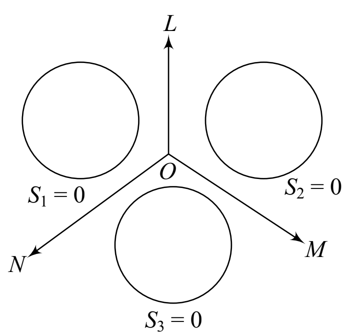
(viii) The centre of a variable circle orthogonal to two fixed circles lies on the radical axis of two circles.

(ix) If two circles are orthogonal, then the polar of a point P on the first circle with respect to the second circle passes through the point Q, which is the other end of the diameter through P.
Hence the locus of a point which moves in such a way that, its polars with respect to the circles $S_1 = 0$, $S_2 = 0$ and $S_3 = 0$ are concurrent in a circle which is orthogonal to all the three circles.

(x) Pairs of circles which do not have radical axis are concurrent.

(xi) A system of circles, every two of which have the same radical axis, is called a coaxial system.

32. Radical Centre



The radical axes of three circles, taken in pairs, meet at a point, which is called their radical centre.

Let $\quad S_1: x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0,$

$\quad S_2: x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

and $\quad S_3: x^2 + y^2 + 2g_3x + 2f_3y + c_3 = 0$

Equations of three radical axes are

$$S_1 - S_2: 2(g_1 - g_2)x + 2(f_1 - f_2)y = c_2 - c_1$$

$$S_2 - S_3: 2(g_2 - g_3)x + 2(f_2 - f_3)y = c_3 - c_2$$

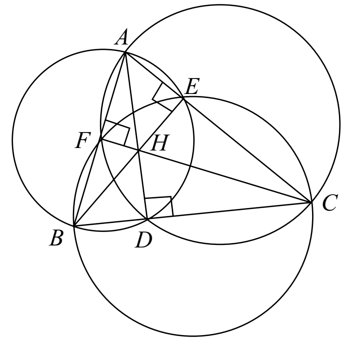
and

$$S_3 - S_1: 2(g_3 - g_1)x + 2(f_3 - f_1)y = c_1 - c_3$$

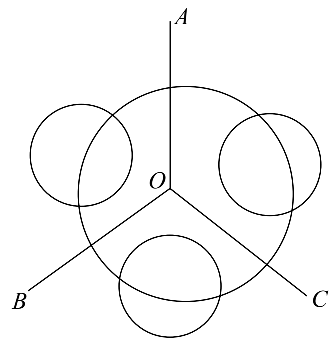
Solving the three equations of radical axes, we get the required radical centre.

Properties of the Radical Centres

1. The radical centre of the three circles described on the sides of a triangle as diameters is the orthocentre of the triangle.

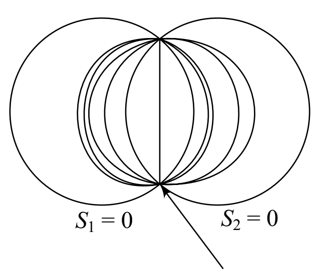


2. The radical centre of the three given circles will be the centre of a fourth circle, which cuts all the three circles orthogonally and the radius of the fourth circle is the length of the tangent drawn from the radical centre of the three given circles to any of these circles.



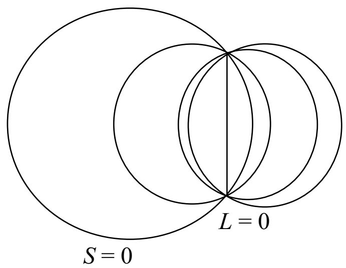
Let the fourth circle be $(x - h)^2 + (y - k)^2 = r^2$, where (h, k) is the centre of the circle and r be the radius. The centre of the circle is the radical centre of the given circles and r is the length of the tangent from (h, k) to any of the given three circles.

33. Family of Circles



Family of circles

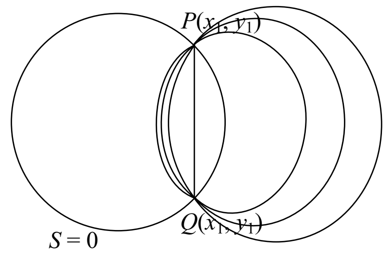
1. The equation of the family of circles passing through the point of intersection of two given circles $S_1 = 0$ and $S_2 = 0$ is given by
$S_1 + \lambda S_2 = 0$, where λ is a parameter and $\lambda \neq -1$.
2. The equation of the family of circles passing through the point of intersection of a circle $S = 0$ and a line $L = 0$ is given by
$S + \lambda L = 0$, where λ is a parameter.



3. The equation of the family of circles touching the circle $S = 0$ and the line $L = 0$ is
   $S + \lambda L = 0$, where λ is a parameter.
4. The equation of the family of circles passing through the two given points $P(x_1, y_1)$ and $Q(x_2, y_2)$ can be written in the form

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2)$$

$$+ \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$



EXERCISES

LEVEL I

(Problems Based on Fundamentals)

ABC OF CIRCLES

1. Find the centre and the radius of the circles
   (i) $x^2 + y^2 = 16$
   (ii) $x^2 + y^2 - 8x + 15 = 0$
   (iii) $x^2 + y^2 - x - y = 0$
2. Prove that the radii of the circles
   $x^2 + y^2 = 1, x^2 + y^2 - 2x - 6y = 6$
   and $x^2 + y^2 - 4x - 12y = 9$ are in AP.
3. Find the equation of the circle concentric with the circle $x^2 + y^2 - 8x + 6y - 5 = 0$ and passing through the point $(-2, -7)$.
4. Find the equation of the circle passing through the point of intersection of $x + 3y = 0$ and $2x - 7y = 0$ and whose centre is the point of intersection of lines $x + y + 1 = 0$ and $x - 2y + 4 = 0$.
5. Find the equation of the circle touching the lines $4x - 3y = 30$ and $4x - 3y + 10 = 0$ having the centre on the line $2x + y = 0$.
6. Let the equation of a circle is $x^2 + y^2 - 16x - 24y + 183 = 0$. Find the equation of the image of this circle by the line mirror $4x + 7y + 13 = 0$.
7. Find the area of an equilateral triangle inscribed in the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.
8. A circle has radius 3 units and its centre lies on the line $y = x - 1$. Find the equation of the circle if it passes through (7, 3).
9. Find the point P on the circle $x^2 + y^2 - 4x - 6y + 9 = 0$ such that $\angle POX$ is minimum, where O is the origin and OX is the x-axis.
10. Find the equation of the circle when the end-points of a diameter are (2, 3) and (6, 9).
11. Find the equation of a circle passing through (1, 2), (4, 5) and (0, 9).
12. Find the equation of a circle passing through the points (1, 2) and (3, 4) and touching the line $3x + y = 5$.
13. Find the length of intercepts to the circle $x^2 + y^2 + 6x + 10y + 8 = 0$.
14. Find the length of y-intercept to the circle $x^2 + y^2 - x - y = 0$.
15. Show that the circle $x^2 + y^2 - 2ax - 2ay + a^2 = 0$ touches both the co-ordinate axes.

POSITION OF A POINT WITH RESPECT TO A CIRCLE

16. Discuss the position of the points (1, 2) and (6, 0) with respect to the circle $x^2 + y^2 - 4x + 2y - 11 = 10$.
17. If the point $(\lambda, -\lambda)$ lies inside the circle
    $x^2 + y^2 - 4x + 2y - 8 = 0$, then find range of λ.

SHORTEST AND LONGEST DISTANCE OF A CIRCLE FROM A POINT

18. Find the shortest and the longest distance from the point $(2, -7)$ to the circle
    $x^2 + y^2 - 14x - 10y - 151 = 0$

INTERSECTION OF A LINE AND A CIRCLE

19. Prove that the line $y = x + 2$ touches the circle $x^2 + y^2 = 2$. Find its point of contact.
20. Find the equation of the tangent to the circle $x^2 + y^2 = 4$ parallel to the line $3x + 2y + 5 = 0$.
21. Find the equation of the tangent to the circle $x^2 + y^2 = 9$ perpendicular to the line $4x + 3y = 0$.
22. Find the equation of the tangent to the circle $x^2 + y^2 + 4x + 3 = 0$, which makes an angle of 60° with the positive direction of x-axis.
23. If a tangent is equally inclined with the co-ordinate axes to the circle $x^2 + y^2 = 4$, find its equation.
24. Find the equation of the tangents to the circle $x^2 + y^2 = 25$ through (7, 1).

TANGENT TO A CIRCLE

25. If the centre of a circle $x^2 + y^2 = 9$ is translated 2 units parallel to the line $x + y = 4$ where x increases, find its equation.
26. Find the equation of the tangents to the circle $x^2 + y^2 = 9$ at $x = 2$.
27. Find the equation of the tangent to the circle $x^2 + y^2 = 16$ at $y = 4$.
28. Find the points of intersection of the line $4x - 3y = 10$ and the circle.
29. Find the equation of the pair of tangents drawn to the circle $x^2 + y^2 - 2x + 4y = 0$ from (0, 1).
30. Find the equation of the common tangent to the curves $x^2 + y^2 = 4$ and $y^2 = 4(x - 2)$.
31. Find the shortest distance between the circle $x^2 + y^2 = 9$ and the line $y = x - 8$.

LENGTH OF THE TANGENT TO A CIRCLE

32. Find the length of the tangent from the point (2, 3) to the circle $x^2 + y^2 = 4$.
33. Find the length of the tangent from any point on the circle $x^2 + y^2 = a^2$ to the circle $x^2 + y^2 = b^2$.
34. Find the length of the tangent from any point on the circle $x^2 + y^2 + 2011x + 2012y + 2013 = 0$ to the circle $x^2 + y^2 + 2011x + 2012y + 2014 = 0$.

POWER OF A POINT W.R.T A CIRCLE

35. Find the power of a point (2, 5) with respect to the circle $x^2 + y^2 = 16$.
36. If a point $P(1, 2)$ is rotated about the origin in an anti-clockwise sense through an angle of 90° say at Q, then find the power of a point Q with respect to the circle $x^2 + y^2 = 4$.

PAIR OF TANGENTS

37. Find the equation of the tangent from the point (1, 2) to the circle $x^2 + y^2 = 4$.
38. Find the equation of the tangent to the circle $x^2 + y^2 - 4x + 3 = 0$ from the point (2, 3).
39. Find the angle between the tangent from the point (3, 5) to the circle $x^2 + y^2 = 25$.
40. If a point (1, 2) is translated 2 units through the positive direction of x-axis and then tangents drawn from that point to the circle $x^2 + y^2 = 9$, find the angle between the tangents.

DIRECTOR CIRCLE

41. Find the locus of the point of intersection of two perpendicular tangents to a circle $x^2 + y^2 = 25$.
42. Tangents are drawn from an arbitrary point on the line $y = x + 1$ to the circle $x^2 + y^2 = 9$. If those tangents are orthogonal to each other, find the locus of that point.
43. Tangents are drawn from any point on the circle $x^2 + y^2 = 20$ to the circle $x^2 + y^2 = 10$. Find the angle between their tangents.
44. Find the equation of the director circle to each of the following given circles.
    (i) $x^2 + y^2 + 2x = 0$
    (ii) $x^2 + y^2 + 10y + 24 = 0$
    (iii) $x^2 + y^2 + 16x + 12y + 99 = 0$
    (iv) $x^2 + y^2 + 2gx + 2fy + c = 0$.
    (v) $x^2 + y^2 - ax - by = 0$.

NORMAL AND NORMALCY

45. Find the equation of the normal to the circle $x^2 + y^2 = 9$ at $x = 2$.
46. Find the equation of the normal to the circle $x^2 + y^2 + 2x + 4y + 4 = 0$ at (–2, 1).
47. Find the equation of a normal to a circle $x^2 + y^2 - 4x - 6y + 4 = 0$, which is parallel to the line $y = x - 3$.
48. Find the equation of the normal to a circle $x^2 + y^2 - 8x - 12y + 99 = 0$, which is perpendicular to the line $2x - 3y + 10 = 0$.

CHORD OF CONTACT

49. Find the equation of the chord of contact of the tangents drawn from (5, 3) to the circle $x^2 + y^2 = 25$.
50. Find the co-ordinates of the point of intersection of the tangents at the points where the line $2x + y + 12 = 0$ meets the circle $x^2 + y^2 - 4x + 3y - 1 = 0$.
51. Find the condition that the chord of contact from an external point (h, k) to the circle $x^2 + y^2 = a^2$ subtends a right angle at the centre.
52. Tangents are drawn from the point (h, k) to the circle $x^2 + y^2 = a^2$. Prove that the area of the triangle formed by the tangents and their chord of contact is $a\left(\frac{(h^2 + k^2 - a^2)^{3/2}}{(h^2 + k^2)}\right)$.
53. The chord of contact of tangents drawn from a point on the circle $x^2 + y^2 = a^2$ to the circle $x^2 + y^2 = b^2$ touches the circle $x^2 + y^2 = c^2$ such that $b^m = a^n c^p$, where $m, n, p \in N$, find the value of $m + n + p + 10$.

CHORD BISECTED AT A POINT

54. Find the equation of the chord of the circle $x^2 + y^2 = 25$, which is bisected at the point (–2, 3).
55. Find the equation of the chord of the circle $x^2 + y^2 + 6x + 8y - 11 = 0$, whose mid-point is (1, –1).
56. Find the locus of the middle points of the chords of the circle $x^2 + y^2 = a^2$ which pass through the fixed point (h, k).
57. Find the locus of the mid-point of a chord of the circle $x^2 + y^2 = 4$ which subtend a right angle at the centre.
58. Let AB be a chord of the circle $x^2 + y^2 = 4$ such that $A = (\sqrt{3}, 1)$. If the chord AB makes an angle of 90° about the origin in anti-clockwise direction, then find the co-ordinates of B.
59. Let CD be a chord of the circle $x^2 + y^2 = 9$ such that $C = (2\sqrt{2}, 1)$. If the chord CD makes an angle of 60°

about the centre of the circle in clockwise direction, find the co-ordinates of D.

60. Find the locus of the mid-points of the chords of the circle $x^2 + y^2 = 9$ which are parallel to the line $2012x + 2013y + 2014 = 0$.
61. Find the locus of the mid-points of the chords of the circle $x^2 + y^2 - 4x - 6y = 0$, which are perpendicular to the line $4x + 5y + 10 = 0$.

COMMON CHORD OF TWO CIRCLES

62. Find the lengths of the common chord of the circles $x^2 + y^2 + 3x + 5y + 4 = 0$ and $x^2 + y^2 + 5x + 3y + 4 = 0$.
63. Find the equation of the circle whose diameter is the common chord of the circles
$$x^2 + y^2 + 2x + 3y + 1 = 0$$
and $x^2 + y^2 + 4x + 3y + 2 = 0$.

INTERSECTION OF TWO CIRCLES

64. Find the number of tangents between the given circles
 (i) $x^2 + y^2 = 4$ and $x^2 + y^2 - 2x = 0$
 (ii) $x^2 + y^2 + 4x + 6y + 12 = 0$ and $x^2 + y^2 - 6x - 4y + 12 = 0$
 (iii) $x^2 + y^2 - 6x - 6y + 9 = 0$ and $x^2 + y^2 + 6x + 6y + 9 = 0$
 (iv) $x^2 + y^2 - 4x - 4y = 0$ and $x^2 + y^2 + 2x + 2y = 0$
 (v) $x^2 + y^2 = 64$ and $x^2 + y^2 - 4x - 4y + 4 = 0$
 (vi) $x^2 + y^2 - 2(1+\sqrt{2})x - 2y + 1 = 0$ and $x^2 + y^2 - 2x - 2y + 1 = 0$.
65. Find all the common tangents to the circles $x^2 + y^2 - 2x - 6y + 9 = 0$ and $x^2 + y^2 + 6x - 2y + 1 = 0$.
66. If two circles $x^2 + y^2 + c^2 = 2$ and $x^2 + y^2 + c^2 = 2by$ touches each other externally, prove that $\frac{1}{a^2} = \frac{1}{b^2} + \frac{1}{c^2}$.
67. If two circles $x^2 + y^2 = 9$ and $x^2 + y^2 - 8x - 6y + n^2 = 0$, where n is any integer, have exactly two common tangents, find the number of possible values of n.

ORTHOGONAL CIRCLES

68. Find the angle at which the circles $x^2 + y^2 + x + y = 0$ and $x^2 + y^2 + x - y = 0$ intersect.
69. If the circles $x^2 + y^2 + 2a_1x + 2b_1y + c_1 = 0$ and $x^2 + y^2 + a_2x + b_2y + c_2 = 0$ intersect orthogonally, prove that $a_1a_2 + b_1b_2 = c_1 + c_2$.
70. If two circles $2x^2 + 2y^2 - 3x + 6y + k = 0$ and $x^2 + y^2 - 4x + 10y + 16 = 0$ cut orthogonally, find the value of k.
71. A circle passes through the origin and centre lies on the line $y = x$. If it cuts the circle $x^2 + y^2 - 4x - 6y + 18 = 0$ orthogonally, find its equation.
72. Find the locus of the centres of the circles which cut the circles $x^2 + y^2 + 4x - 6y + 9 = 0$ and $x^2 + y^2 - 5x + 4y - 2 = 0$ orthogonally.
73. If a circle passes through the point (a, b) and cuts the circle $x^2 + y^2 = 4$ orthogonally, find the locus of its centre.
74. Two circles having radii r_1 and r_2 intersect orthogonally, find the length of the common chord.

RADICAL AXIS

75. Find the radical axis of the two circles
$$x^2 + y^2 + 4x + 6y + 9 = 0$$
and $x^2 + y^2 + 3x + 8y + 10 = 0$
76. Find the image of a point (2, 3) with respect to the radical axis of two circles $x^2 + y^2 + 8x + 2y + 10 = 0$ and $x^2 + y^2 - 2x - y - 8 = 0$.
77. Find the radical centre of the three circles $x^2 + y^2 = 1$, $x^2 + y^2 - 8x + 15 = 0$ and $x^2 + y^2 + 10y + 24 = 0$.
78. Find the equation of a circle which cuts orthogonally every member of the circles
$$x^2 + y^2 - 3x - 6y + 14 = 0,$$
$$x^2 + y^2 - x - 4y + 8 = 0$$
and $x^2 + y^2 + 2x - 6y + 9 = 0$
79 Find the equation of the circle passing through the points of intersection of the circles
$$x^2 + y^2 + 13x - 3y = 0$$
and $2x^2 + 2y^2 + 4x - 7y - 25 = 0$
and the point (1, 1).
80. If the circle $x^2 + y^2 + 2x + 3y + 1 = 0$ cuts the circle $x^2 + y^2 + 4x + 3y + 2 = 0$ in two points, say A and B, find the equation of the circle as AB as a diameter.
81. Find the equation of the smallest circle passing through the intersection of the line $x + y = 1$ and the circle $x^2 + y^2 = 9$.
82. Find the equation of the circle, which is passing through the point of intersection of the circles
$$x^2 + y^2 - 6x + 2y + 4 = 0$$
and $x^2 + y^2 + 2x - 4y - 6 = 0$
and its centre lies on the line $y = x$.
83. Find the circle whose diameter is the common chord of the circles $x^2 + y^2 + 2x + 3y + 1 = 0$ and $x^2 + y^2 + 4x + 3y + 2 = 0$.

LEVEL II

(Mixed Problems)

1. The equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a circle if
 (a) $a = b, c = 0$ (b) $a = b, h = 0$
 (c) $a = b, g = 0$ (d) $a = b, f = 0$
2. The equation of the circle passing through the points (0, 0), (1, 0) and (0, 1) is
 (a) $x^2 + y^2 + x + y = 0$ (b) $x^2 + y^2 - x + y = 0$
 (c) $x^2 + y^2 - x - y = 0$ (d) $x^2 + y^2 - x = 0$
3. The equation $x^2 + y^2 + 4x + 6y + 13 = 0$ represents a
 (a) point (b) circle
 (c) pair of straight lines (d) a pair of coincident lines
4. The circle $x^2 + y^2 + 4x - 7y + 12 = 0$ cuts an intecept on y-axis is
 (a) 7 (b) 4 (c) 3 (d) 1

5. The equation of the diameter of the circle $x^2 + y^2 - 6x + 2y - 8 = 0$ is
   (a) $x + 3y = 0$ (b) $x = 3y$
   (c) $x = 2y$ (d) $x + 2y = 0$.
6. The length of the tangent drawn from any point on the circle $x^2 + y^2 + 4x + 6y + 4 = 0$ to the circle $x^2 + y^2 + 4x + 6y + 11 = 0$ is
   (a) 4 (b) $\sqrt{7}$ (c) $\sqrt{15}$ (d) $\sqrt{17}$
7. The value of c for which the points (2, 0), (0, 1), (0, 5) and (0, c) are concyclic, is
   (a) 1 (b) 2 (c) 3 (d) 4
8. The equation of a circle passing through the origin is $x^2 + y^2 - 4x + 6y = 0$, the equation of the diameter is
   (a) $x = y$ (b) $3x + 2y = 0$
   (c) $y = 3x$ (d) $3x - 4y = 0$
9. The equation of the common chord of the circles $x^2 + y^2 - 4x + 6y = 0$ and $x^2 + y^2 - 6x + 4y - 12 = 0$ is
   (a) $x + y + 6 = 0$ (b) $x - y + 6 = 0$
   (c) $x - y - 6 = 0$ (d) $-x + y + 6 = 0$
10. The circumcentre of the triangle whose vertices are (0, 0), (2, 0) and (0, 2) is
    (a) (1, 2) (b) (2, 2)
    (c) (1, 1) (d) (–2, –2)
11. If the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ cut the x-axis and y-axis in four concyclic points, then
    (a) $a_1a_2 = b_1b_2$ (b) $a_1b_1 = a_2b_2$
    (c) $a_1b_2 = a_2b_1$ (d) None
12. If the circles $x^2 + y^2 + 2g_1x + 2f_1y = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y = 0$ touch each other, then
    (a) $g_1g_2 + f_1f_2 = 0$ (b) $g_1g_2 = f_1f_2$
    (c) $g_1f_2 = f_1g_2$ (d) None
13. Any point on the circle $x^2 + y^2 - 4x - 4y + 4 = 0$ can be taken as
    (a) $(2 + 2\cos\theta, 2 + 2\sin\theta)$
    (b) $(2 - 2\cos\theta, 2 - 2\sin\theta)$
    (c) $(2 - 2\cos\theta, 2 + 2\sin\theta)$
    (d) $(2 + 2\cos\theta, 2 - 2\sin\theta)$
14. The equation of the chord of the circle $x^2 + y^2 = 25$ whose mid-point is (2, –5) is
    (a) $3x - 2y = 11$ (b) $2x - 3y = 13$
    (c) $2x + 3y = 10$ (d) $3x + 2y = 11$
15. The value of λ for which the line $3x - 4\lambda = \lambda$ touches the circle $x^2 + y^2 = 16$ is
    (a) 4 (b) 20 (c) 15 (d) 10
16. The locus of the point of intersection of two perpendicular tangents to the circle $x^2 + y^2 = 1006$ is
    (a) $x^2 + y^2 = 2012$ (b) $x^2 + y^2 = 2020$
    (c) $x^2 + y^2 = 2010$ (d) $x^2 + y^2 = 2000$
17. The equation of the normal to the circle $x^2 + y^2 - 2x - 2y + 1 = 0$, which is parallel to the line $2x + 4y = 3$ is
    (a) $x + 2y = 3$ (b) $x + 2y + 3 = 0$
    (c) $2x + 4y = -3$ (d) none
18. The image of the centre of the circle $x^2 + y^2 - 2x - 6y + 1 = 0$ to the line mirror $y = x$ is
    (a) (1, 3) (b) (3, 1) (c) (1, –3) (d) (–3, 1)
19. The length of the common chord of the circles $x^2 + y^2 + 2x + 3y + 1 = 0$ and $x^2 + y^2 + 3x + 2y + 1 = 0$ is
    (a) $3\sqrt{2}$ (b) $\dfrac{1}{3\sqrt{2}}$ (c) $\dfrac{3}{\sqrt{2}}$ (d) $\dfrac{\sqrt{2}}{3}$
20. The number of common tangents to the circles $x^2 + y^2 + 2x - 8y - 23 = 0$ and $x^2 + y^2 - 4x - 10y + 19 = 0$ is
    (a) 4 (b) 2 (c) 3 (d) 1
21. The angle between the tangents drawn from the origin to the circle $(x - 7)^2 + (y + 1)^2 = 25$ is
    (a) $\pi/3$ (b) $\pi/6$ (c) $\pi/2$ (d) $\pi/4$
22. The ends of a quadrant of a circle have the co-ordinates (1, 3) and (3, 1), the centre of such a circle is
    (a) (1,1) (b) (2, 2) (c) (2, 6) (d) (4, 4)
23. The line $2x - y + 1 = 0$ is a tangent to the circle at the point (2, 5) and the centre of the circles lies on $x - 2y = 4$. The radius of the circle is
    (a) $3\sqrt{5}$ (b) $5\sqrt{3}$ (c) $2\sqrt{5}$ (d) $5\sqrt{2}$
24. Two circles of radii 4 cm and 1 cm touch each other externally and θ is the angle contained by their direct common tangents. Then $\sin\theta$ is
    (a) 24/25 (b) 12/25 (c) 3/4 (d) None
25. The locus of poles whose polar with respect to $x^2 + y^2 = a^2$ always pass through (k, 0) is
    (a) $kx - a^2 = 0$ (b) $kx + a^2 = 0$
    (c) $ky + a^2 = 0$ (d) $ky - a^2 = 0$
26. The locus of the mid-points of the chords of the circle $x^2 + y^2 - ax - by = 0$ which subtend a right angle at $\left(\dfrac{a}{2}, \dfrac{b}{2}\right)$ is
    (a) $ax + by = 0$
    (b) $ax + by = a^2 + b^2$
    (c) $x^2 + y^2 - ax - by + \dfrac{a^2 + b^2}{8} = 0$
    (d) $x^2 + y^2 - ax - by - \dfrac{a^2 + b^2}{8} = 0$

27 From (3, 4), chords are drawn to the circle $x^2 + y^2 - 4x = 0$. The locus of the mid-points of the chords is
   (a) $x^2 + y^2 - 5x - 4y + 6 = 0$
   (b) $x^2 + y^2 + 5x - 4y + 6 = 0$
   (c) $x^2 + y^2 - 5x + 4y + 6 = 0$
   (d) $x^2 + y^2 - 5x - 4y - 6 = 0$

28. The lines $y - y_1 = m(x - x_1) \pm a\sqrt{1 + m^2}$ are tangents to the same circle. The radius of the circle is
    (a) $a/2$ (b) a (c) $2a$ (d) none
29. The centre of the smallest circle touching the circles $x^2 + y^2 - 2y - 3 = 0$ and $x^2 + y^2 - 8x - 18y + 93 = 0$ is
    (a) (3, 2) (b) (4, 4) (c) (2, 7) (d) (2, 5)
30. The ends of the base of an isosceles triangle are at (2, 0) and (0, 1) and the equation of one side is $x = 2$, the orthocentre of the triangle is
    (a) $\left(\dfrac{3}{4}, \dfrac{3}{2}\right)$ (b) $\left(\dfrac{5}{4}, 1\right)$ (c) $\left(\dfrac{3}{4}, 1\right)$ (d) $\left(\dfrac{4}{3}, \dfrac{7}{12}\right)$

31. A rhombus is inscribed in the region common to two circles $x^2 + y^2 - 4x - 12 = 0$ and $x^2 + y^2 + 4x - 12 = 0$ with two of its vertices on the line joining the centres of the circles. Then the area of the rhombus is
(a) $8\sqrt{3}$ sq. units (b) $4\sqrt{3}$ sq. units
(c) $16\sqrt{3}$ sq. units (d) none

32. The angle between the two tangents from the origin to the circles $(x - 7)^2 + (y + 1)^2 = 25$ is
(a) $\pi/4$ (b) $\pi/3$ (c) $\pi/2$ (d) None.

33. The equation of the circle having normal at (3, 3) as the straight line $y = x$ and passing through the point (2, 2) is
(a) $x^2 + y^2 - 5x + 5y + 12 = 0$
(b) $x^2 + y^2 + 5x - 5y + 12 = 0$
(c) $x^2 + y^2 - 5x - 5y - 12 = 0$
(d) $x^2 + y^2 - 5x - 5y + 12 = 0$

34. In a right triangle ABC, right angled at A, on the leg AC as diameter, a semi-circle is described. If the chord joining A with the point of intersection D of the hypotenuse and the semicircle, the length AC equals to
(a) $\dfrac{AB \cdot AD}{\sqrt{AB^2 + AD^2}}$ (b) $\dfrac{AB \cdot AD}{AB + AD}$
(c) $\sqrt{AB \cdot AD}$ (d) $\dfrac{AB \cdot AD}{\sqrt{AB^2 - AD^2}}$

35. If the circle $C_1 : x^2 + y^2 = 16$ intersects another circle C_2 of radius 5 in such a manner that the common chord is of maximum length and has a slope 3/4, the co-ordinates of the centre of C_2 are
(a) $\left(\pm\frac{9}{5}, \pm\frac{12}{5}\right)$ (b) $\left(\pm\frac{9}{5}, \mp\frac{12}{5}\right)$
(c) $\left(\pm\frac{12}{5}, \pm\frac{9}{5}\right)$ (d) $\left(\pm\frac{12}{5}, \mp\frac{9}{5}\right)$

36. Two lines $p_1x + q_1y + r_1 = 0$ and $p_2x + q_2y + r_2 = 0$ are conjugate lines with respect to the circle $x^2 + y^2 = a^2$ if
(a) $p_1p_2 + q_1q_2 = r_1r_2$ (b) $p_1p_2 + q_1q_2 + r_1r_2 = 0$
(c) $a^2(p_1p_2 + q_1q_2) = r_1r_2$ (d) $(p_1p_2 + q_1q_2) = a^2r_1r_2$

37. If a circle passing through the point (a, b) cuts the circle $x^2 + y^2 = k^2$ orthogonally, the equation of the locus of its centre is
(a) $2ax + 2by - (a^2 + b^2 + k^2) = 0$
(b) $2ax + 2by - (a^2 - b^2 - k^2) = 0$
(c) $x^2 + y^2 - 3ax - 4by - (a^2 + b^2 - k^2) = 0$
(d) $x^2 + y^2 - 2ax - 3by - (a^2 - b^2 - k^2) = 0$.

38. Consider the circle
$$S : x^2 + y^2 - 4x - 4y + 4 = 0.$$
If another circle of radius r less than the radius of the circle S is drawn, touching the circle S, and the co-ordinates axes, the value of r is
(a) $3 - 2\sqrt{2}$ (b) $4 - 2\sqrt{2}$
(c) $7 - 4\sqrt{2}$ (d) $6 - 4\sqrt{2}$

39. The distance between the chords of contact of tangents to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ from the origin and the point (g, f) is
(a) $\sqrt{g^2 + f^2}$ (b) $\dfrac{\sqrt{g^2 + f^2 - c}}{2}$
(c) $\dfrac{g^2 + f^2 - c}{2\sqrt{g^2 + f^2}}$ (d) $\dfrac{\sqrt{g^2 + f^2 - c}}{2\sqrt{g^2 + f^2}}$

40. The locus of the centres of the circles which cuts the circles $x^2 + y^2 + 4x - 6y + 9 = 0$ and $x^2 + y^2 - 5x + 4y - 2 = 0$ orthogonally is
(a) $9x + 10y - 7 = 0$ (b) $x - y + 2 = 0$
(c) $9x - 10y + 11 = 0$ (d) $9x + 10y + 7 = 0$.

41. The locus of the centres of the circles such that the point (2, 3) is the mid-point of the chord $5x + 2y + 16 = 0$ is
(a) $2x - 5y + 11 = 0$ (b) $2x + 5y - 11 = 0$
(c) $2x + 5y + 11 = 0$ (d) None.

42. The locus of the mid-points of the chords of the circle $x^2 + y^2 + 4x - 6y - 12 = 0$ which subtends an angle of $\dfrac{\pi}{3}$ radians at its circumcentre is
(a) $(x - 2)^2 + (y + 3)^2 = 6.25$
(b) $(x + 2)^2 + (y - 3)^2 = 6.25$
(c) $(x + 2)^2 + (y - 3)^2 = 18.75$
(d) $(x + 2)^2 + (y + 3)^2 = 18.75$

43. If two chords of the circle $x^2 + y^2 - ax - by = 0$ drawn from the point (a, b) is divided by the x-axis in the ratio 2 : 1 is
(a) $a^2 > 3b^2$ (b) $a^2 < 3b^2$
(c) $a^2 > 4b^2$ (d) $a^2 < 4b^2$

44. The angle at which the circles $(x - 1)^2 + y^2 = 10$ and $x^2 + (y - 2)^2 = 5$ intersect is
(a) $\pi/6$ (b) $\pi/4$ (c) $\pi/3$ (d) $\pi/2$

45. Two congruent circles with centres at (2, 3) and (5, 6) which intersect at right angles has radius equal to
(a) $2\sqrt{2}$ (b) 3 (c) 4 (d) none

46. A circle of radius unity is centred at origin. Two particles start moving at the same time from the point (1, 0) and moves around the circle in opposite direction. One of the particle moves counter-clockwise with constant speed v and the other moves clockwise with constant speed $3v$. After leaving (1, 0), the two particles meet first at a point P and continue until they meet next at point Q. Then the co-ordinates of the point Q are
(a) (1, 0) (b) (0, 1) (c) (0, –1) (d) (–1, 0)

47. The value of c for which the set $\{(x, y) : x^2 + y^2 + 2x \le 1\} \cap \{(x, y) : x - y + c \ge 0\}$ contains only one point in common is
(a) $(-\infty, -1] \cup [3, \infty)$ (b) $\{-1, 3\}$
(c) $\{-3\}$ (d) $\{-1\}$

48. A circle is inscribed into a rhombus $ABCD$ with one angle 60°. The distance from the centre of the circle to the nearest vertex is equal to 1. If P be any point of the circle, then $|PA|^2 + |PB|^2 + |PC|^2 + |PD|^2$ is equal to
(a) 12 (b) 11 (c) 9 (d) none

49. P is a point (a, b) in the first quadrant. If the two circles which pass through P and touch both the co-ordinates axes cut at right angles, then

(a) $a^2 - 6ab + b^2 = 0$ (b) $a^2 + 2ab - b^2 = 0$
(c) $a^2 - 4ab + b^2 = 0$ (d) $a^2 - 8ab + b^2 = 0$

50. The range of value of a such that the angle θ between the pair of tangents drawn from the point $Q(a, 0)$ to the circle $x^2 + y^2 = 1$ satisfies $\pi/2 < \theta < \pi$ is
(a) $(1, 2)$ (b) $(1, \sqrt{2})$
(c) $(-\sqrt{2}, -1)$ (d) $(-\sqrt{2}, -1) \cup (1, \sqrt{2})$

51. Three concentric circles of which the biggest is $x^2 + y^2 = 1$ have their radii in AP. If the line $y = x + 1$ cuts all the circles in real and distinct points the interval in which the common difference of AP will lie is
(a) $\left(0, \frac{1}{4}\right)$ (b) $\left(0, \frac{1}{2\sqrt{2}}\right)$
(c) $\left(0, \frac{2-\sqrt{2}}{4}\right)$ (d) none

52. A tangent is a point on the circle $x^2 + y^2 = a^2$ intersects a concentric circle C at two points P and Q. The tangents to the circle C at P and Q meet at a point on the circle $x^2 + y^2 = b^2$, the equation of the circle is
(a) $x^2 + y^2 = ab$ (b) $x^2 + y^2 = (a - b)^2$
(c) $x^2 + y^2 = (a + b)^2$ (d) $x^2 + y^2 = a^2 + b^2$

53. AB is the diameter of a semicircle k, C is an arbitrary point on the semicircle (other than A or B) and S is the centre of the circle inscribed in triangle ABC, then the measure of
(a) $\angle ASB$ changes as C moves on k
(b) $\angle ASB = 135$ for all C
(c) $\angle ASB = 150$ for all C
(d) $\angle ASB$ is the same for all positions of C, but r cannot be determined without knowing the radius.

54. Tangents are drawn to the circle $x^2 + y^2 = 1$ at the points, where it is met by the circles $x^2 + y^2 - (\lambda + 6)x + (8 - 2\lambda)y - 3 = 0$, λ being the variable. The locus of the point of intersection of these tangents is
(a) $2x - y + 10 = 0$ (b) $x + 2y - 10 = 0$
(c) $x - 2y + 10 = 9$ (d) $2x + y - 10 = 0$

55. Given $\frac{1}{x} + \frac{b}{y} = 1$ and $ax + by = 1$ are two variable lines a and b being the parameters connected by the relation $a^2 + b^2 = ab$. The locus of the point of intersection has the equation
(a) $x^2 + y^2 + xy - 1 = 0$ (b) $x^2 + y^2 - xy + 1 = 0$
(c) $x^2 + y^2 + xy + 1 = 0$ (d) $x^2 + y^2 - xy - 1 = 0$

56. B and C are two fixed points having co-ordinates (3, 0) and (–3, 0) respectively. If the vertical angle BAC is 90°, the locus of the centroid of the ΔABC has the equation
(a) $x^2 + y^2 = 1$ (b) $x^2 + y^2 = 2$
(c) $x^2 + y^2 = 1/9$ (d) $x^2 + y^2 = 4/9$.

57. If $\left(1, \frac{1}{a}\right); \left(b, \frac{1}{b}\right); \left(c, \frac{1}{c}\right)$ and $\left(d, \frac{1}{d}\right)$ are four distinct points on a circle of radius 4 units, then $abcd$ is equal to
(a) 4 (b) ¼ (c) 1 (d) 10

58. The triangle formed by the lines $x + y = 0$, $x - y = 0$ and $lx + my = l$. If l amd m vary subject to the condition $l^2 + m^2 = 1$, the locus of its circumcentre is
(a) $(x^2 - y^2)^2 = x^2 + y^2$ (b) $(x^2 + y^2)^2 = x^2 - y^2$
(c) $(x^2 + y^2) = 4x^2y^2$ (d) $(x^2 + y^2)^2 = x^2 - y^2$

59. Tangents are drawn to a unit circle at the origin from each point on the line $2x + y = 4$. Then the equation to the locus of the mid-point of the chord of contact is
(a) $2(x^2 + y^2) = x + y$ (b) $2(x^2 + y^2) = x + 2y$
(c) $4(x^2 + y^2) = 2x + y$ (d) none

60. $ABCD$ is a square of unit area. A circle is tangent to two sides of $ABCD$ and passes through exactly one of its vertices. The radius of the circle is
(a) $2 - \sqrt{2}$ (b) $\sqrt{2} - 1$ (c) $\frac{1}{2}$ (d) $\frac{1}{\sqrt{2}}$

61. A pair of tangents are drawn to a unit circle with centre at the origin and these tangents intersect at A enclosing an angle of 60°. The area enclosed by these tangents and the arc of the circle is
(a) $\frac{2}{\sqrt{3}} - \frac{\pi}{6}$ (b) $\sqrt{3} - \frac{\pi}{3}$
(c) $\frac{\pi}{3} - \frac{\sqrt{3}}{6}$ (d) $\sqrt{3}\left(1 - \frac{\pi}{6}\right)$.

62. Two circles are drawn through the points (1, 0) and (2, –1) to touch the axis of y. They intersect at angle
(a) $\cot^{-1}\left(\frac{3}{4}\right)$ (b) $\cot^{-1}\left(\frac{4}{5}\right)$
(c) $\frac{\pi}{2}$ (d) $\tan^{-1}(1)$

63. If the line $x \cos\theta + y \sin\theta = 2$ is the equation of transverse common tangent to the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 6\sqrt{3}x - 6y + 20 = 0$, the value of θ is
(a) $5\pi/6$ (b) $2\pi/3$ (c) $\pi/3$ (d) $\pi/6$

64. A circle of constant radius a passes through the origin O and cuts the axis of co-ordinates in points P and Q, the equation of the locus of the foot of perpendicular from O to PQ is
(a) $(x^2 + y^2)\left(\frac{1}{x^2} + \frac{1}{y^2}\right) = 4a^2$
(b) $(x^2 + y^2)^2\left(\frac{1}{x^2} + \frac{1}{y^2}\right) = a^2$
(c) $(x^2 + y^2)^2\left(\frac{1}{x^2} + \frac{1}{y^2}\right) = 4a^2$
(d) $(x^2 + y^2)\left(\frac{1}{x^2} + \frac{1}{y^2}\right) = a^2$.

65. A square is inscribed in the circle $x^2 + y^2 - 2x + 4y + 3 = 0$. Its sides are parallel to the co-ordinate axes. Then one vertex of the square is
(a) $(1 + \sqrt{2}, -2)$ (b) $(1 - \sqrt{2}, -2)$
(c) $(1, -2 + \sqrt{2})$ (d) none

66. The point of contact of the tangent to the circle $x^2 + y^2 = 5$ at the point $(1, -2)$ which touches the circle $x^2 + y^2 - 8x + 6y + 20 = 0$, is
 (a) $(2, -1)$ (b) $(3, -1)$
 (c) $(4, -1)$ (d) $(5, -1)$
67. The centre of the circle passing through the point $(0, 1)$ and touching the curve $y = x^2$ at $(2, 4)$ is
 (a) $(-16/5, 27/10)$ (b) $(-16/7, 53/10)$
 (c) $(-16/5, 53/10)$ (d) none
68. The locus of the mid-points of the chords of the circle $x^2 + y^2 - 2x - 6y - 10 = 0$ which passes through the origin is the circle
 (a) $x^2 + y^2 + x + 3y = 0$ (b) $x^2 + y^2 - x + 3y = 0$
 (c) $x^2 + y^2 + x - 3y = 0$ (d) $x^2 + y^2 - x - 3y = 0$
69. The equation of the circle through the points of intersection of $x^2 + y^2 = 1$, $x^2 + y^2 - 2x - 4y + 1 = 0$ and touching the line $x + 2y = 0$ is
 (a) $x^2 + y^2 + x + 2y = 0$ (b) $x^2 + y^2 - x + 2y = 0$
 (c) $x^2 + y^2 - x - 2y = 0$ (d) $2(x^2 + y^2) - x - 2y = 0$
70. $y - x + 1 = 0$ is the equation of the normal at $\left(3 + \frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$ to which of the following circles?
 (a) $\left(x - 3 - \frac{3}{\sqrt{2}}\right)^2 + \left(y - \frac{3}{\sqrt{2}}\right)^2 = 9$
 (b) $\left(x - 3 - \frac{3}{\sqrt{2}}\right)^2 + y^2 = 6$
 (c) $(x - 3)^2 + y^2 = 6$
 (d) $(x - 3)^2 + (y - 3)^2 = 9$
71. The radical axis of two circles whose centres lie along x and y axes is
 (a) $ax - by - \left(\frac{a^2 + b^2}{4}\right) = 0$
 (b) $2g^2x + 2f^2y - \left(\frac{g^2 + f^2}{4}\right) = 0$
 (c) $gx + fy = g^4 + f^4$
 (d) $2gx - 2fy + g^2 - f^2 = 0$
72. A circle has its centre in the first quadrant and passes through the points of intersection of the lines $x = 2$ and $y = 3$. If it makes intercepts of 3 and 4 units on these lines respectively, its equation is
 (a) $x^2 + y^2 - 3x - 5y + 8 = 0$
 (b) $x^2 + y^2 - 4x - 6y + 13 = 0$
 (c) $x^2 + y^2 - 6x - 8y + 23 = 0$
 (d) $x^2 + y^2 - 8x - 9y + 30 = 0$
73. The radius of the circle passing through the points $(1, 2)$, $(5, 2)$ and $(5, -2)$ is
 (a) $5\sqrt{2}$ (b) $2\sqrt{5}$ (c) $3\sqrt{2}$ (d) $2\sqrt{2}$
74. Lines are drawn from the point $(-2, -3)$ to meet the circle $x^2 + y^2 - 2x - 10y + 1 = 0$. The length of the line that meets the circle at two coincident point is
 (a) $4\sqrt{3}$ (b) 16 (c) 48
 (d) Cannot be calculated unless coincident points are given
75. The equation of the circle, which touches both the axes and the straight line $4x + 3y = 6$ in the first quadrant and lies below it is
 (a) $4(x^2 + y^2) - 4x - 4y + 1 = 0$
 (b) $x^2 + y^2 - 6x - 6y + 9 = 0$
 (c) $x^2 + y^2 - 6x - y + 9 = 0$
 (d) $(x^2 + y^2 - x - 6y) + 4 = 0$
76. Circles are drawn through the point $(2, 0)$ to cut the intercepts of length 5 units on the x-axis. If their centres lie in the first quadrant, their equation is
 (a) $x^2 + y^2 - 9x + 2ky + 14 = 0$
 (b) $3x^2 + 3y^2 + 27x - 2ky + 42 = 0$
 (c) $x^2 + y^2 - 9x - 2ky + 14 = 0$
 (d) $x^2 + y^2 - 2kx - 9y + 14 = 0$,
 where k is a positive real number.
77. If the tangent to the circle $x^2 + y^2 = 5$ at $(1, -2)$ touches the circle $x^2 + y^2 - 8x + 6y + 20 = 0$ at the point
 (a) $(2, -1)$ (b) $(3, -1)$
 (c) $(4, -1)$ (d) $(5, -1)$
78. The centre of the circle passing through the point $(0, 1)$ and touching the curve $y = x^2$ at $(2, 4)$ is
 (a) $(-16/5, 27/10)$ (b) $(-16/7, 53/10)$
 (c) $(-16/5, 53/10)$ (d) none
79. If the lines $2x - 4y = 9$ and $6x - 12y + 7 = 0$ touch a circle, the radius of the circle is
 (a) $\frac{\sqrt{3}}{5}$ (b) $\frac{17}{6\sqrt{5}}$ (c) $\frac{2\sqrt{5}}{3}$ (d) $\frac{17}{3\sqrt{5}}$
80. If the co-ordinates at one end of a diameter of the circle $x^2 + y^2 - 8x - 4y + c = 0$ are $(-3, 2)$, the co-ordinates of the other end are
 (a) $(5, 3)$ (b) $(6, 2)$ (c) $(1, -8)$ (d) $(11, 2)$
81. If a circle is inscribed in an equilateral triangle of side a, the area of the square inscribed in the circle is
 (a) $a^2/6$ (b) $a^2/3$ (c) $2a^2/5$ (d) $2a^2/3$
82. The equations of lines joining the origin to the point of intersection of circle $x^2 + y^2 = 3$ and the line $x + y = 2$ is
 (a) $y - (3 + 2\sqrt{2})x = 0$ (b) $x - (3 + 2\sqrt{2})y = 0$
 (c) $x - (3 - 2\sqrt{2})y = 0$ (d) $y - (3 - 2\sqrt{2})x = 0$
83. Two circles $x^2 + y^2 - 10x + 16 = 0$ and $x^2 + y^2 = r^2$ intersect each other at two distinct points if
 (a) $r < 2$ (b) $r > 8$
 (c) $2 < r < 8$ (d) $2 \le r \le 8$
84. If the equations of the tangents drawn from the origin to the circle $x^2 + y^2 - 2rx - 2hy + h^2 = 0$ are perpendicular, then
 (a) $h = r$ (b) $h^2 = r^2$
 (c) $h = -r$ (d) $r^2 + h^2 = 1$
85. If a circle C and $x^2 + y^2 = 1$ are orthogonal and have radical axis parallel to y-axis, then C is
 (a) $x^2 + y^2 - 1 + x = 0$ (b) $x^2 + y^2 - 1 - x = 0$
 (c) $x^2 + y^2 + 1 - y = 0$ (d) $x^2 + y^2 + 1 + x = 0$

86. The equation of the line meeting the circle $x^2 + y^2 = a^2$, two points at equal distances d from a point (x_1, y_1) on the circumference is
(a) $xx_1 + yy_1 - a^2 + \frac{1}{2d^2} = 0$
(b) $xx_1 - yy_1 - a^2 + \frac{1}{2d^2} = 0$
(c) $xx_1 + yy_1 + a^2 - \frac{1}{2d^2} = 0$
(d) $xx_1 + yy_1 - a^2 - \frac{1}{2d^2} = 0$

87. The equations of the tangents drawn from the origin to the circle $x^2 + y^2 - 2rx - 2hy + h^2 = 0$ are
(a) $x = 0$ (b) $y = 0$
(c) $(h^2 - r^2)x - 2rhy = 0$ (d) $(h^2 - r^2)x + 2rhy = 0$

88. Two circles $x^2 + y^2 = 6$ and $x^2 + y^2 - 6x + 8 = 0$ are given. Then the equation of the circle through their points of intersection and the point (1, 1) is
(a) $x^2 + y^2 - 6x + 4 = 0$ (b) $x^2 + y^2 - 3x + 1 = 0$
(c) $x^2 + y^2 - 4x + 2 = 0$ (d) none

89. The equation of the circle passing through (1, 1) and the points of intersection of $x^2 + y^2 + 13x - 3y = 0$ and $2x^2 + 2y^2 + 4x - 7y - 25 = 0$ is
(a) $4x^2 + 4y^2 - 30x - 10y - 25 = 0$
(b) $4x^2 + 4y^2 + 30x - 13y - 25 = 0$
(c) $4x^2 + 4y^2 - 17x - 10y - 25 = 0$
(d) none

90. The centre of the circle passing through the point (0, 1) and touching the curve $y = x^2$ at (2, 4) is
(a) $\left(-\frac{16}{5}, \frac{27}{10}\right)$ (b) $\left(-\frac{16}{7}, \frac{53}{10}\right)$
(c) $\left(-\frac{16}{5}, \frac{53}{10}\right)$ (d) none

91. AB is a diameter of a circle and C is any point on the circumference of the circle. Then
(a) the area of the triangle ABC is maximum when it is isosceles.
(b) the area of the triangle ABC is minimum when it is isosceles.
(c) the area of the triangle ABC is minimum when it is isosceles.
(d) None

92. The locus of the mid-point of a chord of the circle $x^2 + y^2 = 4$, which subtends a right angle at the origin is
(a) $x + y = 2$ (b) $x^2 + y^2 = 1$
(c) $x^2 + y^2 = 2$ (d) $x + y = 1$

93. If a circle passes through the point (a, b) and cuts the circle $x^2 + y^2 = 2$ orthogonally, then the equation of the locus of its centre is
(a) $2ax + 2by - (a^2 + b^2 + k^2) = 0$
(b) $2ax + 2by - (a^2 - b^2 + k^2) = 0$
(c) $x^2 + y^2 - 3ax - 4by + (a^2 + b^2 - k^2) = 0$
(d) $x^2 + y^2 - 2ax - 3by + (a^2 + b^2 - k^2) = 0$

94. The equation of the tangents drawn from the origin to the circle $x^2 + y^2 + 2rx - 2hy + h^2 = 0$ are
(a) $x = 0$ (b) $y = 0$
(c) $(h^2 - r^2)x - 2rhy = 0$ (d) $(h^2 - r^2)x + 2rhy = 0$

95. If two circles $(x - 1)^2 + (y - 3)^2 = r^2$ and $x^2 + y^2 - 8x + 2y + 8 = 0$ intersect in two distinct points, then
(a) $2 < r < 8$ (b) $r < 2$
(c) $r = 2$ (d) $r > 2$

96. The lines $2x - 3y = 5$ and $3x - 4y = 7$ are diameters of a circle of area 154 sq. units. Then the equation of the circle is
(a) $x^2 + y^2 + 2x - 2y - 62 = 0$
(b) $x^2 + y^2 + 2x - 2y - 47 = 0$
(c) $x^2 + y^2 - 2x + 2y - 47 = 0$
(d) $x^2 + y^2 - 2x + 2y - 62 = 0$

97. The centre of a circle passing through the points (0, 0), (1, 0) and touching the circle $x^2 + y^2 = 9$ is
(a) (3/2, ½) (b) (1/2, 3/2)
(c) (1/2, ½) (d) $(1/2, 1/\sqrt{2})$

98. The locus of the centre of a circle which touches externally the circle $x^2 + y^2 - 6x - 6y + 14 = 0$ and also touches the y-axis is given by the equation
(a) $x^2 - 6x - 10y + 14 = 0$
(b) $x^2 - 10x - 6y + 14 = 0$
(c) $y^2 - 6x - 10y + 14 = 0$
(d) $y^2 - 10x + 6y + 14 = 0$

99. The circles $x^2 + y^2 - 10x + 16 = 0$ and $x^2 + y^2 = r^2$ intersect each other in distinct points if
(a) $r < 2$ (b) $r > 8$
(c) $2 < r < 8$ (d) $2 \le r \le 8$

100. The angle between a pair of tangents drawn from a point P to the circle $x^2 + y^2 + 4x - 6y + 9\sin^2\alpha + 13\cos^2\alpha = 0$ is 2α. Then the locus of P is
(a) $x^2 + y^2 + 4x - 6y + 4 = 0$
(b) $x^2 + y^2 + 4x - 6y - 9 = 0$
(c) $x^2 + y^2 + 4x - 6y - 4 = 0$
(d) $x^2 + y^2 + 4x - 6y + 9 = 0$

101. The number of common tangents to the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 6x - 8y - 24 = 0$ is
(a) 0 (b) 1 (c) 3 (d) 4

102. Let $A_0A_1A_2A_3A_4A_5$ be a regular hexagon inscribed in a unit circle. Then the product of the lengths of the line segments A_0A_1, A_0A_2 and A_0A_4 is
(a) 3/4 (b) $3\sqrt{3}$
(c) 3 (d) $3\sqrt{3}/2$

103. If two distinct chords, drawn from the point (p, q) on the circle $x^2 + y^2 = px + qx$, where $pq \ne 0$ are bisected by the x-axis, then
(a) $p^2 = q^2$ (b) $p^2 = 8q^2$
(c) $p^2 < 8q^2$ (d) $p^2 > 8q^2$

104. The triangle PQR is inscribed in the circle $x^2 + y^2 = 25$. If Q and R have co-ordinates (3, 4) and (–4, 3), respectively, then $\angle QPR$ is
(a) $\pi/2$ (b) $\pi/3$ (c) $\pi/4$ (d) $\pi/6$.

105. If the circles $x^2 + y^2 + 2x + 2ky + b = 0$ and $x^2 + y^2 + 2ky + k = 0$ intersect orthogonally, then k is
(a) 2 or $-3/2$ (b) -2 or $-3/2$
(c) 2 or $3/2$ (d) -2 or $3/2$

106. Let AB be a chord of the circle $x^2 + y^2 = r^2$ subtending right angle at the centre, the locus of the centroid of the triangle PAB as P moves on the circle is
(a) a parabola (b) a circle
(c) an ellipse (d) a pair of straight lines

107. Let PQ and RS be tangents at the extremities of the diameter PR of a circle of radius r. If PS and RQ intersect a point x on the circumference of the circle, then $2r$ equals
(a) $\sqrt{PQ \cdot RS}$ (b) $\dfrac{PQ + RS}{2}$
(c) $\dfrac{2PQ + RS}{PQ + RS}$ (d) $\sqrt{\dfrac{PQ^2 + RS^2}{2}}$

108. If the tangent at the point P on the circle $x^2 + y^2 + 6x + 6y = 2$ meets the straight line $5x - 2y + 6 = 0$ at a point Q on the y-axis, the length of PQ is
(a) 4 (b) $2\sqrt{5}$ (c) 5 (d) $2\sqrt{5}$

109. If $a > 2b > 0$, the positive value of m for which $y = mx - b\sqrt{1 + m^2}$ is a common tangent to $x^2 + y^2 = b^2$ and $(x - a)^2 + y^2 = b^2$ is
(a) $\dfrac{2b}{\sqrt{a^2 - 4b^2}}$ (b) $\dfrac{\sqrt{a^2 - 4b^2}}{2a}$
(c) $\dfrac{2b}{a - 2b}$ (d) $\dfrac{b}{a - 2b}$

110. The centre of the circle inscribed in a square formed by the lines $x^2 - 8x + 12 = 0$ and $y^2 - 14y + 45 = 0$ is
(a) (4, 7) (b) (7, 4) (c) (9, 4) (d) (4, 9)

111. If one of the diameters of the circle $x^2 + y^2 - 2x - 6y + 6 = 0$ is a chord to the circle with centre (2, 1), the radius of the circle is
(a) $\sqrt{3}$ (b) $\sqrt{2}$ (c) 3 (d) 2

112. A circle is given by $x^2 + (y - 1)^2 = 1$, another circle C touches it externally and also the x-axis, the locus of its centre is
(a) $\{(x, y) : x^2 = 4y\} \cup \{(x, y) : y \le 0\}$
(b) $\{(x, y) : x^2 + (y - 1)^2 = 4\} \cup \{(x, y) : y \le 0\}$
(c) $\{(x, y) : x^2 = y\} \cup \{(0, y) : y < 0\}$
(d) $\{(x, y) : x^2 = y\} \cup \{(0, y) : y \le 0\}$

113. The tangent to the curve $y = x^2 + 6$ at a point $P(1, 7)$ touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at a point Q. Then the co-ordinates of Q are
(a) $(-6, -7)$ (b) $(-10, -15)$
(c) $(-9, -13)$ (d) $(-6, -11)$

114. Let $ABCD$ be a quadrilateral with area 18, with side AB parallel to the side CD and $AB = 2\,CD$. Let AD be a perpendicular to AB and CD. If a circle is drawn inside the quadrilateral $ABCD$ touching all the sides, then its radius is
(a) 3 (b) 2 (c) 3/2 (d) 1

115. Tangents drawn from the point $P(1, 8)$ to the circle $x^2 + y^2 - 6x - 4y - 11 = 0$ touch the circle at A and B. The equation of the circumcircle of the triangle PAB is
(a) $x^2 + y^2 + 4x - 6y + 19 = 0$
(b) $x^2 + y^2 - 4x - 10y + 19 = 0$
(c) $x^2 + y^2 - 2x + 6y - 29 = 0$
(d) $x^2 + y^2 - 6x - 4y + 19 = 0$

116. Two parallel chords of a circle of radius 2 are at a distance $\sqrt{3} + 1$ apart. If the chords subtend, angles of $\dfrac{\pi}{k}$ and $\dfrac{2\pi}{k}$ at the centre, where $k > 0$, the value of $[k]$ is, where [,] = GIF
(a) 1 (b) 2 (c) 3 (d) 4

117. The circle passing through the point $(-1, 0)$ and touching the y-axis at (0, 2), also passes through the point
(a) $\left(-\dfrac{3}{2}, 0\right)$ (b) $\left(-\dfrac{5}{2}, 2\right)$
(c) $\left(-\dfrac{3}{2}, \dfrac{5}{2}\right)$ (d) $(-4, 0)$

Level III

(Problems for JEE Advanced)

1. Find the number of points with integral co-ordinates that are interior to the circle $x^2 + y^2 = 16$.
2. If the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ cuts each of the circles $x^2 + y^2 = 4$, $x^2 + y^2 - 6x - 8y + 10 = 0$ and $x^2 + y^2 + 2x - 4y - 2 = 0$ at the extremities of a diameter, find the equation of the circle.
3. The equations of four circles are $(x \pm a)^2 + (y \pm a)^2 = a^2$. Find the radius of a circle which touches all the four circles.
4. A square is inscribed in the circle $x^2 + y^2 - 10x - 6y + 30 = 0$. One side of the square is parallel to $y = x + 3$, then one vertex of the square is ….
5. If a chord of the circle $x^2 + y^2 = 8$ makes equal intercepts of length a on the co-ordinate axes, then $|a| <$ ….
6. The equation of a circle and a line are $x^2 + y^2 - 8x + 2y + 12 = 0$ and $x - 2y - 1 = 0$. Determine whether the line is a chord or a tangent or does not meet the circle at all.
7. If the circles $(x - a)^2 + (y - b)^2 = c^2$ and $(x - b)^2 + (y - a)^2 = c^2$ touch each other, find the value of a.
8. Find the locus of the mid-points of the chords of the circle $x^2 + y^2 + 4x - 6y - 12 = 0$, which subtends an angle of $\dfrac{\pi}{3}$ radians at its circumference.
9. Find the equation of a circle, which touches the axis of y at (0, 3) and cuts an intercept of 8 units on the axis of x.
10. Find the distance between the chord of contact of tangents to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ from the origin and the point (g, f).
11. If $(1 + \alpha x)^n = 1 + 8x + 24x^2 + \ldots$ and a line through $P(\alpha, n)$ cuts the circle $x^2 + y^2 = 4$ in A and B, then $PA.PB =$ ….

12. The circle $x^2 + y^2 = 4$ cuts the circle $x^2 + y^2 + 2x + 3y - 5 = 0$ in A and B, the centre of the circle AB as diameter is ... .
13. Find the length of the tangents from any point of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ to the circle $x^2 + y^2 + 2gx + 2fy + d = 0$, $(d > c)$.
14. Find the equation of the circle whose diameter is the chord $x + y = 1$ of the circle $x^2 + y^2 = 4$.
15. Find the equation of the image of the circle $(x - 3)^2 + (y - 2)^2 = 1$ by the line mirror $x + y = 19$.
16. Two circles $(x - 1)^2 + (y - 3)^2 = r^2$ and $x^2 + y^2 - 8x + 2y + 8 = 0$ intersect in two distinct points, prove that $2 < r < 8$.
17. If exactly two real tangents can be drawn to the circles $x^2 + y^2 - 2x - 2y = 0$ and $x^2 + y^2 - 8x - 8y + \lambda = 0$, prove that $0 < \lambda < 24$.
18. Two vertices of an equilateral triangle are $(-1, 0)$ and $(1, 0)$ and its third vertex lies above the x-axis. Prove that the equation of the circumcircle is $x^2 + y^2 - \frac{2}{\sqrt{3}}y - 1 = 0$.
19. Find the equation of the circumcircle of the triangle formed by the lines $y + \sqrt{3}x = 6$, $y - \sqrt{3}x = 6$ and $y = 0$.
20. If the equation of incircle of an equilateral triangle is $x^2 + y^2 + 4x - 6y + 4 = 0$, prove that the equation of the circumcircle of the triangle is $x^2 + y^2 + 4x - 6y - 23 = 0$.
21. If a square is inscribed in the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ of radius r, prove that the length of its side is $r\sqrt{2}$.
22. If r be the radius, prove that the area of the equilateral triangle inscribed in the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $\frac{3\sqrt{3}}{4} \times r^2$.
23. Prove that the equation of a circle with centre at the origin and passing through the vertices of an equilateral triangle whose median is of length $3a$ is $x^2 + y^2 = 4a^2$.
24. A circle is inscribed in an equilateral triangle of side a. Prove that the area of any square inscribed in the circle is $(a^2, 6)$.
25. Find the co-ordinates of the point on the circle $x^2 + y^2 - 12x + 4y + 30 = 0$ which is farthest from the origin.
26. A diameter of $x^2 + y^2 - 2x - 6y + 6 = 0$ is a chord to the circle with centre $(2, 1)$, find the radius of it.
27. Prove that angle between two tangents from the origin to the circle $(x - 7)^2 + (y + 1)^2 = 25$ is $\frac{\pi}{2}$.
28. Prove that the tangents are drawn from the point $(4, 3)$ to the circle $x^2 + y^2 - 2x - 4y = 0$ are inclined at an angle is $\frac{\pi}{2}$.
29. Find the number of tangents that can be drawn from the point $(0, 1)$ to the circle $x^2 + y^2 - 2x - 4y = 0$.
30. Prove that the locus of the point of intersection of tangents to the circle $x^2 + y^2 = a^2$ at the points whose parametric angles differ by $\frac{\pi}{3}$ is $x^2 + y^2 = \frac{4a^2}{3}$.
31. Prove that the area of the triangle formed by positive axis and the normal and the tangent to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$ is $2\sqrt{3}$.

Level IV

(Tougher Problems for JEE Advanced)

1. A circle of diameter 13 m with centre O coinciding with the origin of co-ordinates axes has diameter AB on the x-axis. If the length of the chord AC be 5 m, find the following:
   (i) Equations of the pair of lines BC and BC'.
   (ii) The area of the smaller portion bounded between the circle and the chord AC.
   [Roorkee, 1983]
2. A circle I of radius 5 m is having its centre A at the origin of the co-ordinate axes. Two circles II and III with centres at B and C and radii 3 and 4 m, respectively, touch the circle I and also touch the x-axis to the right of A. Find the equations of any two common tangents to the circles II and III. **[Roorkee, 1983]**
3. Find the equation of a circle which is co-axial with the circles $2x^2 + 2y^2 - 2x + 6y - 3 = 0$ and $x^2 + y^2 + 4x + 2y + 1 = 0$ **[Roorkee, 1984]**
4. Find the condition such that the four points in which the circles $x^2 + y^2 + ax + by + c = 0$ and $x^2 + y^2 + a'x + b'y + c' = 0$ are intersected by the straight lines $Ax + By + C = 0$ and $A'x + B'y + C' = 0$, respectively lie on another circle. **[Roorkee, 1986]**
5. Obtain the equation of the straight lines passing through the point $A(2, 0)$ and making an angle of 45° with the tangent A to the circle $(x + 2)^2 + (y + 3)^2 = 25$.
   Find the equations of the circles each of radius 3 whose centres are on these straight lines at a distance of $5\sqrt{2}$ from A. **[Roorkee, 1987]**
6. A circle has radius 3 units and its centre lies on the line $y = x - 1$. Find the equation of this circle if it passes through $(7, 3)$. **[Roorkee, 1988]**
7. Find the equation of the circles passing through the point $(2, 8)$ touching the lines $4x - 3y - 24 = 0$ and $4x + 3y - 42 = 0$ and having x co-ordinate of the centre of the circle less than or equal to 8.
   [Roorkee, 1989]
8. The abscissa of two points A and B are the roots of the equation $x^2 + 2x - a^2 = 0$ and the ordinates are the roots of the equation $y^2 + 4y - b^2 = 0$. Find the equation of the circle with AB as its diameter. Also find the co-ordinates of the centre and the length of the radius of the circle. **[Roorkee, 1989]**

9. The point of contact of the tangent to the circle $x^2 + y^2 = 5$ at the point $(1, -2)$ which touches the circle $x^2 + y^2 - 8x + 6y + 20 = 0$, is
(a) $(2, -1)$ (b) $(3, -1)$
(c) $(4, -1)$ (d) $(5, -1)$
[Roorkee, 1989]

10. The centre of the circle passing through the point $(0, 1)$ and touching the curve $y = x^2$ at $(2, 4)$ is
(a) $(-16/5, 27/10)$ (b) $(-16/7, 53/10)$
(c) $(-16/5, 53/10)$ (d) none

11. The locus of the mid-points of the chords of the circle $x^2 + y^2 - 2x - 6y - 10 = 0$ which passes through the origin is the circle
(a) $x^2 + y^2 + x + 3y = 0$ (b) $x^2 + y^2 - x + 3y = 0$
(c) $x^2 + y^2 + x - 3y = 0$ (d) $x^2 + y^2 - x - 3y = 0$
[Roorkee, 1989]

12. The equation of the circle through the points of intersection of $x^2 + y^2 = 1$ and $x^2 + y^2 - 2x - 4y + 1 = 0$ and touches the line $x + 2y = 0$ is
(a) $x^2 + y^2 + x + 2y = 0$ (b) $x^2 + y^2 - x + 2y = 0$
(c) $x^2 + y^2 - x - 2y = 0$ (d) $2(x^2 + y^2) - x - 2y = 0$
[Roorkee, 1989]

13. $y - x + 3 = 0$ is the equation of the normal at $\left(3 + \frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$ to which of the following circles?
(a) $\left(x - 3 - \frac{3}{\sqrt{2}}\right)^2 + \left(y - \frac{3}{\sqrt{2}}\right)^2 = 9$
(b) $\left(x - 3 - \frac{3}{\sqrt{2}}\right)^2 + y^2 = 9$
(c) $(x - 3)^2 + y^2 = 6$
(d) $(x - 3)^2 + (y - 3)^2 = 9$ **[Roorkee, 1990]**

14. The radical axis of two circles whose centres lie along x and y axes is
(a) $ax - by - \left(\frac{a^2 + b^2}{4}\right) = 0$
(b) $2g^2x + 2f^2y - \left(\frac{f^2 + g^2}{4}\right) = 0$
(c) $gx + fy = g^4 + f^4$
(d) $2gx - 2fy + g^2 - f^2 = 0$ **[Roorkee, 1990]**

15. Find the equations of the circle having the lines $x + 2xy + 3x + 6y = 0$ as its normals and having size just sufficient to contain the circle $x(x - 4) + y(y - 3) = 0$.
[Roorkee Main, 1990]

16. A circle has its centre in the first quadrant and passes through the points of intersection of the lines $x = 2$ and $y = 3$. If it makes intercepts of 3 and 4 units on these lines respectively, its equation is
(a) $x^2 + y^2 + 3x - 5y + 8 = 0$
(b) $x^2 + y^2 - 4x - 6y + 13 = 0$
(c) $x^2 + y^2 - 6x - 8y + 23 = 0$
(d) $x^2 + y^2 - 8x - 9y + 30 = 0$ **[Roorkee, 1991]**

17. The radius of the circle passing through the points $(1, 2)$, $(5, 2)$ and $(5, -2)$ is
(a) $5\sqrt{2}$ (b) $2\sqrt{5}$ (c) $3\sqrt{2}$ (d) $2\sqrt{2}$
[Roorkee, 1991]

18. Find the equations of the circle passing through the points $A(4, 3)$ and $B(2, 5)$ and touching the axis of y. Also find the point P on the y-axis such that the angle APB has largest magnitude.
[Roorkee Main, 1991]

19. Find the radius of the smallest circle which touches the straight line $3x - y = 6$ at $(1, -3)$ and also touches the line $y = x$. Compute up to one place of decimal only.
[Roorkee Main, 1991]

20. Lines are drawn from the point $(-2, -3)$ to meet the circle $x^2 + y^2 - 2x - 10y + 1 = 0$. The length of the line that meets the circle at two coincident points is
(a) $4\sqrt{3}$ (b) 16
(c) 48
(d) Cannot be calculated unless coincident points are given **[Roorkee, 1992]**

21. The equation of the circle which touches both the axes and the straight line $4x + 3y = 6$ in the first quadrant and lies below it, is
(a) $4(x^2 + y^2) - 4x - 4y + 1 = 0$
(b) $x^2 + y^2 - 6x - 6y + 9 = 0$
(c) $49(x^2 + y^2) - 420(x + y) + 900 = 0$
(d) $x^2 + y^2 - x - 6 + 4 = 0$ **[Roorkee, 1992]**

22. Circles are drawn through the point $(2, 0)$ to cut intercepts of length 5 units on the x-axis. If their centres lie in the first quadrant, their equation is
(a) $x^2 + y^2 - 9x + 2ky + 14 = 0$
(b) $3x^2 + 3y^2 + 27x - 2ky + 42 = 0$
(c) $x^2 + y^2 - 9x - 2ky + 14 = 0$
(d) $x^2 + y^2 - 2kx - 9y + 14 = 0$
where k is a positive real number. **[Roorkee, 1992]**

23. From a point P, tangents are drawn to the circles $x^2 + y^2 + x - 3 = 0$, $3x^2 + 3y^2 - 5x + 3y = 0$ and $4x^2 + 4y^2 + 8x + 7y + 9 = 0$ are of equal lengths. Find the equation of the circle through P which touches the line $x + y = 5$ at the point $(6, -1)$. **[Roorkee Main, 1992]**

24. Find the equation of the system of co-axial circles that are the tangent at $(\sqrt{2}, 4)$ to the locus of the point of intersection of mutually perpendicular tangents to the conic $x^2 + y^2 = 9$. **[Roorkee Main, 1993]**

25. If the tangent to the circle $x^2 + y^2 = 5$ at $(1, -2)$ touches the circle $x^2 + y^2 - 8x + 6y + 20 = 0$ at the point
(a) $(2, -1)$ (b) $(3, -1)$
(c) $(4, -1)$ (d) $(5, -1)$
[Roorkee, 1994]

26. The centre of the circle passing through the point $(0, 1)$ and touching the curve $y = x^2$ at $(2, 4)$ is
(a) $(-16/5, 27/10)$ (b) $(-16/7, 53/10)$
(c) $(-16/5, 53/10)$ (d) none
[Roorkee, 1994]

27. Find the equation of the circle which touches the circle $x^2 + y^2 - 6x + 6y + 17 = 0$ externally and to which the lines $x^2 - 3xy - 3x + 9y = 0$ are normal.
[Roorkee Main, 1994]

28. If the lines $2x - 4y = 9$ and $6x - 12y + 7 = 0$ touch a circle, the radius of the circle is
(a) $\frac{\sqrt{3}}{5}$ (b) $\frac{17}{6\sqrt{5}}$ (c) $\frac{2\sqrt{5}}{3}$ (d) $\frac{17}{3\sqrt{5}}$
[Roorkee, 1995]

29. If the co-ordinates at one end of a diameter of the circle $x^2 + y^2 - 8x - 4y + c = 0$ are $(-3, 2)$, the co-ordinates of the other end are
(a) (5, 3) (b) (6, 2)
(c) (1, –8) (d) (11, 2)
[Roorkee, 1995]

30. If a circle is inscribed in an equilateral triangle of side a, the area of the square inscribed in the circle is
(a) $\frac{a^2}{6}$ (b) $\frac{a^2}{3}$ (c) $\frac{2a^2}{5}$ (d) $\frac{2a^2}{3}$
[Roorkee, 1995]

31. The equations of lines joining the origin to the point of intersection of circle $x^2 + y^2 = 3$ and the line $x + y = 2$ is
(a) $y - (3 + 2\sqrt{2})x = 0$ (b) $x - (3 + 2\sqrt{2})y = 0$
(c) $x - (3 - 2\sqrt{2})y = 0$ (d) $y - (3 - 2\sqrt{2})x = 0$
[Roorkee, 1995]

32. From a point on the line $4x - 3y = 6$, tangents are drawn to the circle $x^2 + y^2 - 6x - 4y + 4 = 0$ which make an angle of $\tan^{-1}\left(\frac{24}{7}\right)$ between them. Find the co-ordinates of all such points and the equations of tangents.
[Roorkee Main, 1995]

33. Two circles $x^2 + y^2 - 10x + 16 = 0$ and $x^2 + y^2 = r^2$ intersect each other at two distinct points if
(a) $r < 2$ (b) $r > 8$
(c) $2 < r < 8$ (d) $2 \le r \le 8$
[Roorkee, 1996]

34. A tangent is drawn from the point (4, 0) to the circle $x^2 + y^2 = 8$ touches at a point A in the first quadrant. Find the co-ordinates of another point B on the circle such that $AB = 4$. **[Roorkee Main, 1996]**

35. If the equations of the tangents drawn from the origin to the circle $x^2 + y^2 - 2rx - 2hy + h^2 = 0$ are perpendicular, then
(a) $h = r$ (b) $h^2 = r^2$
(c) $h = -r$ (d) $h^2 + r^2 = 1$
[Roorkee, 1997]

36. If a circle C and $x^2 + y^2 = 1$ are orthogonal and have radical axis parallel to y-axis, then C is
(a) $x^2 + y^2 - 1 + x = 0$ (b) $x^2 + y^2 - 1 - x = 0$
(c) $x^2 + y^2 + 1 - y = 0$ (d) $x^2 + y^2 + 1 + x = 0$
[Roorkee, 1997]

37. The equation of the line meeting the circle $x^2 + y^2 = a^2$, two points at equal distances d from a point (x_1, y_1) on the circumference is
(a) $xx_1 + yy_1 - a^2 + \frac{1}{2d^2} = 0$
(b) $xx_1 - yy_1 - a^2 + \frac{1}{2d^2} = 0$
(c) $xx_1 + yy_1 + a^2 - \frac{1}{2d^2} = 0$
(d) $xx_1 + yy_1 - a^2 + \frac{1}{2d^2} = 0$
[Roorkee, 1998]

38. The equations of the tangents drawn from the origin to the circle $x^2 + y^2 + 2rx - 2hy + h^2 = 0$ are
(a) $x = 0$ (b) $y = 0$
(c) $(h^2 - r^2)x - 2rhy = 0$ (d) $(h^2 - r^2)x + 2rhy = 0$
[Roorkee, 1998]

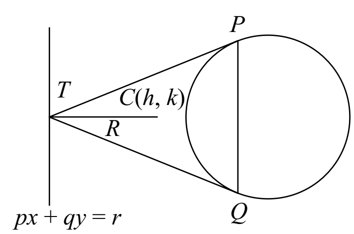
39. Find the equation of a circle which touches the line $x + y = 5$ at the point (–2, 7) and cuts the circle $x^2 + y^2 + 4x - 6y + 9 = 0$ orthogonally.
[Roorkee Main, 1998]

40. Extremities of a diagonal of a rectangle are (0, 0) and (4, 3). Find the equations of the tangents to the circumcircle of the rectangle which are parallel to this diagonal. **[Roorkee Main, 2000]**

41. Find the point on the straight line $y = 2x + 11$ which is nearest to the circle $16(x^2 + y^2) + 32x - 8y - 50 = 0$.
[Roorkee Main, 2000]

42. A circle of radius 2 units rolls on the outer side of the circle $x^2 + y^2 + 4x = 0$, touching it externally. Find the locus of the centre of this outer circle. Also find the equations of the common tangents of the two circles when the line joining the centres of the two circles makes an angle of 60° with x-axis.
[Roorkee Main, 2000]

43.



Tangents TP and TQ are drawn from a point T to the circle $x^2 + y^2 = a^2$. If the point T lies on the line $px + qy = r$, find the locus of the centre of the circumference of the triangle TPQ.
[Roorkee Main, 2001]

44. Find the equation of the circle which passes through the points of intersection of circles $x^2 + y^2 - 2x - 6y + 6 = 0$ and $x^2 + y^2 + 2x - 6y + 6 = 0$ and intersects the circle $x^2 + y^2 + 4x + 6y + 6 = 0$ orthogonally.
[Roorkee Main, 2001]

Integer Type Questions

1. Find the number of common tangents between the circles $x^2 + y^2 = 10$ and $x^2 + y^2 - 6x - 8y = 0$.
2. If the equations $x^2 + y^2 + 2x + 2ky + 6 = 0$ and $x^2 + y^2 + 2ky + k = 0$ intersect orthogonally, find the number of values of k.
3. If $\left(m_i, \dfrac{1}{m_i}\right), i = 1, 2, 3, 4$ are four distinct points on a circle, find the value of $(m_1 m_2 m_3 m_4 + 4)$.
4. If the circumference of the circle $x^2 + y^2 - 2x + 8y - q = 0$ is bisected by the circle $x^2 + y^2 + 4x + 22y + p = 0$, find the value of $\left(\dfrac{p+q}{10} + 2\right)$.
5. If the two circles $(x - 1)^2 + (y - 3)^2 = r^2$ and $x^2 + y^2 - 8x + 2y + 8 = 0$ intersect in two distinct points such that $n < r < m$ where $m, n \in N$, find the value of $(m - n)$.
6. If a straight line through $P(-2\sqrt{2}, 2\sqrt{2})$ making an angle of 135° with x-axis cuts the circle $x = 4\cos\theta$, $y = 4\sin\theta$ in points A and B respectively, find the length of the segment AB.
7. If a circle passes through the point of intersection of the co-ordinate axes with the lines $\lambda x - y + 1 = 0$ and $x - 2y + 3 = 0$, find the integral value of λ.
8. Find the radius of the circumcircle of the triangle formed by the lines $y + \sqrt{3}x = 6$, $y - \sqrt{3}x = 6$ and $y = 0$.
9. If the circle $x^2 + y^2 - 4x - 6y + \lambda = 0$ touches the axis of x, find the value of λ.
10. If the straight lines $y = m_1 x + c_1$ and $y = m_2 x + c_2$ meet the co-ordinate axes in concyclic points, find the value of $(m_1 m_2 + 4)$.

Comprehensive Link Passage

Passage 1

If $7l^2 - 9m^2 + 8l + 1 = 0$ and we have to find the equation of circle having $lx + my + 1 = 0$ is a tangent and we can adjust the given condition as

$$16l^2 + 8l + 1 = 9(l^2 + m^2) \text{ or } (4l + 1)^2 = 9(l^2 + m^2)$$

$$\Rightarrow \quad \left|\frac{(4l+1)}{\sqrt{(l^2 + m^2)}}\right| = 3$$

Thus centre of a circle = (4, 0) and radius = 3.

Also when two non-parallel lines touching a circle, the centre of circle lies on angle bisector of lines.

1. If $16m^2 - 8l - 1 = 0$, the equation of a circle having $lx + my + 1 = 0$ is a tangent is
   (a) $x^2 + y^2 + 8x = 0$ (b) $x^2 + y^2 - 8x = 0$
   (c) $x^2 + y^2 + 8y = 0$ (d) $x^2 + y^2 - 8y = 0$
2. If $4l^2 - 5m^2 + 6l + 1 = 0$, the centre and the radius of the circle, which have $lx + my + 1 = 0$ as a tangent, is
   (a) $(0, 4); \sqrt{5}$ (b) $(4, 0); \sqrt{5}$
   (c) $(0, 3); \sqrt{5}$ (d) $(3, 0); \sqrt{5}$
3. If $16l^2 + 9m^2 = 24lm + 6l + 8m + 1$ and if S be the equation of the circle having $lx + my + 1 = 0$ as a tangent, when the equation of director circle of S is
   (a) $x^2 + y^2 + 6x + 8y = 25$
   (b) $x^2 + y^2 - 6x + 8y = 25$
   (c) $x^2 + y^2 - 6x - 8y = 25$
   (d) $x^2 + y^2 + 6x - 8y = 25$

Passage II

A circle C of radius 1 is inscribed in an equilateral triangle PQR. The points of contact of C with the sides PQ, QR, RP are D, E, F respectively. The line PQ is given by the equation $\sqrt{3}x + y = 6$ and the point D is $\left(\dfrac{3\sqrt{3}}{2}, \dfrac{3}{2}\right)$. Further it is given that the origin and the centre C are on the same side PQ.

1. The equation of the circle C is
   (a) $(x - 2\sqrt{3})^2 + (y - 1)^2 = 1$
   (b) $(x - 2\sqrt{3})^2 + \left(y - \dfrac{1}{2}\right)^2 = 1$
   (c) $(x - \sqrt{3})^2 + (y + 1)^2 = 1$
   (d) $(x - \sqrt{3})^2 + (y - 1)^2 = 1$
2. Points E and F are given by
   (a) $\left(\dfrac{\sqrt{3}}{2}, \dfrac{3}{2}\right), (\sqrt{3}, 0)$ (b) $\left(\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right), (\sqrt{3}, 0)$
   (c) $\left(\dfrac{\sqrt{3}}{2}, \dfrac{3}{2}\right), \left(\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right)$ (d) $\left(\dfrac{3}{2}, \dfrac{\sqrt{3}}{2}\right), \left(\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right)$
3. The equation of the sides QR, PR are
   (a) $y = \dfrac{2}{\sqrt{3}}x + 1, y = -2\sqrt{3}x + 1$
   (b) $y = \dfrac{1}{\sqrt{3}}x + 1; y = 0$
   (c) $y = \left(\dfrac{\sqrt{3}}{2}\right)x + 1; y = \left(-\dfrac{\sqrt{3}}{2}\right)x - 1$
   (d) $y = \sqrt{3}x, y = 0$

Passage III

The equation of the tangent and the normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at (x_1, x_2) are

tangent : $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

Normal: $\dfrac{x - x_1}{x_1 + g} = \dfrac{y - y_1}{y_1 + f}$.

Clearly, the normal always passes through the centre of the circle.

1. The equation of the tangent to the circle $x^2 + y^2 + 4x + 6y - 12 = 0$ at (1, 1) is
   (a) $3x + 4y = 7$ (b) $3x - 4y = 7$
   (c) $-3x + 4y = 7$ (d) $-3x - 4y = 7$
2. The tangent to the circle $x^2 + y^2 = 5$ at (1, –2) also touches the circle
   (a) $x^2 + y^2 - 8x + 6y - 20 = 0$
   (b) $x^2 + y^2 + 8x + 6y - 20 = 0$

(c) $x^2 + y^2 - 8x + 6y + 20 = 0$
(d) $x^2 + y^2 - 8x + 9y + 20 = 0$

3. The area of the triangle formed by the positive x-axis, the normal and the tangent to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$ is
(a) $3\sqrt{3}$ s.u. (b) $2\sqrt{3}$ s.u.
(c) $4\sqrt{3}$ s.u. (d) $5\sqrt{3}$ s.u.

4. The extremities of a diagonal of a rectangle are (–4, 4) and (6, –1). A circle circumscribes the rectangle and cuts an intercept AB on the y-axis. The area of the triangle formed by AB and the tangents to the circle at A and B is
(a) $\frac{363}{8}$ s.u. (b) $\frac{365}{8}$ s.u.
(c) $\frac{363}{4}$ s.u. (d) $\frac{365}{4}$ s.u.

5. The equation of the normal to the circle $x^2 + y^2 - 5x + 2y - 48 = 0$ at the point (5, 6) is
(a) $14x + 5y = 10$ (b) $14x - 5y = 40$
(c) $5x - 14y = 40$ (d) none

6. If the normal to the circle $x^2 + y^2 - 6x + 4y - 50 = 0$ is parallel to the line $3x + 4y + 5 = 0$, the equation of the normal is
(a) $3x + 4y + 1 = 0$ (b) $3x + 4y = 2$
(c) $3x - 4y = 1$ (d) $3x + 4y + 2 = 0$

Passage IV

Let $P(x_1, y_1)$ be a point lying inside the circle $S : x^2 + y^2 + 2gx + 2fy + c = 0$. If the tangents from P to the circle $S = 0$ at A and B, then AB is called the chord of contact.

The equation of the chord of contact is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

Let the tangents PA and PB are drawn from $P(0, -2)$ to the circle $x^2 + y^2 + 2x - 4y = 0$.

1. The equation of the chord of contact is
(a) $x - 4y + 4 = 0$ (b) $x - 3y - 3 = 0$
(c) $x + 4y + 4 = 0$ (d) $x + 5y + 6 = 0$.

2. The length of the chord AB is
(a) $2\sqrt{\frac{60}{17}}$ (b) $2\sqrt{\frac{65}{17}}$ (c) $2\sqrt{\frac{12}{17}}$ (d) $2\sqrt{\frac{37}{17}}$

3. The area of a triangle PAB is
(a) $12\sqrt{\frac{5}{17}}$ (b) $12\sqrt{\frac{7}{17}}$ (c) $12\sqrt{\frac{11}{17}}$ (d) $6\sqrt{\frac{5}{17}}$

4. The area of a quadrilateral $PACB$, where C is the centre of the circle, is
(a) $2\sqrt{15}$ (b) $3\sqrt{15}$ (c) $4\sqrt{15}$ (d) $5\sqrt{15}$

5. The angle between PA and PB is
(a) $\tan^{-1}\left(\frac{4\sqrt{15}}{7}\right)$ (b) $\tan^{-1}\left(\frac{8\sqrt{15}}{7}\right)$
(c) $\tan^{-1}\left(\frac{16\sqrt{15}}{7}\right)$ (d) $\tan^{-1}\left(\frac{6\sqrt{15}}{7}\right)$

6. The chord of contact of the tangents drawn from a point on the circle $x^2 + y^2 = a^2$ to the circle $x^2 + y^2 = b^2$ touches the circle $x^2 + y^2 = c^2$. Then a, b, c are in
(a) AP (b) GP (c) HP (d) AGP

Passage V

The line $y = mx + c$ will be a tangent to the circle $x^2 + y^2 = a^2$ if $c = \pm a\sqrt{1 + m^2}$.

The equation of any tangent to the circle $x^2 + y^2 = a^2$ can be considered as $y = mx + a\sqrt{1 + m^2}$ and the co-ordinates of the points of contact are $\left(\pm\frac{am}{\sqrt{1 + m^2}}, \mp\frac{a}{\sqrt{1 + m^2}}\right)$ and the length of the tangent from the point $P(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$.

1. The equation of tangent to the circle $x^2 + y^2 + 4x + 2y = 0$ from the point $P(1, -2)$ is
(a) $x - 2y - 5 = 0$ (b) $x + 2y + 5 = 0$
(c) $x - 2y = 0$ (d) $y - 3x = 0$

2. The equation of the tangents from the origin to the circle $x^2 + y^2 - 2x - 4y = 0$ is
(a) $3x - 4y = 0$ (b) $4x - 3y = 0$
(c) $3x + 4y = 0$ (d) $4x + 3y = 0$.

3. The length of the tangent from any point on the circle $x^2 + y^2 - 2009x - 2010y + 2012 = 0$ to the circle $x^2 + y^2 - 2009x - 2010y + 2020 = 0$ is
(a) $2\sqrt{2}$ (b) $3\sqrt{2}$ (c) $5\sqrt{2}$ (d) $\sqrt{2}$

4. If the angle between a pair of tangents from a point P to the circle $x^2 + y^2 + 4x - 6y + 13\cos^2\alpha + 9\sin^2\alpha = 0$ is 2α. Then the equation of the locus of P is
(a) $(x + 2)^2 + (y - 3)^2 = 1$
(b) $(x - 2)^2 + (y - 3)^2 = 1$
(c) $(x - 2)^2 + (y + 3)^2 = 1$
(d) $(x + 2)^2 + (y + 3)^2 = 1$

5. Tangents PA and PB are drawn from $P(-1, 2)$ to the circle $x^2 + y^2 - 2x - 4y + 2 = 0$. The area of a triangle PAB is
(a) $\frac{\sqrt{3}}{4}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{2\sqrt{3}}{5}$ (d) $\frac{4\sqrt{3}}{5}$

Passage VI

The equation of the chord of the circle $x^2 + y^2 = a^2$ bisected at the point (x_1, y_1) is $T = S_1$, i.e. $xx_1 + yy_1 - a^2 = x_1^2 + y_1^2 - a^2$.

1. The equation of the chord of the circle $x^2 + y^2 - 6x + 10y - 9 = 0$ bisected at the point (–2, 4) is
(a) $5x - 9y + 40 = 0$ (b) $5x - 9y + 46 = 0$
(c) $3x - 4y + 46 = 0$ (d) $4x - 5y + 46 = 0$.

2. The locus of the mid-point of a chord of the circle $x^2 + y^2 = 4$, which subtends a right angle at the origin is
(a) $x + y = 1$ (b) $x + y = 2$
(c) $x^2 + y^2 = 2$ (d) $x^2 + y^2 = 1$

3. The equation of the locus of the mid-points of the chords of the circle $4x^2 + 4y^2 - 12x + 4y + 1 = 0$ that subtends an angle of $\frac{2\pi}{3}$ at its centre is
   (a) $16(x^2 + y^2) - 48x + 16y + 31 = 0$
   (b) $16(x^2 + y^2) - 16x + 48y + 31 = 0$
   (c) $16(x^2 + y^2) - 16x - 48y + 31 = 0$
   (d) $16(x^2 + y^2) - 16x - 48y - 31 = 0$
4. If two distinct chords, drawn from the point (p, q) on the circle $x^2 + y^2 = px + qy$ (where $pq \neq 0$) are bisected by the x-axis, then
   (a) $p^2 = q^2$ (b) $p^2 = 8q^2$
   (c) $p^2 > 8q^2$ (d) $p^2 < 8q^2$
5. Let a circle be given by $2x(x - a) + 2y(y - b) = 0$, $(a, b \neq 0)$. If two chords are bisected by the x-axis, can be drawn to the circle from the point $\left(a, \frac{b}{2}\right)$
   (a) $a^2 > b^2$ (b) $a^2 > 2b^2$
   (c) $a^2 < 2b^2$ (d) $a^2 < b^2$
6. The locus of the mid-points of the chords of the circle $x^2 + y^2 = a^2$, which subtend right angle at the point $(c, 0)$ is
   (a) $2(x^2 + y^2) - 2cx = a^2 - c^2$
   (b) $2(x^2 + y^2) - 2cx = a^2 + c^2$
   (c) $2(x^2 + y^2) + 2cx = a^2 + c^2$
   (d) $2(x^2 + y^2) + 2cx = a^2 - c^2$

Passage VII

The equation of the family of circles passing through the point of intersection of two given circles $S_1 = 0$ and $S_2 = 0$ is given by $S_1 + \lambda S_2 = 0$, $\lambda \neq -1$, where λ is a parameter.

The equation of the family of circles passing through the point of intersection of circle $S = 0$ and a line $L = 0$ is given by $S + \lambda L = 0$, where λ is a parameter. The equation of the family of circles touching the circle $S = 0$ and the line $L = 0$ at their point of contact P is $S + \lambda L = 0$, where λ is a parameter.

1. The equation of the circle passing through (1, 1) and the points of intersection of the circles $x^2 + y^2 + 13x - 3y = 0$ and $2(x^2 + y^2) + 4x - 7y - 25 = 0$ is
   (a) $4(x^2 + y^2) + 30x - 13y = 25$
   (b) $4(x^2 + y^2) - 30x - 13y = 25$
   (c) $4(x^2 + y^2) - 30x + 13y = 25$
   (d) $4(x^2 + y^2) + 30x + 13y = 25$
2. The equation of the circle passing through the point of intersection of the circles $x^2 + y^2 - 6x + 2y + 4 = 0$, $x^2 + y^2 + 2x - 4y - 6 = 0$ and with its centre on the line $y = x$ is
   (a) $7(x^2 + y^2) - 10(x + y) = 12$
   (b) $7(x^2 + y^2) - 10(x - y) = 12$
   (c) $7(x^2 + y^2) + 10(x - y) = 12$
   (d) $7(x^2 + y^2) + 10(x + y) = 12$
3. The equation of the circle through the points of intersection of the circle $x^2 + y^2 - 2x - 4y + 4 = 0$ and the line $x + 2y = 4$ which touches the line $x + 2y = 0$ is
   (a) $x^2 + y^2 - x - 2y = 0$ (b) $x^2 + y^2 + x - 2y = 0$
   (c) $x^2 + y^2 + x + 2y = 0$ (d) $x^2 + y^2 - x + 2y = 0$
4. The equation of the circle whose diameter is the common chord of the circles $x^2 + y^2 + 2x + 3y + 1 = 0$ and $x^2 + y^2 + 4x + 3y + 2 = 0$ is
   (a) $2(x^2 + y^2) + 2x + 6y + 1 = 0$
   (b) $2(x^2 + y^2) + 3x + 6y + 1 = 0$
   (c) $2(x^2 + y^2) + 2x + 5y + 1 = 0$
   (d) $2(x^2 + y^2) + x + 6y + 1 = 0$

Matrix Match
(For JEE-Advanced Examination Only)

1. Match the following columns:

Column I		Column II	
(A)	If the shortest and the longest distance from the point (10, 7) to the circle $x^2 + y^2 - 4x - 2y - 2 = 0$ are L and M respectively, then	(P)	$L + M = 10$
(B)	If the shortest and the longest distance from the point (3, –6) to the circle $x^2 + y^2 - 16x - 12y - 125 = 0$ are L and M respectively, then	(Q)	$L + M = 20$
(C)	If the shortest and the longest distance from the point (6, –6) to the circle $x^2 + y^2 - 4x + 6y - 12 = 0$ are L and M respectively, then	(R)	$L + M = 30$
		(S)	$M - L = 10$
		(T)	$M - L = 26$

2. Match the following columns:

Column I		Column II	
(A)	The radius of the circle $x^2 + y^2 - 2x - 2y = 0$ is	(P)	2
(B)	The radius of the circle $x^2 + y^2 - 4x - 4y + 4 = 0$ is	(Q)	4
(C)	The radius of the circle $x^2 + y^2 - 6x - 10y + 30 = 0$ is	(R)	$\sqrt{2}$
(D)	The radius of the circle $2(x^2 + y^2) - 4x - 6y + 16 = 0$ is	(S)	3
(E)	The radius of the circle $x^2 + y^2 - 10x = 0$ is	(T)	5

3. Match the following columns:

Column I		Column II	
(A)	The point $(\lambda, \lambda + 2)$ lies inside the circle $x^2 + y^2 = 4$, the value of λ can be	(P)	–1
(B)	The point $(\lambda, \lambda + 2)$ lies outside the circle $x^2 + y^2 - 2x - 4y = 0$, the value of λ can be	(Q)	–1/2
(C)	If both the equations $x^2 + y^2 + 2\lambda\lambda + 4 = 0$ and $x^2 + y^2 - 4\lambda\lambda + 8 = 0$ represent real circles, the value of λ can be	(R)	3/2
		(S)	3
		(T)	5

4. Match the following columns:

Column I		Column II	
(A)	If the straight lines $y = a_1x + b$ and $y = a_2x + b$, $(a_1 \neq a_2)$ and $b \in R$ meet the co-ordinate axes in concyclic points, then	(P)	$a_1^2 + a_2^2 = 4$
(B)	If the chord of contact of the tangents drawn from any point on $x^2 + y^2 = a_1^2$ to $x^2 + y^2 = b^2$ touches the circle $x^2 + y^2 = a_2^2$, where $(a_1 \neq a_2)$, then	(Q)	$a_1 + a_2 = 3$
		(R)	$a_1a_2 = b$
		(S)	$a_1a_2 = 1$
(C)	If the circles $x^2 + y^2 + 2a_1x + b = 0$ and $x^2 + y^2 + 2a_2x + b = 0$ where $a_1 \neq a_2$ and $b \in R$ cuts orthogonally, then	(T)	$a_1a_2 = b^2$

5. Match the following columns:

Column I		Column II	
(A)	The lines $3x - 4y + 4 = 0$ and $6x - 8y - 7 = 0$ are the tangents to a circle, its radius is	(P)	2
(B)	The radius of the circle inscribed in the triangle formed by the lines $x = 0$, $y = 0$ and $4x + 3y = 24$ is	(Q)	3/4
(C)	The radius of the circle $3x(x - 2) + 3y(y + 1) = 4$ is	(R)	7
(D)	The lines $2x - 3y = 5$ and $3x - 4y = 7$ are the diameters of a circle of area 154 s.u., its radius is	(S)	$\sqrt{\frac{31}{12}}$

6. Match the following Columns:

Column I		Column II	
(A)	The length of the tangents from any point on the circle $x^2 + y^2 + 4x + 6y + 2008 = 0$ to the circle $x^2 + y^2 + 4x + 6y + 2012 = 0$ is	(P)	$3/\sqrt{2}$
		(Q)	2
(B)	The lengths of the common chord of the circles $x^2 + y^2 + 2x + 3y + 1 = 0$ and $x^2 + y^2 + 3x + 2y + 1 = 0$ is	(R)	1
(C)	The number of tangents which can be drawn from the point (2, 3) to the circle $x^2 + y^2 = 13$ is	(S)	5/3
(D)	The radius of the circle $ax^2 + (2a - 3)y^2 - 4x - 7 = 0$ is	(T)	4/3

7. Match the following columns:

Column I		Column II	
(A)	Number of common tangents to the circles $x^2 + y^2 - 2x = 0$ and $x^2 + y^2 + 6x - 6y + 2 = 0$ is	(P)	1
(B)	Number of common tangents to the circles $x^2 + y^2 - 4x - 10y + 4 = 0$ and $x^2 + y^2 - 6x - 12y - 55 = 0$ is	(Q)	2
(C)	Number of common tangents to the circles $x^2 + y^2 - 2x - 4y = 0$ and $x^2 + y^2 - 8y - 4 = 0$ is	(R)	3
(D)	Number of common tangents to the circles $x^2 + y^2 + 2x - 8y + 13 = 0$ and $x^2 + y^2 - 6x - 2y + 6 = 0$ is	(S)	0

8. Match the following columns:

Column I		Column II	
(A)	If $\left(a, \frac{1}{a}\right), \left(b, \frac{1}{b}\right), \left(c, \frac{1}{c}\right)$ and $\left(d, \frac{1}{d}\right)$ are four distinct points on a circle of radius 2012 units, the value of $abcd$ is	(P)	2
(B)	If a circle passes through the point of intersection of axes with the lines $\lambda x - y + 1 = 0$ and $x - 2y + 3 = 0$, then the value of λ is	(Q)	1
		(R)	–5
(C)	If the curves $ax^2 + 4xy + 2y^2 + x + y + 5 = 0$ and $ax^2 + 6xy + 5y^2 + 2x + 3y + 8 = 0$ intersect at four concyclic points, the value of a is	(S)	–4
		(T)	3

9. Match the following columns:

Column I		Column II	
(A)	If one of the diameters of the circle $x^2 + y^2 - 2x - 6y + 6 = 0$ is a chord to the circle with centre (2, 1), the radius of the circle is	(P)	7
		(Q)	9
(B)	If the tangent at the point P on the circle $x^2 + y^2 + 6x + 6y = 2$ meets the straight line $5x - 2y + 6 = 0$ at a point Q on the y-axis, the length of PQ is	(R)	5
		(S)	2
(C)	If the angle between the tangents from a point P to the circle $x^2 + y^2 + 4x - 6y + 4\cos^2\alpha + 9 = 0$ is 2α, the radius of the locus of P is	(T)	3

10. Match the following columns:

Column I		Column II	
(A)	The locus of the point of intersection of two perpendicular tangents to a circle $x^2 + y^2 = 10$ is	(P)	$x^2 + y^2 - 10x - 25 = 0$
(B)	The equation of the director circle of the circle $x^2 + y^2 - 10x = 0$ is	(Q)	$x^2 + y^2 - 6y + 1 = 0$
(C)	The equation of the director circle of the circle $x^2 + y^2 - 6y + 5 = 0$ is	(R)	$x^2 + y^2 = 20$

Questions asked in Previous Years' JEE-Advanced Examinations

1. Find the equation of the circle which passes through the point (2, 0) and whose centre is the limit of the point of intersection of the lines $3x + 5y = 1$, $(2 + c)x + 5c^2y = 1$ as c tends to 1. **[IIT-JEE, 1979]**
2. Two circles $x^2 + y^2 = 6$ and $x^2 + y^2 - 6x + 8 = 0$ are given. The equation of the circle through their points of intersection and the point (1, 1) is
   (a) $x^2 + y^2 - 6x + 4 = 0$ (b) $x^2 + y^2 - 3x + 1 = 0$
   (c) $x^2 + y^2 - 4x + 2 = 0$ (d) none **[IIT-JEE, 1980]**
3. Let A be the centre of the circle $x^2 + y^2 - 2x - 4y - 20 = 0$. Suppose that the tangents at the points $B(1, 7)$ and $D(4, -2)$ on the circle meet at the point C, find the area of the quadrilateral $ABCD$. **[IIT-JEE, 1981]**
4. Find the equations of the circles passing through (–4, 3) and touching the lines $x + y = 4$ and $x - y = 2$. **[IIT-JEE, 1982]**
5. If A and B are points in the plane such that $\frac{PA}{PB} = k$ (constant) for all P on a given circle, the value of k cannot be equal to.... **[IIT-JEE, 1982]**
6. The points of intersection of the line $4x - 3y - 10 = 0$ and the circle $x^2 + y^2 - 2x + 4y - 20 = 0$, is ... **[IIT-JEE, 1983]**
7. The equation of the circle passing through (1, 1) and the points of intersection of $x^2 + y^2 + 13x - 3y = 0$, $2x^2 + 2y^2 + 4x - 7y - 25 = 0$ is
   (a) $4x^2 + 4y^2 - 30x - 10y - 25 = 0$
   (b) $4x^2 + 4y^2 + 30x - 13y - 25 = 0$
   (c) $4x^2 + 4y^2 - 17x - 10y - 25 = 0$
   (d) none **[IIT-JEE, 1983]**
8. The centre of the circle passing through the point (0, 1) and touching the curve $y = x^2$ at (2, 4) is
   (a) $\left(-\frac{16}{5}, \frac{27}{10}\right)$ (b) $\left(-\frac{16}{7}, \frac{53}{10}\right)$
   (c) $\left(-\frac{16}{5}, \frac{53}{10}\right)$ (d) none **[IIT-JEE, 1983]**
9. AB is a diameter of a circle and C is any point on the circumference of the circle. Then
   (a) The area of triangle ABC is maximum when it is isosceles.
   (b) The area of triangle ABC is minimum when it is isosceles.
   (c) The area of triangle ABC is minimum when it is isosceles.
   (d) None **[IIT-JEE, 1983]**
10. Through a fixed point (h, k) secants are drawn to the circle $x^2 + y^2 = r^2$. Show that the locus of the mid-points of the secants intercepted by the circle is $x^2 + y^2 = hx + ky$ **[IIT-JEE, 1983]**
11. The locus of the mid-point of a chord of the circle $x^2 + y^2 = 4$, which subtends a right angle at the origin is
   (a) $x + y = 2$ (b) $x^2 + y^2 = 1$
   (c) $x^2 + y^2 = 2$ (d) $x + y = 1$ **[IIT-JEE, 1984]**
12. The abscissa of the two points A and B are the roots of the equation $x^2 + 2ax - b^2 = 0$ and their ordinates are the roots of the equation $y^2 + 2py - q^2 = 0$. Find the equation of the circle on AB as diameter. **[IIT-JEE, 1984]**
13. The lines $3x - 4y + 4 = 0$ and $6x - 8y - 7 = 0$ are tangents to the same circle. The radius of the circle is… **[IIT-JEE, 1984]**
14. From the origin chords are drawn to the circle $(x - 1)^2 + y^2 = 1$. The equation of the locus of the mid-points of these chords is… **[IIT-JEE, 1984]**
15. Let $x^2 + y^2 - 4x - 2y - 11 = 0$ be a circle. A pair of tangents from (4, 5) with a pair of radii form a quadrilateral of area… **[IIT-JEE, 1985]**
16. The equation of the line passing through the points of intersection of the circles $3x^2 + 3y^2 - 2x + 12y - 9 = 0$ and $x^2 + y^2 + 6x + 2y - 15 = 0$ is… **[IIT-JEE, 1986]**
17. From the point $A(0, 3)$ on the circle $x^2 + 4x + (y - 3)^2 = 0$. A chord AB is drawn and extended to a point M such that $AM = 2AB$. The equation of the locus of M is… **[IIT-JEE, 1986]**
18. Lines $5x + 12y - 10 = 0$ and $5x - 12y - 40 = 0$ touch a circle C_1 of diameter 6. If the centre of C_1 lies in the first quadrant, find the equation of the circle C_2 which is concentric with C_1 and cuts intercepts of length 8 on these lines. **[IIT-JEE, 1986]**
19. Let a given line L_1 intersects the x and y axes at P and Q, respectively. Let another line L_2 perpendicularly cuts the x and y axes at R and S, respectively. Show that the locus of the point of intersection of the lines PS and QR is a circle passing through the origin. **[IIT-JEE, 1987]**
20. The circle $x^2 + y^2 - 4x - 4y + 4 = 0$ is inscribed in a triangle which has two of its sides along the co-ordinate axes. The locus of the circumcentre of the triangle is $x + y - xy + k(x^2 + y^2) = 0$, find the value of k. **[IIT-JEE, 1987]**

21. The area of the triangle formed by tangents from the points (4, 3) to the circle $x^2 + y^2 = 9$ and the line joining their points of contact is… **[IIT-JEE, 1987]**
22. A polygon of nine sides, each of length 2, is inscribed in a circle. The radius of the circle is… **[IIT-JEE, 1987]**
23. If the circle $C_1 : x^2 + y^2 = 16$ intersects another circle C_2 of radius 5 in such a manner that the common chord is of maximum length and has a slope equal to 3/4, the co-ordinates of the centre C_2 are… **[IIT-JEE, 1988]**
24. If a circle passes through the point (a, b) and cuts the circle $x^2 + y^2 = k^2$ orthogonally, the equation of the locus of its centre is
    (a) $2ax + 2by - (a^2 + b^2 + k^2) = 0$
    (b) $2ax + 2by - (a^2 - b^2 - k^2) = 0$
    (c) $x^2 + y^2 - 3ax - 4by + (a^2 + b^2 - k^2) = 0$
    (d) $x^2 + y^2 - 2ax - 3by + (a^2 + b^2 - k^2) = 0$
    [IIT-JEE, 1988]
25. The equation of the tangents drawn from the origin to the circle $x^2 + y^2 + 2rx - 2hy + h^2 = 0$ are
    (a) $x = 0$ (b) $y = 0$
    (c) $(h^2 - r^2)x - 2rhy = 0$ (d) $(h^2 - r^2)x + 2rhy = 0$
    [IIT-JEE, 1988]
26. Let $S = x^2 + y^2 + 2gx + 2fy + c = 0$ be a given circle. Find the locus of the foot of the perpendicular drawn from the origin upon any chord of S which subtends a right angle at the origin. **[IIT-JEE, 1988]**
27. If the two circles $(x - 1)^2 + (y - 3)^2$ and $x^2 + y^2 - 8x + 2y + 8 = 0$ intersect in two distinct points, then
    (a) $2 < r < 8$ (b) $r < 2$
    (c) $r = 2$ (d) $r > 2$
    [IIT-JEE, 1989]
28. The lines $2x - 3y = 5$ and $3x - 4y = 7$ are diameters of a circle of area 154 sq. units. Then the equation of the circle is
    (a) $x^2 + y^2 + 2x - 2y - 62 = 0$
    (b) $x^2 + y^2 + 2x - 2y - 47 = 0$
    (c) $x^2 + y^2 - 2x + 2y - 47 = 0$
    (d) $x^2 + y^2 - 2x + 2y - 62 = 0$ **[IIT-JEE, 1989]**
29. If $\left(m_i, \dfrac{1}{m_i}\right)$ $i = 1, 2, 3, 4$ are four distinct points on a circle, show that $m_1m_2m_3m_4 = 1$
    [IIT-JEE, 1989]
30. The line $x + 3y = 0$ is a diameter of the circle $x^2 + y^2 - 6x + 2y = 0$. Is it true/false?
    [IIT-JEE, 1989]
31. The area of the triangle formed by the positive x-axis and the normal and the tangent to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$ is… **[IIT-JEE, 1989]**
32. A circle touches the line $y = x$ at a point P such that $OP = 4\sqrt{2}$, where O is the origin. The circle contains the point (–10, 2) in its interior and the length of its chord on the line $x + y = 0$ is $6\sqrt{2}$. Determine the equation of the circle. **[IIT-JEE, 1990]**
33. A point P is given on the circumference of a circle of radius r, a chord QR is parallel to the tangent at P. Determine the maximum possible area of the triangle PQR. **[IIT-JEE, 1990]**
34. Two circles each of radius 5 units, touch each other at (1, 2). If the equation of their common tangent is $4x + 3y = 10$. Find the equations of the circles.
    [IIT-JEE, 1991]
35. If a circle passes through the points of intersection of the co-ordinate axes with the lines $\lambda x - y + 1 = 0$ and $x - 2y + 3 = 0$, the value of λ is…
    [IIT-JEE, 1991]
36. Three circles, each of radius 5 units, touch each other externally. The tangent at their points of contact meet at a point whose distance from a point of contact is 4. Find the ratio of the product of the radii to the sum of the radii of the circles. **[IIT-JEE, 1992]**
37. Let a circle be given by $2x(x - a) + y(2y - b) = 0$, $(a, b \neq 0)$, find the condition on a and b if two chords, each bisected by the x-axis, can be drawn to the circle from $\left(a, \dfrac{b}{2}\right)$. **[IIT-JEE, 1992]**
38. The centre of a circle passing through the points (0, 0), (1, 0) and touching the circle $x^2 + y^2 = 9$ is
    (a) (3/2, ½) (b) (1/2, 3/2)
    (c) (1/2, ½) (d) $\left(\dfrac{1}{2}, \sqrt{2}\right)$
    [IIT-JEE, 1992]
39. The locus of the centre of a circle which touches externally the circle $x^2 + y^2 - 6x - 6y + 14 = 0$ and also touches the y-axis is given by the equation
    (a) $x^2 - 6x - 10y + 14 = 0$
    (b) $x^2 - 10x - 6y + 14 = 0$
    (c) $y^2 - 6x - 10y + 14 = 0$
    (d) $y^2 - 10x + 6y + 14 = 0$ **[IIT-JEE, 1993]**
40. Consider a family of circles passing through two fixed points $A(3, 7)$ and $B(6, 5)$. Show that the chords in which the circle $x^2 + y^2 - 4x - 6y - 3 = 0$ cuts the members of the family are concurrent at a point. Find the co-ordinates of this point. **[IIT-JEE, 1993]**
41. Find the co-ordinates of the point at which the circle $x^2 + y^2 - 4x - 2y + 4 = 0$ and $x^2 + y^2 - 12x - 8y + 36 = 0$ touch each other. Also find equations of common tangents touching the circles in distinct points.
    [IIT-JEE, 1993]
42. The equation of the locus of the mid-points of chords of the circle $4x^2 + 4y^2 - 12x + 4y + 1 = 0$ that subtend an angle of $\dfrac{2\pi}{3}$ at its centre is… **[IIT-JEE, 1993]**
43. The circles $x^2 + y^2 - 10x + 16 = 0$ and $x^2 + y^2 = r^2$ intersect each other in distinct points if
    (a) $r < 2$ (b) $r > 8$
    (c) $2 < r < 8$ (d) $2 \leq r \leq 8$
    [IIT-JEE, 1994]

44. A circle is inscribed in an equilateral triangle of side a. The area of any square inscribed in this circle is… **[IIT-JEE, 1994]**
45. The angle between a pair of tangents drawn from a point P to the circle $x^2 + y^2 + 4x - 6y + 9\sin^2\alpha + 13\cos^2\alpha = 0$ is 2α. Then the locus of P is
 (a) $x^2 + y^2 + 4x - 6y + 4 = 0$
 (b) $x^2 + y^2 + 4x - 6y - 9 = 0$
 (c) $x^2 + y^2 + 4x - 6y - 4 = 0$
 (d) $x^2 + y^2 + 4x - 6y + 9 = 0$
 [IIT-JEE, 1996]
46. A circle passes through three points A, B and C with the line segment AC as its diameter. A line passing through A intersects the chord BC at a point D inside the circle. If angles DAB and CAB are α and β respectively and the distance between the point A and the mid-point of the line segment DC is d. Prove that the area of the circle is $\dfrac{\pi d \times \cos^2\alpha}{\cos^2\alpha + \cos^2\beta + 2\cos\alpha\cos\beta\cos(\beta - \alpha)}$
 [IIT-JEE, 1996]
47. Find the interval in which a lies for which the line $y + x = 0$ bisects two chords drawn from the point $\left(\dfrac{1+a\sqrt{2}}{2}, \dfrac{1-a\sqrt{2}}{2}\right)$ to the circle $2(x^2 + y^2) - (1 + a\sqrt{2})x - (1 - a\sqrt{2})y = 0$. **[IIT-JEE, 1996]**
48. Intercepts on the line $y = x$ by the circle $x^2 + y^2 - 2x = 0$ is AB. Equation of the circle with AB as diameters is… **[IIT-JEE, 1996]**
49. Let C be any circle with the centre $(0, \sqrt{2})$. Prove that at the most two rational points can be there on C. (A rational point is a point for which both the co-ordinates are rational numbers.) **[IIT-JEE, 1997]**
50. Consider a curve $ax^2 + 2hxy + by^2 = 1$ and a point P on the curve. A line drawn from the point P intersects the curve at point Q and R. If the product $PQ \cdot PR$ is independent of the slope of the line, show that the curve is a circle. **[IIT-JEE, 1997]**
51. For each natural number k, let C_k denotes the circle with radius k centimetres and centre at the origin. On the circle C_k, a particle moves k centimetres in the counter-clockwise direction. After completing its motion on C_k, the particle moves to C_{k+1} in the radial direction. The particle starts at (1, 0). If the particle crosses the positive direction of the x-axis for the first time on the circle C_n, then $n = \ldots$ **[IIT-JEE, 1997]**
52. The chords of contact of the pair of tangents drawn from each point on the line $2x + y = 4$ to the circle $x^2 + y^2 = 1$ pass through the point… **[IIT-JEE, 1997]**
53. Two vertices of an equilateral triangle are (–1, 0) and (1, 0) and its third vertex lies above the x-axis, the equation of its circumcircle is… **[IIT-JEE, 1997]**
54. The number of common tangents to the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 6x - 8y - 24 = 0$ is
 (a) 0 (b) 1 (c) 3 (d) 4
 [IIT-JEE, 1998]
55. C_1 and C_2 are two concentric circles, the radius of C_2 being twice that of C_1. From a point P on C_2, tangents PA and PB are drawn to C_1. Prove that the centroid of the triangle PAB lies on C_1. **[IIT-JEE, 1998]**
56. Let $A_0A_1A_2A_3A_4A_5$ be a regular hexagon inscribed in a unit circle. Then the product of the lengths of line segments A_0A_1, A_0A_2, A_0A_4 is
 (a) 3/4 (b) $3\sqrt{3}$ (c) 3 (d) $3\sqrt{3}/2$
 [IIT-JEE, 1998]
57. If two distinct chords, drawn from the point (p, q) on the circle $x^2 + y^2 = px + qy$, where $pq \neq 0$ are bisected by the x-axis, then
 (a) $p^2 = q^2$ (b) $p^2 = 8q^2$
 (c) $p^2 < 8q^2$ (d) $p^2 > 8q^2$
 [IIT-JEE, 1999]
58. Let T_1 and T_2 be two tangents drawn from (–2, 0) on the circle C: $x^2 + y^2 = 1$. Determine the circles touching C and having T_1, T_2 as their pairs of tangents. Further find the equations of all possible common tangents to these circles, taken two at a time. **[IIT-JEE, 1999]**
59. The triangle PQR is inscribed in the circle $x^2 + y^2 = 25$. If Q and R have co-ordinates (3, 4) and (–4, 3) respectively, then $\angle QPR$ is … **[IIT-JEE, 2000]**
 (a) $\pi/2$ (b) $\pi/3$
 (c) $\pi/4$ (d) $\pi/6$
60. If the circles $x^2 + y^2 + 2x + 2ky + 6 = 0$ and $x^2 + y^2 + 2ky + k = 0$ intersect orthogonally, then k is
 (a) 2 or –3/2 (b) –2 or –3/2
 (c) 2 or 3/2 (d) –2 or 3/2
 [IIT-JEE, 2000]
61. Let AB be a chord of the circle $x^2 + y^2 = r^2$ subtending right angle at the centre, the locus of the centroid of the triangle PAB as P moves on the circle is
 (a) a parabola (b) a circle
 (c) an ellipse (d) a pair of straight lines
 [IIT-JEE, 2001]
62. Let $2x^2 + y^2 - 3xy = 0$ be the equation of pair of tangents drawn from the origin O to a circle of radius 3 with centre is in the first quadrant. If A is one of the points of contact, find the length of OA. **[IIT-JEE, 2001]**
63. Let PQ and RS be tangents at the extremities of the diameter PR of a circle of radius r. If PS and RQ intersect a point x on the circumference of the circle, then $2r$ equals
 (a) $\sqrt{PQ \cdot RS}$ (b) $\dfrac{PQ + RS}{2}$
 (c) $\dfrac{2PQ \cdot RS}{PQ + RS}$ (d) $\sqrt{\dfrac{PQ^2 + RS^2}{2}}$
 [IIT-JEE, 2001]

64. Let C_1 and C_2 be two circles with C_2 lying inside C_1. A circle C lying inside C_1 touches C_1 internally and C_2 externally. Identify the locus of the centre of C. **[IIT-JEE, 2001]**
65. If the tangent at the point P on the circle $x^2 + y^2 + 6x + 6y = 2$ meets the straight line $5x - 2y + 6 = 0$ at a point Q on the y-axis, the length of PQ is
    (a) 4 (b) $2\sqrt{5}$ (c) 5 (d) $3\sqrt{5}$
    [IIT-JEE, 2002]
66. If $a > 2b > 0$, the positive value of m for which $y = mx - b\sqrt{1+m^2}$ is a common tangent to $x^2 + y^2 = b^2$ and $(x - a)^2 + y^2 = b^2$ is
    (a) $\dfrac{2b}{\sqrt{a^2-4b^2}}$ (b) $\dfrac{\sqrt{a^2-4b^2}}{2a}$
    (c) $\dfrac{2b}{a-2b}$ (d) $\dfrac{b}{a-2b}$
    [IIT-JEE, 2002]
67. Tangents are drawn from $P(6, 8)$ to the circle $x^2 + y^2 = r^2$. Find the radius of the circle such that the area of the triangle formed by the tangents and the chord of contact is maximum. **[IIT-JEE, 2003]**
68. If I_n represents area of n-sided regular polygon inscribed in a unit circle and O_n be the area of the n-sided regular polygon circumscribing it, prove that $I_n = \dfrac{O_n}{2}\left[1+\sqrt{1-\left(\dfrac{2I_n}{n}\right)^2}\right]$. **[IIT-JEE, 2003]**
69. The centre of the circle inscribed in a square formed by the lines $x^2 - 8x + 12 = 0$ and $y^2 - 14y + 45 = 0$ is
    (a) (4, 7) (b) (7, 4)
    (c) (9, 4) (d) (4, 9)
    [IIT-JEE, 2004]
70. If one of the diameters of the circle $x^2 + y^2 - 2x - 6y + 6 = 0$ is a chord to the circle with centre (2, 1), the radius of the circle is
    (a) $\sqrt{3}$ (b) $\sqrt{2}$ (c) 3 (d) 2
    [IIT-JEE, 2004]
71. Find the centre and the radius of the circle formed by all the points represented by $z = x + iy$ satisfying the relation $\dfrac{|z-\alpha|}{|z-\beta|} = k(k \neq 1)$, where α and β are constant complex numbers given by $\alpha = \alpha_1 + i\alpha_2$, and $\beta = \beta_1 + i\beta_2$. **[IIT-JEE, 2004]**
72. $|z-1| = \sqrt{2}$ is a circle inscribed in a square whose one vertex is $2 + i\sqrt{3}$. Find the remaining vertices. **[IIT-JEE, 2005]**
73. Find the equation of the circle touches the line $2x + 3y + 1 = 0$ at the point $(1, -1)$ and is orthogonal to the circle which has the line segment having end-points $(0, -1)$ and $(-2, 3)$ as the diameter. **[IIT-JEE, 2004]**
74. Three circles of radii 3, 4 and 5 units touches each other externally and the tangents drawn at the point of contact intersect at P. Find the distance between point P and the point of contact. **[IIT-JEE, 2005]**
75. A circle is given by $x^2 + (y - 1)^2 = 1$, another circle C touches it externally and also the x-axis, the locus of its centre is
    (a) $\{(x, y) : x^2 = 4y\} \cup \{(x, y) : y \leq 0\}$
    (b) $\{(x, y) : x^2 + (y - 1)^2 = 4\} \cup \{(x, y) : y \leq 0\}$
    (c) $\{(x, y) : x^2 = y\} \cup \{(0, y) : y < 0\}$
    (d) $\{(x, y) : x^2 = y\} \cup \{(0, y) : y \leq 0\}$
    [IIT-JEE, 2005]
76. The tangent to the curve $y = x^2 + 6$ at a point $P(1, 7)$ touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at a point Q. Then the co-ordinates of Q are
    (a) (–6, –7) (b) (–10, –15)
    (c) (–9, –13) (d) (–6, –11)
    [IIT-JEE, 2005]

Comprehension Link Passage

Let $ABCD$ be a square of side 2 units. C_2 is the circle through vertices A, B, C, D and C_1 in the circle touching all the sides of the square $ABCD$, L is a line through A.

77. If P is a point on C_1 and Q in another point on C_2, then $\dfrac{PA^2+PB^2+PC^2+PD^2}{QA^2+QB^2+QC^2+QD^2}$ is equal to
    (a) 0.75 (b) 1.25 (c) 1 (d) 0.5
78. A circle touches the line L and the circle C_1 externally such that both the circles are on the same side of the line, the locus of the centre of the circle is
    (a) ellipse (b) hyperbola
    (c) parabola (d) pairs of straight lines
79. A line M through A is drawn parallel to BD. Point S moves such that its distances from the line BD and the vertex A are equal. If locus of S cuts M at T_2 and T_3 and AC at T_1, the area of $\Delta T_1T_2T_3$ is
    (a) 1/2 sq. unit (b) 2/3 sq. units
    (c) 1 sq. unit (d) 2 sq. units
    [IIT-JEE, 2006]
80. Let $ABCD$ be a quadrilateral with area 18, with side AB parallel to the side CD and $AB = 2CD$. Let AD be perpendicular to AB and CD. If a circle is drawn inside the quadrilateral $ABCD$ touching all the sides, its radius is **[IIT-JEE, 2007]**
    (a) 3 (b) 2 (c) 3/2 (d) 1.
81. Tangents are drawn from the point (17, 7) to the circle $x^2 + y^2 = 169$.
    Statement 1: The tangents are mutually perpendicular.
    Statement 2: The locus of the point from which mutually perpendicular tangents can be drawn to the given circle is $x^2 + y^2 = 338$. **[IIT-JEE, 2007]**
82. Consider L_1: $2x + 3y + (p - 3) = 0$
    L_2: $2x + 3y + (p + 3) = 0$,
    where p is a real number and
    C: $x^2 + y^2 + 6x - 10y + 30 = 0$

Statement 1: If L_1 is a chord of circle C, then L_2 is not always the diameter of the circle C.
Statement 2: If L_1 is a diameter of circle C, the line L_2 is not a chord of circle C. **[IIT-JEE, 2008]**

Comprehension

A circle C of radius 1 is inscribed in an equilateral triangle PQR. The points of contact of C with the side PQ, QR, RP are D, E, F, respectively. The line PQ is given by the equation $\sqrt{3}x + y = 6$ and the point D is $(3\sqrt{3/2}, 3/2)$. Further it is given that the origin and the centre C are on the same side PQ.

83. The equation of the circle C is
    (a) $(x-2\sqrt{3})^2+(y-1)^2=1$
    (b) $(x-2\sqrt{3})^2+(y-1/2)^2=1$
    (c) $(x-\sqrt{3})^2+(y+1)^2=1$
    (d) $(x-\sqrt{3})^2+(y-1)^2=1$ **[IIT-JEE, 2008]**
84. Points E and F are given by
    (a) $(\sqrt{3}/2, 3/2), (\sqrt{3}, 0)$
    (b) $(\sqrt{3}/2, 1/2), (\sqrt{3}, 0)$
    (c) $(\sqrt{3}/2, 3/2), (\sqrt{3}/2, 1/2)$
    (d) $(3/2, \sqrt{3}/2), (\sqrt{3}/2, 1/2)$ **[IIT-JEE, 2008]**
85. Equations of the sides QR, PR are
    (a) $y=(2/\sqrt{3})x+1,\ y=-(2/\sqrt{3})x-1$
    (b) $y=(1/\sqrt{3})x+1,\ y=0$
    (c) $y=(\sqrt{3}/2)x+1,\ y=(-\sqrt{3}/2)x-1$
    (d) $y=\sqrt{3}x,\ y=0$ **[IIT-JEE-2008]**
86. Tangents are drawn from a point $P(1, 8)$ to the circle $x^2+y^2-6x-4y-11=0$ touch the circle at the points A and B. The equation of the circumcircle of the triangle PAB is
    (a) $x^2+y^2-6x-4y-19=0$
    (b) $x^2+y^2-6x-10y-19=0$
    (c) $x^2+y^2-2x+6y-29=0$
    (d) $x^2+y^2-6x-4y+19=0$ **[IIT-JEE, 2009]**
87. The locus of the point (h, k) for which the line $hx + ky = 1$ touches the circle $x^2+y^2=4$ is.... **[IIT-JEE, 2009]**
88. Two parallel chords of a circle of the radius 2 are at a distance $\sqrt{3}+1$ apart. If the chords subtend, at the centre angles of $\frac{\pi}{k}$ and $\frac{2\pi}{k}$, where $k > 0$, the value of $[k]$ is…, where [,] = GIF **[IIT-JEE, 2010]**
89. The straight line $2x-3y=1$ divides the circular region $x^2+y^2 \le 6$ into two parts. If
    $$S=\left\{\left(2,\frac{3}{4}\right),\left(\frac{5}{2},\frac{3}{4}\right),\left(\frac{1}{4},-\frac{1}{4}\right),\left(\frac{1}{8},\frac{1}{4}\right)\right\},$$
    the number of point(s) in S lying inside the smaller part is… **[IIT-JEE, 2011]**
90. The locus of the mid-point of the chord of contact of tangents drawn from points lying on the straight line $4x-5y=20$ to the circle $x^2+y^2=9$ is
    (a) $20(x^2+y^2)-36x+45y=0$
    (b) $20(x^2+y^2)+36x-45y=0$
    (c) $20(x^2+y^2)-20x+45y=0$
    (d) $20(x^2+y^2)+20x-45y=0$ **[IIT-JEE, 2012]**
91. A tangent PT is drawn to the circle $x^2+y^2=4$ at the point $P(\sqrt{3}, 1)$. A straight line L, perpendicular to PT is a tangent to the circle $(x-3)^2+y^2=1$
    (i) A possible equation of L is
    (a) $x-\sqrt{3}y=1$ (b) $x+\sqrt{3}y=1$
    (c) $x-\sqrt{3}y=-1$ (d) $x+\sqrt{3}y=5$
    (ii) A common tangent to the two circles is
    (a) $x=4$ (b) $y=2$
    (c) $x+\sqrt{3}y=4$ (d) $x+2\sqrt{2}y=6$
    [IIT-JEE, 2012]

No questions asked in 2013.

92. A circle S passes through the point (0, 1) and is orthogonal to the circles $(x-1)^2+y^2=16$ and $x^2+y^2=1$. Then the
    (a) radius of S is 8 (b) radius of S is 7
    (c) centre of S is (–7, 1) (d) centre of S is (–8, 1)
    [IIT-JEE, 2014]
93. The common tangents to the circle $x^2+y^2=2$ and the parabola $y^2=8x$ touch the circle at the points P, Q and the parabola at the points R, S. Then the area of the quadrilateral $PQRS$ is
    (a) 3 (b) 6 (c) 9 (d) 15
    [IIT-JEE, 2014]
94. Let RS be the diameter of the circle $x^2+y^2=2$, where S is the point (1, 0).
    Let P be a variable point (other than R and S) on the circle and tangents to the circle at S and P meet at the point Q. The normal to the circle at P intersects a line drawn through Q parallel to RS at point E. Then the locus of E passes through the point(s)
    (a) $\left(\frac{1}{3},\frac{1}{\sqrt{3}}\right)$ (b) $\left(\frac{1}{4},\frac{1}{2}\right)$
    (c) $\left(\frac{1}{3},-\frac{1}{\sqrt{3}}\right)$ (d) $\left(\frac{1}{4},-\frac{1}{2}\right)$
    [IIT-JEE-2016]

ANSWERS

LEVEL I

1. (i) centre is (0, 0) and radius is 4
(ii) centre is (4, 0) and radius is 1
(iii) centre is $\left(\frac{1}{2}, \frac{1}{2}\right)$ and radius is $\frac{1}{\sqrt{2}}$
3. $x^2 + y^2 - 8x + 6y = 27$
4. $x^2 + y^2 + 4x - 2y = 0$
5. $\left(x - \frac{5}{7}\right)^2 + \left(y + \frac{10}{7}\right)^2 = 16$
6. $x^2 + y^2 + 4y = 23$
7. $\frac{3\sqrt{3}}{4} \times (g^2 + f^2 - c)$
8. $x^2 + y^2 - 8x - 6y + 16 = 0$
and $x^2 + y^2 - 14x - 12y + 76 = 0$.
9. $\left(\frac{36}{13}, \frac{15}{13}\right)$
10. $x^2 + y^2 - 8x - 12y + 39 = 0$
11. $2(x^2 + y^2) - 13x - 33y - 135 = 0$
12. $x^2 + y^2 - 5x - 5y + 10 = 0$
13. $2\sqrt{17}$
14. 1
15. the centre is (a, a) and the radius is a.
16. the point (1, 2) lies inside of the circle and (6, 0) lies outside of the circle.
17. (–1, 4)
18. 2, 28
19. (–1, 1)
20. $3x + 2y \pm 2\sqrt{13} = 0$
21. $3x - 4y + 15 = 0$ and $3x - 4y - 15 = 0$
22. $y = \sqrt{3}x + (2\sqrt{3} \pm 2)$
23. $y = \pm x \pm 4$
24. $4x - 3y = 25$ and $3x + 4y = 25$
25. $x^2 + y^2 - 2\sqrt{2}x - 2\sqrt{2}y - 5 = 0$
26. $2x + \sqrt{5}y = 9$ and $2x - \sqrt{5}y = 9$
27. $y = 4$
28. $\left(-2, -\frac{11}{10}\right)$ and $\left(4, \frac{13}{10}\right)$
29. $x + 2y - 1 = 0$ and $2x - y + 1 = 0$
30. $x = 2$
31. $(4\sqrt{2} - 3)$
32. 3
33. $\sqrt{a^2 - b^2}$
34. 1
35. 13
36. 1
37. $y = 2$, $4x - 3y + 2 = 0$
38. $y = 3 \pm 2\sqrt{2}(x - 2)$
39. $2\tan^{-1}\left(\frac{5}{3}\right)$
40. $2\tan^{-1}\left(\frac{3}{2}\right)$
41. $x^2 + y^2 = 50$
42. $x^2 + y^2 = 18$
43. 90°
44. (i) $x^2 + y^2 + 2x - 1 = 0$
(ii) $x^2 + y^2 + 10y + 23 = 0$
(iii) $x^2 + y^2 + 16x + 12y + 98 = 0$
(iv) $x^2 + y^2 + 2gx + 2fy - g^2 - f^2 + 2c = 0$
(v) $x^2 + y^2 - ax - by - \frac{1}{4}(a^2 + b^2) = 0$
45. $y = \frac{\sqrt{5}}{2}x$ and $y = -\frac{\sqrt{5}}{2}x$
46. $3x + y + 5 = 0$
47. $x - y + 1 = 0$
48. $3x + 2y = 24$
49. $5x + 3y = 25$
50. (1, –2)
51. $a^2(x^2 + y^2) = (hx + ky)^2$
52. $a\left(\frac{(h^2 + k^2 - a^2)^{3/2}}{(h^2 + k^2)}\right)$
53. $m = 2$, $n = 1$ and $p = 1$
54. $2x - 3y + 13 = 0$
55. $4x + 3y = 17$
56. $x^2 + y^2 = hx + ky$
57. $x^2 + y^2 = 2$
58. $(-1, \sqrt{3})$
59. $\left(\sqrt{2} + \frac{\sqrt{3}}{2}, \frac{1}{2} - \sqrt{3}\right)$
60. $2012x + 2013y = 0$
61. $5x - 4y + 2 = 0$
62. 4
63. $2(x^2 + y^2) + 2x + 6y + 1 = 0$
64. (i) 1 (ii) 4 (iii) 4 (iv) 3 (v) 0
65. D.C.T: $y = 4$, $4x - 3y = 0$
T.C.T: $x = 0$ and $3x + 4y = 10$
66. $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$
67. 4
68. 90°
70. $k = 4$
71. $x^2 + y^2 - \frac{18}{5}x - \frac{18}{5}y = 0$
72. $9x - 10y + 11 = 0$
73. $2ax + 2by - (a^2 + b^2 + 4) = 0$

74. $\dfrac{2r_1r_2}{\sqrt{r_1^2+r_2^2}}$
75. $x - 2y - 1 = 0$
76. (3, 2)
77. $\left(2, -\dfrac{5}{2}\right)$
78. $x^2 + y^2 - 2x - y - 8 = 0$
79. $(4x^2 + 4y^2 + 30x - 13y - 25) = 0$
80. $(x^2 + y^2 + 2x + 6y + 1) = 0$
81. $x^2 + y^2 - x - y - 8 = 0$
82. $x^2 + y^2 - \dfrac{10}{7}x - \dfrac{10}{7}y - \dfrac{12}{7} = 0$
83. $2(x^2 + y^2) + 2x + 6y + 1 = 0.$

Level II

1. (b)	2. (c)	3. (b)	4. (b)	5. (a)
6. (b)	7. (a)	8. (b)	9. (a)	10. (c)
11. (a)	12. (a)	13. (a)	14. (b)	15. (b)
16. (a)	17. (a)	18. (b)	19. (c)	20. (c)
21. (c)	22. (b)	23. (a)	24. (a)	25. (a)
26. (d)	27. (a)	28. (b)	29. (d)	30. (b)
31. (a)	32. (c)	33. (d)	34. (b)	35. (b)
36. (a)	37. (a)	38. (d)	39. (c)	40. (c)
41. (a)	42. (b)	43. (a)	44. (b)	45. (b)
46. (d)	47. (d)	48. (b)	49. (c)	50. (d)
51. (c)	52. (a)	53. (b)	54. (a)	55. (b)
56. (a)	57. (c)	58. (a)	59. (c)	60. (c)
61. (b)	62. (c)	63. (c)	64. (c)	65. (d)
66. (b)	67. (c)	68. (d)	69. (c)	70. (c)
71. (d)	72. (a)	73. (d)	74. (a)	75. (b)
76. (d)	77. (b)	78. (c)	79. (b)	80. (d)
81. (a)	82. (b)	83. (c)	84. (c)	85. (b, d)
86. (a)	87. (a, c)	88. (d)	89. (b)	90. (c)
91. (a)	92. (c)	93. (a)	94. (a, c)	95. (a)
96. (c)	97. (d)	98. (b)	99. (c)	100. (d)
101. (b)	102. (c)	103. (d)	104. (c)	105. (a)
106. (b)	107. (a)	108. (c)	109. (a)	110. (a)
111. (c)	112. (d)	113. (a)	114. (b)	115. (b)
116. (c)	117. (d)			

Level III

1. 45
3. $a(\sqrt{2} - 1), 2\sqrt{2}a$
4. (3, 3); (7, 3)
5. 4
6. (3/25, 4/25)
7. $a = b \pm c\sqrt{2}$
8. $(x + 2)^2 + (y - 3)^2 = 6.25$
9. $x^2 + y^2 - 2x - 4y + 4 = 0$
10. $\dfrac{(g^2 + f^2 - c)}{2\sqrt{g^2 + f^2}}$
11. 16
12. (2/13, 3/13)
13. $\sqrt{d - c}$
14. –6
15. $(x - 17)^2 + (y - 16)^2 = 1$
19. $x^2 + y^2 - 4y - 12 = 0$
25. (9, –3)
26. 3
29. 0

Level IV

1. (i) $24y = \pm 5(2x + 13)$
   (ii) $\left\{\dfrac{169}{8}\sin^{-1}\left(\dfrac{120}{169}\right) - 15\right\}$ sq. m.
2.
3. $4x^2 + 4y^2 + 6x + 10y - 1 = 0$
4. $(a - a')(BC' - CB') + (b - b')(CA' - AC') + (c - c')(AB' - BA') = 0$
5. $(x - 1)^2 + (y - 7)^2 = 9$
   or $(x - 7)^2 + (y - 1)^2 = 9$
6. $x^2 + y^2 - 8x - 6y + 16 = 0$
   and $x^2 + y^2 - 14x - 12y + 76 = 0$
7. $x^2 + y^2 - 2x - 6y - 12 = 0$
8. $\sqrt{a^2 + b^2 + 5}$
9. (3, –1)
10. (c)
11. (d)
12. (c)
13. (c)
14. (d)
15. $x^2 + y^2 + 6x - 3y - 45 = 0$
16. $x^2 + y^2 - 8x - 9y + 30 = 0$
17. $2\sqrt{2}$
18. (0, 3), $x^2 + y^2 - 4x - 6y + 4 = 0$
19. $B = (3 - 2\sqrt{5}, 3 - 2\sqrt{5})$
20. (a)
21. (c)
22. $49(x^2 + y^2) - 420(x + y) + 900 = 0$
22. (c)
23. $x^2 + y^2 - 7x + 7y + 12 = 0$
24. $x^2 + y^2 - 4\sqrt{2}x - 42 = 0$
25. (b)
26. (c)
27. $x^2 + y^2 - 6x - 2y + 1 = 0$
28. (b)
29. (d)
30. $\dfrac{a^2}{6}$
31. (d)
32. (0, –2), (6, 6)

33. (c)
34. $B = (2, -2)$ or $(-2, 2)$
35. (a)
36. (a, d)
37.
38. (d)
39. $x^2 + y^2 + 7x - 11y + 38 = 0$
40. $8y - 6x \pm 25 = 0$
41. $\left(-\frac{9}{2}, 2\right)$
42. $y - (\sqrt{3} - 1) = \sqrt{3}(x - (\sqrt{3} - 1))$
    and $y - (\sqrt{3} - 1) = \sqrt{3}(x + (\sqrt{3} + 1))$
43. $x^2 + y^2 + \left(\frac{10}{3}\right)x - 6y + 6 = 0$

INTEGER TYPE QUESTIONS

1. 1	2. 2	3. 5	4. 7	5. 6
6. 8	7. 2	8. 4	9. 4	10. 5

COMPREHENSIVE LINK PASSAGE

Passage I: 1. (b) 2. (a) 3. (b)
Passage II: 1. (d) 2. (a)
Passage III: 1. (a) 2. (c) 3. (b) 4. (a) 5. (b) 6. (a)
Passage IV: 1. (a) 2. (a) 3. (a) 4. (a) 5. (a) 6. (b)
Passage V: 1. (a) 2. (a) 3. (a) 4. (a) 5. (a)
Passage VI 1. (b) 2. (c) 3. (a) 4. (c) 5. (b) 6. (a)
Passage VII: 1. (a) 2. (a) 3. (a) 4. (a)

MATRIX MATCH

1. (A) $\to$ (Q); (B) $\to$ (T); (C) $\to$ (P, S)
2. (A) $\to$ (R); (B) $\to$ (P); (C) $\to$ (P); (D) $\to$ (R); (E) $\to$ (T)
3. (A) $\to$ (P, Q); (B) $\to$ (S, T); (C) $\to$ (R)
4. (A) $\to$ (S); (B) $\to$ (T); (C) $\to$ (R);
5. (A) $\to$ (Q); (B) $\to$ (P); (C) $\to$ (S); (D) $\to$ (R)
6. (A) $\to$ (Q); (B) $\to$ (P); (C) $\to$ (R); (D) $\to$ (R)
7. (A) $\to$ (R); (B) $\to$ (S); (C) $\to$ (P); (D) $\to$ (Q)
8. (A) $\to$ (Q); (B) $\to$ (P); (C) $\to$ (S)
9. (A) $\to$ (T); (B) $\to$ (R); (C) $\to$ (S)
10. (A) $\to$ (R); (B) $\to$ (P); (C) $\to$ (Q)

HINTS AND SOLUTIONS

LEVEL I

1. (i) The given equation of circle is $x^2 + y^2 = 16$
   Hence, the centre is (0, 0) and the radius = 4.
   (ii) The given equation of a circle is
   $$x^2 + y^2 - 8x + 15 = 0$$
   $$\Rightarrow (x - 4)^2 + y^2 = 16 - 15 = 1$$
   Hence, the centre is (4, 0) and the radius = 1.
   (iii) The given equation of a circle is
   $$x^2 + y^2 - x - y = 0$$
   $$\Rightarrow \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} = \left(\frac{1}{\sqrt{2}}\right)^2$$
   the centre is $\left(\frac{1}{2}, \frac{1}{2}\right)$ and the radius is $\frac{1}{\sqrt{2}}$.
2. The equations of given circles are
   $$x^2 + y^2 = 1 \quad \text{...(i)}$$
   $$x^2 + y^2 - 2x - 6y = 6 \quad \text{...(ii)}$$
   and
   $$x^2 + y^2 - 4x - 12y = 9 \quad \text{...(iii)}$$
   Let r_1, r_2 and r_3 are the radii of the circles (i), (ii) and (iii).
   Then $r_1 = 1$,
   $$r_2 = \sqrt{g^2 + f^2 - c} = \sqrt{1 + 9 + 6} = 4$$
   and $r_3 = \sqrt{4 + 36 + 9} = 7$
   Therefore, $r_1 + r_3 = 7 + 1 = 8 = 2(4) = 2r_2$.
   Thus, r_1, r_2, r_3 are in AP.
3. Since the circle is concentric, so the centre of the circle is the same as
   $$x^2 + y^2 - 8x + 6y - 5 = 0$$
   Let CP is the radius.
   Then $CP = \sqrt{(-4 - 2)^2 + (-7 + 3)^2}$
   $$= \sqrt{36 + 16} = \sqrt{52} = 2\sqrt{13}$$
   Hence, the equation of the circle is
   $$(x - 4)^2 + (y + 3)^2 = (2\sqrt{13})^2$$
   $$\Rightarrow x^2 + y^2 - 8x + 6y = 27$$
4. The point of intersection of $x + 3y = 0$ and $2x - 7y = 0$ is $P(0, 0)$ and the point of intersection of $x + y + 1 = 0$ and
   $$x - 2y + 4 = 0 \text{ is } C(-2, 1).$$
   Thus, CP is the radius, i.e. $CP = \sqrt{4 + 1} = \sqrt{5}$
   Hence, the equation of the circle is
   $$(x + 2)^2 + (y - 1)^2 = (\sqrt{5})^2$$
   $$\Rightarrow x^2 + y^2 + 4x - 2y = 0$$
5. Clearly, the radius of the circle is
   $$= \frac{1}{2}\left|\frac{30 - (-10)}{\sqrt{4^2 + 3^2}}\right| = \frac{1}{2} \times \frac{40}{5} = 4$$

Let (h, k) be the centre of the circle.

Thus $2h + k = 0$

$\Rightarrow \quad k = -2h$

Also, $\left|\frac{8h - 3k - 30}{\sqrt{4^2 + 3^2}}\right| = 4$

$\Rightarrow \quad \left|\frac{8h - 3(-2h) - 30}{5}\right| = 4$

$\Rightarrow \quad \left|\frac{14h - 30}{5}\right| = 4$

$\Rightarrow \quad 14h - 30 = \pm 20$

$\Rightarrow \quad 14h = 30 \pm 20 = 50, 10$

$\Rightarrow \quad h = \frac{25}{7}, \frac{5}{7}$

So, $\quad k = -\frac{50}{7}, -\frac{10}{7}$

Hence, the equation of the circle is

$$\left(x - \frac{25}{7}\right)^2 + \left(y + \frac{50}{7}\right)^2 = 16$$

or $$\left(x - \frac{5}{7}\right)^2 + \left(y + \frac{10}{7}\right)^2 = 16$$

6. The centre of the circle

$$x^2 + y^2 - 16x - 24y + 183 = 0 \text{ is } C(8, 12).$$

Let the centre of the new circle be $C'(h, k)$.

Now, $\frac{h - 8}{-4} = \frac{k - 12}{7} = -\frac{2(-32 + 84 + 13)}{16 + 49}$

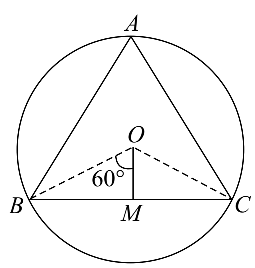
$\Rightarrow \quad h = 0, k = -2$

Hence, the equation of the new circle is

$$x^2 + (y + 2)^2 = (3\sqrt{3})^2$$

$\Rightarrow \quad x^2 + Y^2 + 4Y = 23$

7. Let ABC be an equilateral triangle such that OM is perpendicular on BC.



Here, $OB = \sqrt{g^2 + f^2 - c}$

Clearly, $\angle BOM = 60°$.

Now, $\sin\left(\frac{\pi}{3}\right) = \frac{BM}{OB}$

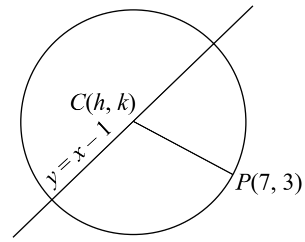
$\Rightarrow \quad BM = \frac{\sqrt{3}}{2} OB = \frac{\sqrt{3}}{2} \times \sqrt{g^2 + f^2 - c}$

Thus, $ar\ (\Delta ABC) = \frac{\sqrt{3}}{4} \times (BC)^2$

$$= \frac{\sqrt{3}}{4} \times (2BM)^2$$

$$= \frac{3\sqrt{3}}{4} \times (g^2 + f^2 - c)$$

8. Let the centre of the circle be (h, k) such that $k = h - 1$.



We have,

$(h - 7)^2 + (k - 3)^2 = 3^2$

$\Rightarrow \quad (h - 7)^2 + (h - 4)^2 = 3^2$

$\Rightarrow \quad h^2 - 11h + 28 = 0$

$h = 7, 4$ and $k = 6, 3$

Hence, the equation of a circle can be

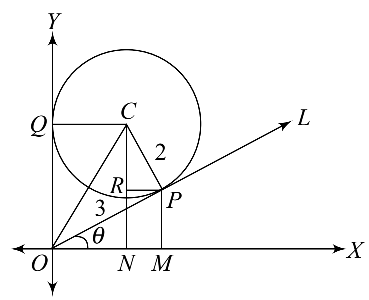
$(x - 7)^2 + (y - 6)^2 = 3^2$

& $(x - 4) + (y - 3)^2 = 3^2$

$\Rightarrow \quad x^2 + y^2 - 8x - 6y + 16 = 0$

and $\quad x^2 + y^2 - 14x - 12y + 76 = 0$

9.



Let $\quad \angle POX = \theta$,

then $\quad \angle NCP = \theta$

Here, $CP = 2$, $OC = \sqrt{2^2 + 3^2} = \sqrt{13}$

$$OP = \sqrt{OC^2 - OP^2} = \sqrt{13 - 4} = \sqrt{9} = 3$$

$C = (2, 3)$ and $ON = 2$

Now, $OM = ON + MN = ON + PR$

$\Rightarrow \quad OP \cos\theta = 2 + 2\sin\theta$

$\Rightarrow \quad 3\cos\theta = 2 + 2\sin\theta$

$\Rightarrow \quad 9\,\text{ocs}^2\theta = 4 + 4\sin^2\theta + 8\sin\theta$

$\Rightarrow \quad 9 - 9\sin^2\theta = 4 + 4\sin^2\theta + 8\sin\theta$

$\Rightarrow \quad 13\sin^2\theta + 8\sin\theta - 5 = 0$

$\Rightarrow \quad 13\sin^2\theta + 13\sin\theta - 5\sin\theta - 5 = \theta$

$\Rightarrow \quad 13\sin\theta(\sin\theta + 1) - 5(\sin\theta + 1) = 0$

$\Rightarrow \quad (\sin\theta + 1)(13\sin\theta - 5) = 0$

$\Rightarrow \quad \sin\theta = \frac{5}{13}$

$\Rightarrow \quad \cos\theta = \frac{12}{13}$

Therefore,

$$P = (OP\cos\theta, OP\sin\theta) = \left(\frac{36}{13}, \frac{15}{13}\right)$$

10. Hence, the equation of the circle is
$$(x-2)(x-6)+(y-3)(y-9)=0$$
$\Rightarrow \quad x^2+y^2-8x-12y+39=0$

11. Hence, the equation of the circle be
$$x^2+y^2+2gx+2fy+c=0 \qquad \ldots\text{(i)}$$
which is passing through (1, 2), (4, 5) and (0, 9).
Thus when $(x, y) = (1, 21)$ $1+4+2g+8f+c=0$
$\Rightarrow \quad 2g+8f+c=-5 \qquad \ldots\text{(ii)}$
when $(x, y) = (4, 5)$
$16+25+8g+10f+c=0$
$\Rightarrow \quad 8g+10f+c=-41 \qquad \ldots\text{(iii)}$
when $(x, y) = (0, 9)$
$0+81+18f+c=0$
$\Rightarrow \quad 18f+c=-81 \qquad \ldots\text{(iv)}$
Eqs (iii) – Eqs (ii), we get
$6g+2f=-36$
$\Rightarrow \quad 3g+f=-18 \qquad \ldots\text{(v)}$
Eqs (iii) – Eqs (iv), we get
$8g-8f=40$
$\Rightarrow \quad g-f=5 \qquad \ldots\text{(vi)}$
From Eqs (v) and (vi), we get
$4g=-13$
$\Rightarrow \quad g=-13/4$
and $\quad f=g-5=-13/4-5$
$\Rightarrow \quad f=-33/4$
Now, from Eq. (iv), we get,
$c=-81-18f$
$=-81-297/2=-135/2$
Put all these values of f, g and c in Eq. (i), we get
$$x^2+y^2-\frac{13}{2}x-\frac{33}{2}y-\frac{135}{2}=0$$
$\Rightarrow \quad 2(x^2+y^2)-13x-33y-135=0$

12. Let the equation of the circle is
$$x^2+y^2+2gx+2fy+c=0$$
which is passing through (1, 2) and (3, 4).
Now, when $(x, y) = (1, 2)$ $1+4+2g+4f+c=0$
$\Rightarrow \quad 2g+4f+c=-5 \qquad$ (i)
And, when $(x, y) = (3, 4)$ $9+16+6g+8f+c=0$
$\Rightarrow \quad 6g+8f+c=-25 \qquad$ (ii)
Eq. (ii) – Eq. (i), we get
$4g+4f=-20$
$\Rightarrow \quad g+f=-5 \qquad$ (ii)
Also, $\left|\dfrac{-3g-f-5}{\sqrt{3^2+1^2}}\right|=\sqrt{g^2+f^2-c}$
$$=\sqrt{(g+1)^2+(f+2)^2}$$
$\Rightarrow \quad (3g+f+5)^{20}=10\{(g+1)^2+(g+3)^2\}$
$\Rightarrow \quad (2g-5+5)^2=10(g^2+2g+1+g^2+6g+9))$
$\Rightarrow \quad 4g^2=10(2g^2+8g+10)$
$\Rightarrow \quad 4g^2+20g+25=0$
$\Rightarrow \quad (2g+5)^2=0$
$\Rightarrow \quad g=-5/2$

Thus, $f=-5-g=-5+\dfrac{5}{2}=-\dfrac{5}{2}$ and $c=10$
Hence, the equation of the required circle is
$$x^2+y^2-2\left(\frac{5}{2}\right)x-2\left(\frac{5}{2}\right)+10=0$$
$\Rightarrow \quad x^2+y^2-5x-5y+10=0$

13. The lengths of the x-intercept
$$=2\sqrt{g^2-c}=2\sqrt{9-8}=2$$
and the y-intercept
$$=2\sqrt{f^2-c}=2\sqrt{25-8}=2\sqrt{17}$$

14. The length of the y-intercept
$$=2\sqrt{f^2-c}=2\sqrt{\frac{1}{4}-0}=2\times\frac{1}{2}=1$$

15. The given equation of a circle is
$$x^2+y^2-2ax-2ay+a^2=0$$
$\Rightarrow \quad (x-a)^2+(y-a)^2=a^2.$
Thus, the centre is (a, a) and the radius is a.

16. Let $N=x^2+y^2-4x+2y-11$
The values of N at (1, 2) is
$1+4-4+4-11=5-11=-6<0$
and at (6, 0) is
$36+0-0+12-11=37>0$
Thus, the point (1, 2) lies inside of the circle and (6, 0) lies outside of the circle.

17. Since, the point $(\lambda, -\lambda)$ lies inside the circle $x^2+y^2-4x+2y-8=0$, so
$\lambda^2+\lambda^2-4\lambda-2\lambda-8<0$
$\Rightarrow \quad 2\lambda^2-6\lambda-8<0$
$\Rightarrow \quad \lambda^2-3\lambda-4<0$
$\Rightarrow \quad (\lambda-4)(\lambda+1)<0$
$\Rightarrow \quad -1<\lambda<4$
Thus, the range of λ is $(-1, 4)$.

18. Let $N=x^2+y^2-14x-10y-151$
The value of N at $(2, -7)$ is
$4+49-28+98-151=153-151-28$
$=2-28=-26.$
Thus, the point $P(2, -7)$ lies inside of the circle.
The centre of the circle is $C(7, 5)$ and the radius is
$=CA=CB=15.$
Now,
$$CP=\sqrt{(7-2)^2+(5+7)^2}$$
$$=\sqrt{25+144}=\sqrt{169}=13$$
Hence, the shortest distance,
$PA=CA-CP=15-13=2$
and the longest distance
$=PB=CP+CB=13+15=28.$

19. The line $y=x+2$ touches the circle $x^2+y^2=2$, if the length of the perpendicular from the centre to the given line is equal to the radius of a circle.

Thus, the radius of the circle is $\sqrt{2}$.
The length of the perpendicular from the centre (0, 0) to the line $x - y + 2 = 0$ is

$$\left|\frac{0-0+2}{\sqrt{1^2+1^2}}\right| = \frac{2}{\sqrt{2}} = \sqrt{2} = \text{radius.}$$

Hence, the line $y = x + 2$ touches the circle $x^2 + y^2 = 2$.
On solving $y = x + 2$ and $x^2 + y^2 = 2$, we get $x^2 + (x + 2)^2 = 2$

$\Rightarrow \quad x^2 + x^2 + 4x + 4 - 2 = 0$
$\Rightarrow \quad 2x^2 + 4x + 2 = 0$
$\Rightarrow \quad x^2 + 2x + 1 = 0$
$\Rightarrow \quad (x + 1)^2 = 0$
$\Rightarrow \quad x = -1, y = 1 + 2 = 1$

Thus, the co-ordinates of the point of contact is (–1, 1).

20. The equation of any line parallel to the line
$3x + 2y + 5 = 0$ is $3x + 2y + k = 0$ …(i)
The line (i) will be a tangent to the circle $x^2 + y^2 = 4$, if the length of the perpendicular from the centre to the line (i) is equal to the radius of the circle.
Therefore, $\left|\dfrac{3.0+2.0+k}{\sqrt{3^2+2^2}}\right| = 2$
$\Rightarrow \quad k = \pm 2\sqrt{13}$
Hence, the equation of the tangents to the given circle are $3x + 2y \pm 2\sqrt{13} = 0$.

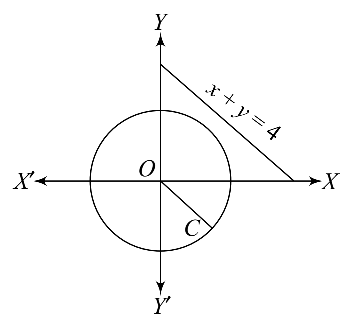
21. The equation of any line perpendicular to the line
$4x + 3y = 0$ is $3x - 4y + k = 0$ …(i)
The line (i) will be a tangent to the circle $x^2 + y^2 = 9$, if the length of the perpendicular from the centre to the line (i) is equal to the radius of the circle.
Therefore, $\left|\dfrac{3.0-4.0+k}{\sqrt{3^2+4^2}}\right| = 3$
$\Rightarrow \quad k = \pm 5$
Hence, the equation of the tangents are
$3x - 4y + 15 = 0$ and $3x - 4y - 15 = 0$

22. Let the equation of line be
$y = mx + c = \sqrt{3}x + c$ …(i)
$(\because m = \tan(60°) = \sqrt{3})$
The line (i) will be a tangent to the circle $x^2 + y^2 + 4x + 3 = 0$, if the length of the perpendicular from the centre (–2, 0) to the line (i) is equal to the radius (= 1) of a circle.
Therefore, $\left|\dfrac{-2\sqrt{3}-0+c}{\sqrt{3+1}}\right| = 1$
$\Rightarrow \quad c - 2\sqrt{3} = \pm 2$
$\Rightarrow \quad c = 2\sqrt{3} \pm 2$
Hence, the equation of the tangents becomes
$y = \sqrt{3}x + (2\sqrt{3} \pm 2)$

23. A line is equally inclined to the co-ordinate axes is
$y = \pm x + c$ …(i)
The line (i) will be tangent to the circle $x^2 + y^2 = 4$, if the length of the perpendicular from the centre (0, 0) to the circle is equal to the radius (= 2) of the given circle.
Therefore, $\left|\dfrac{0+0+c}{\sqrt{1^2+1^1}}\right| = 2$
$\Rightarrow \quad c = \pm 4$
Hence, the equation of the tangents becomes
$y = \pm x \pm 4$

24. The equation of any line passing through (7, 1) is
$y - 1 = m(x - 7)$ …(i)
The line (i) will be a tangent to the circle $x^2 + y^2 = 25$, if the length of the perpendicular from the centre of the given circle is equal to the radius of the same circle.
Therefore, $\left|\dfrac{0-0-(7m-1)}{\sqrt{m^2+1}}\right| = 5$
$\Rightarrow \quad (7m - 1)^2 = 25(m^2 + 1)$
$\Rightarrow \quad 12m^2 - 7m - 12= 0$
$\Rightarrow \quad 12m^2 - 16m + 9m - 12 = 0$
$\Rightarrow \quad (3m - 4)(4m + 3) = 0$
$\Rightarrow \quad m = \dfrac{4}{3}, -\dfrac{3}{4}$
Hence, the equation of the tangents are
$$y - 1 = \frac{4}{3}(x - 7) \text{ and } y - 1 = -\frac{3}{4}(x - 7)$$
$\Rightarrow \quad 4x - 3y = 25$ and $3x + 4y = 25$

25. The centre of a circle is (0, 0) and the radius is 3. Here $OC = 2$.



We shall find the co-ordinates of C.
Let C be (x, y).
Then $x = 2\cos(-45°) = \sqrt{2}$
and $y = 2\sin(-45°) = -\sqrt{2}$.
Thus, the co-ordinates of C becomes $(\sqrt{2}, -\sqrt{2})$.
Hence, the equation of a circle is
$$(x - \sqrt{2})^2 + (y + \sqrt{2})^2 = 3^2$$
$\Rightarrow \quad x^2 + y^2 - 2\sqrt{2}x - 2\sqrt{2}y - 5 = 0$

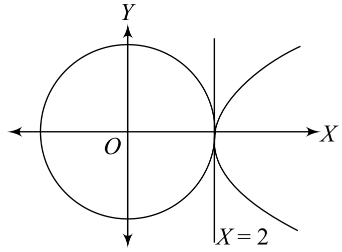
26. When $x = 2$, then $y = \pm\sqrt{5}$.
Therefore, the points are $(2, \sqrt{5})$ and $(2, -\sqrt{5})$.
Thus, the equation of the tangents at $(2, \sqrt{5})$ and $(2, -\sqrt{5})$ are $2x + \sqrt{5}y = 9$ and $2x - \sqrt{5}y = 9$.

27. When $y = 4$, then $x = 0$.
Therefore, the point is (0, 4).
Hence, the equation of the tangent to the given circle at (0, 4) is $x \cdot 0 + y \cdot 4 = 16$
$\Rightarrow \quad y = 4$.

28. The given equations are
$$4x - 3y = 10 \quad \ldots(i)$$
and $$x^2 + y^2 - 2x + 4y - 20 = 0 \quad \ldots(ii)$$
On solving Eqs (i) and (ii), we get
$$x^2 + \left(\frac{4x-3}{10}\right)^2 - 2x + 4\left(\frac{4x-3}{10}\right) - 20 = 0$$
$\Rightarrow \quad 9x^2 + (4x - 10)^2 - 18x + 48x - 120 - 180 = 0$
$\Rightarrow \quad 25x^2 - 50x - 200 = 0$
$\Rightarrow \quad x^2 - 2x - 8 = 0$
$\Rightarrow \quad (x - 4)(x + 2) = 0$
$\Rightarrow \quad x = -2, 4$ and $y = -\frac{11}{10}, \frac{13}{10}$.
Hence, the points of intersection are
$$\left(-2, -\frac{11}{10}\right) \text{ and } \left(4, \frac{13}{10}\right)$$

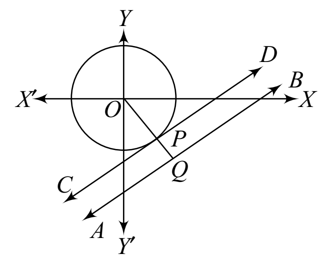
29. The given circle is $x^2 + y^2 - 2x + 4y = 0$
$\Rightarrow \quad (x - 1)^2 + (y + 2)^2 = 5 \quad \ldots(i)$
The equation of any line passing through (0, 1) is
$$y - 1 = m(x - 0) \quad \ldots(ii)$$
The line (ii) will be a tangent to the circle (i), then the length of the perpendicular from the centre (1, –2) to the line (ii) is equal to the radius of a circle (i).
Therefore, $\left|\frac{m+2+1}{\sqrt{m^2+1}}\right| = \sqrt{5}$
$\Rightarrow \quad (m + 3)^2 = 5(m^2 + 1)$
$\Rightarrow \quad m^2 + 5m + 9 - 5m^2 - 5 = 0$
$\Rightarrow \quad 2m^2 - 3m - 2 = 0$
$\Rightarrow \quad (m - 2)(2m + 1) = 0$
$\Rightarrow \quad m = -\frac{1}{2}, 2$
Hence, the equation of the tangents are
$$y - 1 = -\frac{1}{2}x \text{ and } y - 1 = 2x$$
$\Rightarrow \quad x + 2y - 1 = 0$ and $2x - y + 1 = 0$

30. The given curves are $x^2 + y^2 = 4$ and $y^2 = 4(x - 2)$.



From the graph, it is clear that the equation of the common tangent is $x = 2$.

31. $$x^2 + y^2 - 2x + 4y - 20 = 0$$



Let AB: $y = x - 4$ and CD be a tangent parallel to AB.
Now $OQ = \left|\frac{0-0-8}{\sqrt{1^2+1^2}}\right| = \frac{8}{\sqrt{2}} = 4\sqrt{2}$
and OP = radius of a circle = 3.
Thus, the shortest distance = PQ
$= OQ - OP$
$= (4\sqrt{2} - 3)$

32. Hence, the length of the tangent from the point (2, 3) to the circle $x^2 + y^2 = 4$ is
$$\sqrt{4+9-4} = \sqrt{9} = 3$$

33. Let $P(x_1, y_1)$ be any point on the circle $x^2 + y^2 = a^2$.
Therefore, $x_1^2 + y_1^2 = a^2$.
Now, the length of the tangent from (x_1, y_1) to the circle $x^2 + y^2 = b^2$ is
$$\sqrt{x_1^2 + y_1^2 - b^2} = \sqrt{a^2 - b^2}.$$

34. Let $P(x_1, y_1)$ be any point on
$$x^2 + y^2 + 2011x + 2012y + 2013 = 0$$
Therefore,
$$x_1^2 + y_1^2 + 2011x_1 + 2012y_1 + 2013 = 0$$
Now, the length of the tangent from (x_1, y_1) to the circle $x^2 + y^2 + 2011x + 2012y + 2014 = 0$ is
$$\sqrt{x_1^2 + y_1^2 + 2011x_1 + 2012y_1 + 2014}$$
$$= \sqrt{-2013 + 2014}$$
$$= 1$$

35. The power of a point (2, 5) with respect to a circle $x^2 + y^2 = 16$ is $(4 + 25 - 16) = 13$.

36. Let the point P represents a complex number Z. i.e. $Z = 1 + 2i$
Then, from the rotation theorem,
$$Q = iZ = i(1 + 2i) = i - 2 = -2 + 1.i = (-2, 1)$$
Thus, the power of a point Q with respect to a circle is $4 + 1 - 4 = 1$.

37. Hence, the equation of a pair of tangents to a circle $x^2 + y^2 = 4$ from a point (1, 2) is $SS_1 = T^2$
$\Rightarrow \quad (x^2 + y^2 - 4)(1 + 4 - 4) = (x + 2y - 4)^2$
$\Rightarrow \quad (x^2 + y^2 - 4) = (x^2 + 4y^2 + 16 + 4xy - 8x - 16y)$

$\Rightarrow \quad 3y^2 + 4(x-4)y + 4(5-2x) = 0$

$\Rightarrow \quad y = \dfrac{-4(x-4) \pm \sqrt{16(x-4)^2 - 48(5-2x)}}{6}$

$\Rightarrow \quad y = \dfrac{-4(x-4) \pm 4\sqrt{x^2 - 8x + 16 - 15 + 6x}}{6}$

$\Rightarrow \quad y = \dfrac{-4(x-4) \pm 4(x-1)}{6}$

$\Rightarrow \quad y = 2, 4x - 3y + 2 = 0$

38. Hence, the equation of the pair of tangents are $SS_1 = T^2$

$\Rightarrow \quad (x^2 + y^2 - 4x + 3)(4 + 9 - 8 + 3)$
$\quad = (2x + 3y - 2(x + 2) + 3)^2$

$\Rightarrow \quad 8(x^2 + y^2 - 4x + 3) = (3y - 1)^2$

$\Rightarrow \quad 8(x^2 + y^2 - 4x + 3) = 9y^2 - 6y + 1$

$\Rightarrow \quad y^2 - 8x^2 + 32x - 6y - 23 = 0$

$\Rightarrow \quad y^2 - 6y - (8x^2 - 32x + 23) = 0$

$\Rightarrow \quad y = \dfrac{6 \pm \sqrt{36 + 4(8x^2 - 32x + 23)}}{2}$

$\Rightarrow \quad y = \dfrac{6 \pm \sqrt{32x^2 - 128x + 128}}{2}$

$= \dfrac{6 \pm \sqrt{32(x^2 - 4x + 4)}}{2}$

$= \dfrac{6 \pm 4\sqrt{2}\,(x-2)}{2}$

$\Rightarrow \quad y = 3 \pm 2\sqrt{2}(x-2)$

39. Hence, the angle between the tangent from the point (3, 5) to the circle $x^2 + y^2 = 25$ is

$$2\tan^{-1}\left(\frac{a}{\sqrt{S_1}}\right) = 2\tan^{-1}\left(\frac{5}{\sqrt{9+25-25}}\right)$$
$$= 2\tan^{-1}\left(\frac{5}{3}\right)$$

40. After translation the point (1, 2) becomes
(1+2, 2) = (3, 2).
Hence, the angle between the tangent from the point (3, 2) to the circle $x^2 + y^2 = 9$ is

$$2\tan^{-1}\left(\frac{a}{\sqrt{S_1}}\right) = 2\tan^{-1}\left(\frac{3}{\sqrt{9+4-9}}\right)$$
$$= 2\tan^{-1}\left(\frac{3}{2}\right)$$

41. The locus of the point of intersection of two perpendicular tangents to a circle is the director circle of the given circle.
Thus, the equation of the director circle to the circle $x^2 + y^2 = 25$ is $x^2 + y^2 = 2 \times 25 = 50$

42. Clearly, the locus of the arbitrary point is the director circle of the given circle $x^2 + y^2 = 9$.
Hence, the equation of the director circle to the circle $x^2 + y^2 = 9$ is $x^2 + y^2 = 2 \times 9 = 18$.

43. Clearly, the equation $x^2 + y^2 = 20$ is the director circle of the circle $x^2 + y^2 = 10$.
Hence, the angle between the pair of tangents is 90°.

44. (i) The given circle $x^2 + y^2 + 2x = 0$ can be reduced to $(x+1)^2 + y^2 = 1$.
Hence, the equation of the director circle to the circle $(x+1)^2 + y^2 = 1$ is
$(x+1)^2 + y^2 = 2 \times 1 = 2$
$\Rightarrow \quad x^2 + y^2 + 2x - 1 = 0$

(ii) The given circle $x^2 + y^2 + 10y + 24 = 0$ can be reduced to $x^2 + (y+5)^2 = 1$
Hence, the equation of the director circle to the circle $x^2 + (y+5)^2 = 1$ is
$x^2 + (y+5)^2 = 2 \times 1 = 1$
$\Rightarrow x^2 + y^2 + 10y + 23 = 0$

(iii) The given circle
$x^2 + y^2 + 16x + 12y + 99 = 0$
can be reduced to
$(x+8)^2 + (y+6)^2 = 64 + 36 - 99 = 1$
Hence, the equation of the director circle to the circle $(x+8)^2 + (y+6)^2 = 1$
is $(x+8)^2 + (y+6)^2 = 2 \times 1 = 2$
$\Rightarrow \quad x^2 + y^2 + 16x + 12y + 98 = 0$

(iv) The given circle
$x^2 + y^2 + 2gx + 2fy + c = 0$ can be reduced to
$(x+g)^2 + (y+f)^2 = (\sqrt{g^2 + f^2 - c})^2$
Hence, the equation of the director circle to the circle $(x+g)^2 + (y+f)^2 = (\sqrt{g^2 + f^2 - c})^2$ is
$(x+g)^2 + (y+f)^2 = 2(g^2 + f^2 - c)$
$\Rightarrow \quad x^2 + y^2 + 2gx + 2fy - g^2 - f^2 + 2c = 0.$

(v) The given circle $x^2 + y^2 - ax - by = 0$ can be reduced to $\left(x - \dfrac{a}{2}\right)^2 + \left(y - \dfrac{b}{2}\right)^2 = \left(\dfrac{\sqrt{a^2+b^2}}{2}\right)^2$

Hence, the equation of the director circle to the circle

$$\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b}{2}\right)^2 = \left(\frac{\sqrt{a^2+b^2}}{2}\right)^2 \text{ is}$$

$$\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b}{2}\right)^2 = \left(\frac{a^2+b^2}{2}\right)$$

$$\Rightarrow x^2 + y^2 - ax - by - \frac{1}{4}(a^2 + b^2) = 0$$

45. When $x = 2,\ y = \pm\sqrt{5}$.
So, the points are $(2, \sqrt{5})$ and $(2, -\sqrt{5})$.
The equation of the normal to the circle at $(2, \sqrt{5})$ and $(2, -\sqrt{5})$ are

$$\frac{x}{2} = \frac{y}{\sqrt{5}} \text{ and } \frac{x}{2} = -\frac{y}{\sqrt{5}}$$

$\Rightarrow \quad y = \dfrac{\sqrt{5}}{2}x$ and $y = -\dfrac{\sqrt{5}}{2}x$

46. The equation of the normal to the given circle at (–2, 1) is

$$\Rightarrow \quad \frac{x-x_1}{x_1+g}=\frac{y-y_1}{y_1+f}$$

$$\frac{x+2}{-2+1}=\frac{y-1}{1+2}$$

$\Rightarrow \quad 3x+6=-y+1$

$\Rightarrow \quad 3x+y+5=0$

47. The centre of a circle $x^2+y^2-4x-6y+4=0$ is (2, 3). The equation of a normal parallel to $x-y-3=0$ is

$$x-y+k=0 \qquad \ldots\text{(i)}$$

As we know that the normal always passes through the centre of a circle.

Therefore, $2-3+k=0 \Rightarrow k=1$.

Hence, the equation of the normal is

$$x-y+1=0$$

48. The centre of a circle

$$x^2+y^2-8x-12y+99=0 \text{ is } (4, 6).$$

The equation of the normal which is perpendicular to

$$2x-3y+10=0 \text{ is } 3x+2y-k=0 \qquad \ldots\text{(i)}$$

which is passing through (4, 6).

Therefore, $12+12-k=0 \Rightarrow k=24$.

Hence, the equation of the normal is $3x+2y=24$.

49. The equation of the chord of contact of tangents from (5, 3) to the circle $x^2+y^2=25$ is $5x+3y=25$.

50. Let the point be (x_1, y_1).

Clearly, $2x+y+12=0$ ...(i)

is the chord of contact of the tangents to the circle $x^2+y^2-4x+3y-1=0$ from (x_1, y_1).

The equation of the chord of contact of tangents to the circle $x^2+y^2-4x+3y-1=0$ from the point (x_1, y_1) is

$$xx_1+yy_1-2(x+x_1)+\frac{3}{2}(y+y_1)-1=0 \qquad \ldots\text{(ii)}$$

Clearly Eqs (i) and (ii) are identical.

$$\text{Therefore, } \frac{x_1-2}{2}=\frac{y_1+\frac{3}{2}}{1}=\frac{\frac{3}{2}y_1-2x_1-1}{12}$$

$\Rightarrow \quad x_1-2y_1=5$ and $4x_1+21y_1=-38$

$\Rightarrow \quad x_1=1, y_1=-2$

Hence, the point is (1, –2).

51. The equation of the chord of contact QR is

$$hx+ky=a^2 \qquad \ldots\text{(i)}$$

The chord of contact subtends a right angle at the centre of the circle $x^2+y^2=a^2$.

$$\text{Therefore, } \frac{x^2+y^2}{a^2}=\left(\frac{hx+ky}{a^2}\right)^2$$

$\Rightarrow \quad a^2(x^2+y^2)=(hx+ky)^2$

$\Rightarrow \quad x^2(a^2-h^2)+y^2(a^2-k^2)-2hkxy=0$

Since, the line (i) makes a right angle at the centre, so $(a^2-h^2)+(a^2-k^2)=0$

$\Rightarrow \quad h^2+k^2=2a^2$

which is the required condition.

52. Let AB be the chord of contact.

Then $AB: hx+ky=a^2$

$$\text{Now, } PM=\left|\frac{h^2+k^2-a^2}{\sqrt{h^2+k^2}}\right|$$

The length of the chord of contact,

$$AB=\frac{2LR}{\sqrt{L^2+R^2}},$$

where L = the length of the tangent and R be the radius of the circle.

Therefore,

$$AB=\frac{2a\times\sqrt{h^2+k^2-a^2}}{\sqrt{(h^2+k^2-a^2)+a^2}}$$

$$=\frac{2a\times\sqrt{h^2+k^2-a^2}}{\sqrt{h^2+k^2}}$$

Thus, the area of the $\Delta\, PAB$

$$=\frac{1}{2}\times\left(\frac{2a\times\sqrt{h^2+k^2-a^2}}{\sqrt{h^2+k^2}}\right)\times\left(\frac{h^2+k^2-a^2}{\sqrt{h^2+k^2}}\right)$$

$$=a\left(\frac{(h^2+k^2-a^2)^{3/2}}{(h^2+k^2)}\right)$$

53. Let $P(a\cos\theta, a\sin\theta)$ be any point on the circle $x^2+y^2=a^2$.

Therefore, QR:

$$ax\cos\theta+ay\sin\theta=b^2 \qquad \ldots\text{(i)}$$

Since the line (i) be a tangent to the circle $x^2+y^2=c^2$, so

$$\left|\frac{0+0-b^2}{\sqrt{a^2\cos^2\theta+a^2\sin^2\theta}}\right|=c$$

$\Rightarrow \quad b^2=ac$

Thus, $m=2$, $n=1$ and $p=1$.

Hence, the value of $m+n+p+10=14$.

54. The equation of the chord of the circle $x^2+y^2=25$ bisected at (–2, 3) is

$T=S_1$

$\Rightarrow \quad -2x+3y-25=4+9-25$

$\Rightarrow \quad -2x+3y=13$

$\Rightarrow \quad 2x-3y+1\,3=0$

55. Hence, the equation of the chord of the circle $x^2+y^2+6x+8y-11=0$, whose mid-point is (1, –1) is $T=S_1$

$\Rightarrow \quad x-y+3(x+1)+4(y-1)-11$

$$=1+1+6+8-11$$

$\Rightarrow \quad 4x+3y-1=16$

$\Rightarrow \quad 4x+3y=17$

56. Let the chord AB is bisected at $C(x_1, y_1)$

The equation of AB is $xx_1 + yy_1 = x_1^2 + y_1^2$

which is passing through (h, k), so

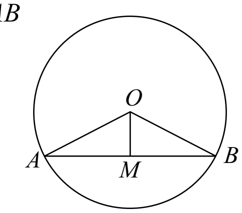
$$hx_1 + ky_1 = x_1^2 + y_1^2$$

Thus, the locus of (x_1, y_1) is

$$x^2 + y^2 = hx + ky$$

57. Let $M(h, k)$ is the mid-point of AB

In ΔOAM,



$$\sin(45°) = \frac{OM}{AM} = \frac{OM}{2}$$

$\Rightarrow \quad OM = 2\sin(45°)$

$$= 2 \times \frac{1}{\sqrt{2}} = \sqrt{2}$$

$\Rightarrow \quad OM^2 = 2$

$\Rightarrow \quad h^2 + k^2 = 2$

Hence, the locus of (h, k) is $x^2 + y^2 = 2$.

58. Let A represents a complex number Z.

i.e. $\quad Z = \sqrt{3} + i$

Here, $\angle AOB = 90°$.

By rotation theorem, we can say that

$$B = iZ = i(\sqrt{3} + i) = -1 + i\sqrt{3}$$

Hence, the point B is $(-1, \sqrt{3})$.

59. Let C represents a complex number Z and D be Z_1.

Here, $\angle COD = 60°$

By rotation theorem, $\dfrac{Z_1 - 0}{Z - 0} = \left|\dfrac{Z_1 - 0}{Z - 0}\right| \times e^{-i\frac{\pi}{3}}$

$\Rightarrow \quad \dfrac{Z_1}{Z} = \dfrac{|Z_1|}{|Z|} \times e^{-i\frac{\pi}{3}} = e^{-i\frac{\pi}{3}}$

$\Rightarrow \quad Z_1 = Ze^{-i\frac{\pi}{3}}$

$\Rightarrow \quad Z_1 = (2\sqrt{2} + i)\left(\dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}\right)$

$\Rightarrow \quad Z_1 = \left(\sqrt{2} + \dfrac{\sqrt{3}}{2}\right) + i\left(\dfrac{1}{2} - \sqrt{3}\right)$

Hence, the co-ordinates of D are

$$\left(\sqrt{2} + \frac{\sqrt{3}}{2}, \frac{1}{2} - \sqrt{3}\right)$$

60. As we know that the locus of the mid-points of the chord of the circle is the diameter of the circle. Thus, the equation of the diameter which is parallel to

$$2012x + 2013y + 2014 = 0$$

is $\quad 2012x + 2013y + k = 0$

which is passing through the centre of the circle $x^2 + y^2 = 9$.

Therefore, $k = 0$

Hence, the equation of the diameter is

$$2012x + 2013y = 0$$

61. Obviously, the locus of the mid-point of the chords of the circle is the diameter of the given circle.

Here, the centre of the circle $x^2 + y^2 - 4x - 6y = 0$ is $(2, 3)$.

The equation of the diameter of the circle, which is perpendicular to the line $4x + 5y + 10 = 0$ is $5x - 4y + k = 0$ which is passing through $(2, 3)$.

Therefore, $k = 12 - 10 = 2$.

Hence, the equation of the diameter is

$$5x - 4y + 2 = 0$$

62. The equation of the common chord of the circle is

$$2x - 2y = 0$$

$\Rightarrow \quad x = y$

On solving $y = x$ and the circle

$$x^2 + y^2 + 3x + 5y + 4 = 0,$$

we get the points of intersection.

Thus, $x = -2 + \sqrt{2} = y$ and $x = -2 - \sqrt{2} = y$

Let the common chord be PQ, where

$$P = (-2 + \sqrt{2}, -2 + \sqrt{2}) \text{ and}$$

$$Q = (-2 - \sqrt{2}, -2 - \sqrt{2})$$

Thus, the length of the common chord, PQ

$$= \sqrt{(-2 - \sqrt{2} + 2 - \sqrt{2})^2 + (-2 - \sqrt{2} + 2 - \sqrt{2})^2}$$

$$= \sqrt{8 + 8} = \sqrt{16} = 4$$

63. The equation of the common chord of the given circles is $2x + 1 = 0$.

On solving $2x + 1 = 0$ and

$$x^2 + y^2 + 2x + 3y + 1 = 0,$$

we get the points of intersection.

Thus, the points of intersection are

$$x = -\frac{1}{2} \text{ and } y = -\frac{3}{2} \pm \sqrt{2}$$

Let PQ be the common chord, where

$$P = \left(-\frac{1}{2}, -\frac{3}{2} + \sqrt{2}\right) \text{ and}$$

$$Q = \left(-\frac{1}{2}, -\frac{3}{2} - \sqrt{2}\right)$$

The mid-point of P and Q is the centre of the new circle.

Thus, centre is $C = \left(-\dfrac{1}{2}, -\dfrac{3}{2}\right)$

Now, radius $r = CP = \sqrt{2}$.

Hence, the equation of the circle is

$$\left(x + \frac{1}{2}\right)^2 + \left(y + \frac{3}{2}\right)^2 = 2$$

$\Rightarrow \quad 2(x^2 + y^2) + 2x + 6y + 1 = 0$

64. (i) The given circles are $x^2 + y^2 = 4$ ...(i)

and $x^2 + y^2 - 2x = 0$ (ii)

Now, $C_1(0, 0)$ and $C_2(1, 0)$ and $r_1 = 2$ and $r_2 = 1$.

Thus, $C_1C_2 = 1 = r_2 - r_1$

Hence, both the circles touch each other internally.

Thus, the number of common tangent = 1.

(ii) The given circles are $x^2 + y^2 + 4x + 6y + 12 = 0$

and $x^2 + y^2 - 6x - 4y + 12 = 0$

Now, C_1: $(-2, -3)$, C_2: $(3, 2)$, $r_1 = 1$ and $r_2 = 1$

Therefore,

$$C_1C_2 = \sqrt{(3+2)^2 + (-3-2)^2}$$
$$= \sqrt{25+25} = \sqrt{50} = 5\sqrt{2}$$

and $r_1 + r_2 = 1 + 1 = 2$.

Thus, $C_1C_2 > r_1 + r_2$

Hence, the number of common tangents = 4.

(iii) The given circles are

$x^2 + y^2 - 6x - 6y + 9 = 0$

and $x^2 + y^2 + 6x + 6y + 9 = 0$

Now, $C_1 = (3, 3)$, $C_2 = (-3, -3)$, $r_1 = 3$ and $r_2 = 3$.

Therefore,

$$C_1C_2 = \sqrt{(-3-2)^2 + (-3-3)^2}$$
$$= \sqrt{36+36} = \sqrt{72} = 6\sqrt{2}$$

and $r_1 + r_2 = 3 + 3 = 6$

Thus, $C_1C_2 > r_1 + r_2$

Hence, the number of common tangents = 4.

(iv) The given circles are

$x^2 + y^2 - 4x - 4y = 0$

and $x^2 + y^2 + 2x + 2y = 0$

Now, $C_1 = (2, 2)$, $C_2 = (-1, -1)$, $r_1 = 2\sqrt{2}$ and $r_2 = \sqrt{2}$

Therefore,

$$C_1C_2 = \sqrt{(-1-2)^2 + (-1-2)^2}$$
$$= \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$$

and $r_1 + r_2 = 2\sqrt{2} + \sqrt{2} = 3\sqrt{2}$

Thus, $C_1C_2 = r_1 + r_2$

Hence, the number of common tangents = 3.

(v) The given circles are $x^2 + y^2 = 64$

and $x^2 + y^2 - 4x - 4y + 4 = 0$

Now, C_1: $(0, 0)$, C_2: $(2, 2)$, $r_1 = 8$ and $r_2 = 2$

Therefore,

$$C_1C_2 = \sqrt{(2-0)^2 + (2-0)^2}$$
$$= \sqrt{4+4} = 2\sqrt{2}$$

and $r_1 + r_2 = 8 + 2 = 10$

Therefore, $C_1C_2 < r_1 + r_2$

Thus, one circle lies inside of the other.

Hence, the number of common tangents = 0.

(vi) The given circles are

$$x^2 + y^2 - 2(1+\sqrt{2})x - 2y + 1 = 0$$

and $x^2 + y^2 - 2x - 2y + 1 = 0$

Now C_1: $(1+\sqrt{2}, 1)$, C_2: $(1, 1)$,

$r_1 = \sqrt{2} + 1$ and $r_2 = 1$.

Therefore, $C_1C_2 = \sqrt{(\sqrt{2})^2} = \sqrt{2}$

and $r_1 - r_2 = \sqrt{2}$.

Thus, $C_1C_2 = r_1 - r_2$

Hence, the number of common tangents = 1.

65. The given circles are

$x^2 + y^2 - 2x - 6y + 9 = 0$

and $x^2 + y^2 + 6x - 2y + 1 = 0$

Now, C_1: $(1, 3)$, C_2: $(-3, 1)$, $r_1 = 1$ and $r_2 = 3$.

Therefore,

$$C_1C_2 = \sqrt{(-3-1)^2 + (1-3)^2}$$
$$= \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$$

and $r_1 + r_2 = 1 + 3 = 4$

Then, $C_1C_2 > r_1 + r_2$

Thus, both the circles do not intersect.

Hence, the number of common tangents = 4, in which 2 are direct common tangents and another 2 are transverse common tangents.

Here, the point M divides $C_2(-3, 1)$ and $C_1(1, 3)$ internally in the ratio 3 : 1.

Then, the co-ordinates of M are

$$\left(\frac{3.1 - 1.3}{3+1}, \frac{3.3 + 1.1}{3+1}\right) = \left(0, \frac{5}{2}\right)$$

Also, the point P divide $C_2(-3, 1)$ and C_1 $(1, 3)$ externally in the ratio 3 : 1.

Then, the co-ordinates of P are

$$\left(\frac{3.1 + 1.3}{3-1}, \frac{3.3 - 1.1}{3-1}\right) = (3, 4)$$

Case I: Direct common tangents

Any line through (3, 4) is

$y - 4 = m(x - 3)$ ...(i)

$\Rightarrow \quad mx - y + 4 - 3m = 0$

If it is a tangent to the circle,

$$\left|\frac{m - 3 + 4 - 3m}{\sqrt{m^2+1}}\right| = 1$$

$\Rightarrow \quad (1 - 2m)^2 = (m^2 + 1)$

$\Rightarrow \quad 3m^2 - 4m = 0$

$\Rightarrow \quad m(3m - 4) = 0$

$\Rightarrow \quad m = 0, \frac{4}{3}$.

Hence, the direct common tangents are

$y = 4$, $4x - 3y = 0$.

Case II: Transverse common tangents

Any line through (0, 5/2) is

$$y-\frac{5}{2}=m(x-0) \quad \ldots(i)$$

$\Rightarrow \quad 2mx - 2y + 5 = 0$

If it is a tangent to a circle, then

$$\left|\frac{m\cdot 1-3+\frac{5}{2}}{\sqrt{m^2+1}}\right|=1$$

$\Rightarrow \quad (2m-1)^2 = 4(m^2+1)$

$\Rightarrow \quad 4m^2 - 4m + 1 = 4(m^2+1)$

$\Rightarrow \quad 4m + 3 = 0$

$\Rightarrow \quad m=\infty, m=-\frac{3}{4}$

Hence, the equation of transverse tangents are $x = 0$ and $3x + 4y = 10$.

66. Given circles are $x^2 + y^2 + c^2 = 2ax$

$\Rightarrow \quad (x-a)^2+y^2=(\sqrt{a^2-c^2})^2 \quad \ldots(i)$

and $\quad x^2 + y^2 + c^2 = 2by$

$\Rightarrow \quad x^2+(y-b)^2=(\sqrt{b^2-c^2})^2 \quad \ldots(ii)$

Now C_1 : (a, 0), C_2 : (0, b),

$$r_1=\sqrt{a^2-c^2} \text{ and } r_2=\sqrt{b^2-c^2}$$

Since both the circles touch each other externally, then $C_1C_2 = r_1 + r_2$

$\Rightarrow \quad \sqrt{a^2+b^2}=\sqrt{a^2-c^2}+\sqrt{b^2-c^2}$

$\Rightarrow \quad a^2+b^2=a^2-c^2+b^2-c^2+2(\sqrt{a^2-c^2})(\sqrt{b^2-c^2})$

$\Rightarrow \quad 2c^2=2(\sqrt{a^2-c^2})(\sqrt{b^2-c^2})$

$\Rightarrow \quad c^4 = (a^2-c^2)(b^2-c^2) = a^2b^2 - c^2(a^2+b^2) + c^4$

$\Rightarrow \quad a^2b^2 - c^2(a^2+b^2) = 0$

$\Rightarrow \quad \frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}$

67. The given circles are

$$x^2 + y^2 = 9 \quad \ldots(i)$$

and $\quad x^2 + y^2 - 8x - 6y + n^2 = 0$

$\Rightarrow \quad (x-4)^2+(y-3)^2=(\sqrt{25-n^2})^2 \quad \ldots(ii)$

Here, C_1: (0, 0), C_2: (4, 3), $r_1 = 3$ and $r_2=\sqrt{25-n^2}$

Since the circles have only two common tangents, so we can write,

$$|r_1 - r_2| < C_1C_2 < r_1 + r_2$$

$\Rightarrow \quad |3-\sqrt{25-n^2}|<5<3+\sqrt{25-n^2}$

Case I: When $|3-\sqrt{25-n^2}|<5$

Then $-5<3-\sqrt{25-n^2}<5$

$\Rightarrow \quad -5-3<-\sqrt{25-n^2}<5-3$

$\Rightarrow \quad -2<\sqrt{25-n^2}<8$

$\Rightarrow \quad 4 < 25 - n^2 < 64$

$\Rightarrow \quad -21 < -n^2 < 39$

$\Rightarrow \quad -39 < n^2 < 21$

$\Rightarrow \quad 0 < n < \sqrt{21}$

$\Rightarrow \quad n = 1, 2, 3, 4$

Case II: When $5<3+\sqrt{25-n^2}$

Then, $\sqrt{25-n^2}>2$

$\Rightarrow \quad 25 - n^2 > 4$

$\Rightarrow \quad n^2 < 21$

$\Rightarrow \quad n<\sqrt{21}$

$\Rightarrow \quad n = 1, 2, 3, 4.$

Hence, the number of integers = 4.

68. The given circles are

$$x^2 + y^2 + x + y = 0$$

and $\quad x^2 + y^2 + x - y = 0$

Here, $g_1 = -1/2$, $f_1 = -1/2$, $g_2 = -1/2$, $f_2 = 1/2$, $c_1 = 0$, $c_2 = 0$

Now, $2(g_1 g_2 + f_1 f_2) = 2(-1/4 + 1/4) = 0$ and $c_1 + c_2 = 0$.

Hence, the angle between the given circles is 90°.

69. The given circles are

$$x^2 + y^2 + 2a_1x + 2b_1y + c_1 = 0$$

and $\quad x^2 + y^2 + a_2x + b_2y + c_2 = 0$

Here, $g_1 = a_1, f_1 = b_1, g_2 = a_2/2, f_2 = b_2/2, c_1 = c_1$ and $c_2 = c_2$

Since the given two circles are orthogonal, so

$$2(g_1g_2 + f_1f_2) = c_1 + c_2$$

$\Rightarrow \quad 2\left(a_1\cdot\frac{a_2}{2}+b_1\cdot\frac{b_2}{2}\right)=c_1+c_2$

$\Rightarrow \quad (a_1 \cdot a_2 + b_1 \cdot b_2) = c_1 + c_2$

Hence, the result.

70. The given circles are

$$2x^2 + 2y^2 - 3x + 6y + k = 0$$

$\Rightarrow \quad x^2+y^2-\frac{3}{2}x+3y+\frac{k}{2}=0$

and $\quad x^2 + y^2 - 4x + 10y + 16 = 0$

Since, the two given circles are orthogonal, then $2(g_1g_2 + f_1f_2) = c_1 + c_2$

$\Rightarrow \quad 2\left(-\frac{3}{4}\times -2+\frac{3}{2}\times 5\right)=\frac{k}{2}+16$

$\Rightarrow \quad \frac{k}{2}+16=18$

$\Rightarrow \quad \frac{k}{2}=18-16$

$\Rightarrow \quad k = 4$

Hence, the value of k is 4.

71. Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy = 0 \quad \ldots(i)$$

Therefore, the centre of the circle is $(-g, -f)$.

Since, the centre lies on the line $y = x$, so

$$-f = -g$$

$\Rightarrow \quad f = g$

The Eq. (i) is orthogonal to

$x^2 + y^2 - 4x - 6y + 18 = 0,$

so, $2[g(-2) + f(-3)] = 0 + 18$

$\Rightarrow \quad 2(-5g) = 18 \qquad (\because\ f = g)$

$\Rightarrow \quad g = -\frac{9}{5} = f$

Hence, the equation of the circle is

$$x^2 + y^2 - \frac{18}{5}x - \frac{18}{5}y = 0$$

72. Let the equation of the circle be

$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \ldots\text{(i)}$

The equation of the circle (i) is orthogonal to

$x^2 + y^2 + 4x - 6y + 9 = 0$

and $x^2 + y^2 - 5x + 4y - 2 = 0$

Thus, $(4g - 6f) = c + 9 \qquad \ldots\text{(ii)}$

and $(-5g + 4f) = c - 2 \qquad \ldots\text{(iii)}$

Subtracting, we get $-9g + 10f + 11 = 0$

$\Rightarrow \quad 9(-g) - 10(-f) + 11 = 0$

Hence, the locus of the centre is

$9x - 10y + 11 = 0$

73. Let the equation of the circle be

$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \ldots\text{(i)}$

which is passing through (a, b).

Therefore,

$a^2 + b^2 + 2ga + 2fb + c = 0 \qquad \ldots\text{(ii)}$

It is given that the circle (i) is orthogonal to $x^2 + y^2 = 4$,

so $2(g \cdot 0 + f \cdot 0) = c - 4 \Rightarrow c = 4$

From Eq. (ii), we get

$a^2 + b^2 + 2g \cdot a + 2f \cdot b + 4 = 0$

Hence, the locus of $(-g, -f)$ is

$2ax + 2by - (a^2 + b^2 + 4) = 0$

74. Let C_1 and C_2 be the centres of the two orthogonal circles with radii r_1 and r_2, respectively.

Here $\angle C_1PC_2 = 90°$ and let

$\angle PC_1C_2 = \theta$ and $\angle PC_2C_1 = 90° - \theta$

Thus,

$$\sin(\theta) = \frac{PM}{r_1} \quad \text{and} \quad \sin(90° - \theta) = \frac{PM}{r_2}.$$

Squaring and adding, we get

$$\Rightarrow \quad \left(\frac{PM}{r_1}\right)^2 + \left(\frac{PM}{r_2}\right)^2 = 1$$

$$PM^2\left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right) = 1$$

$$\Rightarrow \quad PM^2 = \frac{r_1^2 r_2^2}{r_1^2 + r_2^2}$$

$$\Rightarrow \quad PM = \frac{r_1 r_2}{\sqrt{r_1^2 + r_2^2}}$$

Hence, the length of the common chord,

$$PQ = 2PM = \frac{2r_1 r_2}{\sqrt{r_1^2 + r_2^2}}$$

75. Hence, the equation of the radical axis is

$(x^2 + y^2 + 4x + 6y + 9) -$

$(x^2 + y^2 + 3x + 8y + 10) = 0$

$\Rightarrow \quad (4x - 3x) + (6y - 8y) + (9 - 10) = 0$

$\Rightarrow \quad x - 2y - 1 = 0$

76. Hence, the equation of the radical axis is

$(x^2 + y^2 + 8x + 2y + 10) - (x^2 + y^2 + 7x + 3y + 10) = 0$

$\Rightarrow \quad (8x - 7x) + (2y - 3y) = 0$

$\Rightarrow \quad y = x$

Hence, the image of (2, 3) with respect to the line $y = x$ is (3, 2).

77. Let $S_1 : x^2 + y^2 = 1 \qquad \ldots\text{(i)}$

$S_2 : x^2 + y^2 - 8x + 15 = 0 \qquad \ldots\text{(ii)}$

and $S_3 : x^2 + y^2 + 10y + 24 = 0 \qquad \ldots\text{(iii)}$

Eq. (i) – Eq. (ii), we get

$8x = 16 \Rightarrow x = 2$

Eq. (i) – Eq. (iii), we get

$-10y = 25 \Rightarrow y = -5/2$

Hence, the radical centre is $\left(2, -\frac{5}{2}\right)$.

78. Let the equation of the circle be

$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \ldots\text{(i)}$

The circle (i) is orthogonal to

$x^2 + y^2 - 3x - 6y + 14 = 0,$

$x^2 + y^2 - x - 4y + 8 = 0,$

and $x^2 + y^2 + 2x - 6y + 9 = 0$

Therefore,

$$2\left(g \cdot \left(-\frac{3}{2}\right) - 3f\right) = c + 14$$

$\Rightarrow \quad (-3g - 6f) = c + 14 \qquad \ldots\text{(ii)}$

$$2\left[g \cdot \left(-\frac{1}{2}\right) - 2f\right] = c + 8$$

$\Rightarrow \quad (2g - 4f) = c + 8 \qquad \ldots\text{(iii)}$

and

$2(g \cdot 1 - 3 \cdot f) = c + 9$

$\Rightarrow \quad (2g - 6f) = c + 9 \qquad \ldots\text{(iv)}$

Eq. (iii) – Eq. (ii), we get

$5g + 2f = -6 \qquad \ldots\text{(v)}$

Eq. (iii) – Eq. (iv), we get

$2f = -1 \Rightarrow f = -1/2$

Put the value of f in Eq. (v), we get,

$g = -1$

Also, put the values of f and g in Eq. (iv), we get

$c = -8.$

Hence, the equation of the circle is

$x^2 + y^2 - 2x - y - 8 = 0$

79. Equation of any circle passing through the point of intersection of the circles

$x^2 + y^2 + 13x - 3y = 0$

and $2x^2 + 2y^2 + 4x - 7y - 25 = 0$

is $(x^2 + y^2 + 13x - 3y)$

$+ \lambda(2x^2 + 2y^2 + 4x - 7y - 25) = 0 \qquad \ldots\text{(i)}$

which is passing through (1, 1).

Therefore,
$(1 + 1 + 13 - 3) + \lambda(2 + 2 + 4 - 7 - 25) = 0$
$\Rightarrow \quad 12 - 24\lambda = 0$
$\Rightarrow \quad \lambda = \frac{1}{2}$
Put the value of λ in Eq. (i), we get
$(x^2 + y^2 + 13x - 3y) + \frac{1}{2}(2x^2 + 2y^2 + 4x - 7y - 25) = 0$
$\Rightarrow (2x^2 + 2y^2 + 26x - 6y) + (2x^2 + 2y^2 + 4x - 7y - 25) = 0$
$\Rightarrow (4x^2 + 4y^2 + 30x - 13y - 25) = 0$

80. The equation of radical axis is
$(x^2 + y^2 + 2x + 3y + 1) - (x^2 + y^2 + 4x + 3y + 2) = 0$
$\Rightarrow \quad (2x + 3y + 1) - (4x + 3y + 2) = 0$
$\Rightarrow \quad 2x + 1 = 0$
$\Rightarrow \quad x = -\frac{1}{2}$
Thus, the equation of the circle is
$(x^2 + y^2 + 2x + 3y + 1) + \lambda(2x + 1) = 0$
$\Rightarrow \quad (x^2 + y^2 + 2(\lambda + 1)x + 3y + (1 + \lambda)) = 0$
Since AB is diameter of the circle, so the centre lies on it.
Therefore, $-2\lambda - 2 + 1 = 0$
$\Rightarrow \quad \lambda = -\frac{1}{2}.$
Hence, the equation of the circle is
$(x^2 + y^2 + 2x + 3y + 1) - \frac{1}{2}(2x + 1) = 0$
$\Rightarrow \quad (x^2 + y^2 + 2x + 6y + 1) = 0$

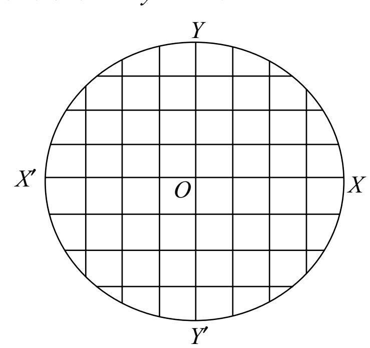
81. Any circle passing through the point of intersection of the given line and the circle is
$x^2 + y^2 - 9 + \lambda(x + y - 1) = 0$
$\Rightarrow \quad x^2 + y^2 + \lambda x + \lambda y - (9 + \lambda) = 0$
So the centre is $\left(-\frac{\lambda}{2}, -\frac{\lambda}{2}\right)$.
Since, the circle is smallest, so the centre $\left(-\frac{\lambda}{2}, -\frac{\lambda}{2}\right)$ lies on the chord $x + y = 1$.
Therefore, $-\frac{\lambda}{2} - \frac{\lambda}{2} = 1 \Rightarrow \lambda = -1.$
Hence, the equation of the smallest circle is
$x^2 + y^2 - 9 - (x + y - 1) = 0$
$\Rightarrow \quad x^2 + y^2 - x - y - 8 = 0$

82. The equation of any circle passing through the point of intersection of the given circle is
$(x^2 + y^2 - 6x + 2y + 4) + \lambda(x^2 + y^2 + 2x - 4y - 6) = 0$
$\Rightarrow \quad (1 + \lambda)x^2 + (1 + \lambda)y^2 + 2(\lambda - 3)x + 2(1 - 2\lambda)y + 2(2 - 3\lambda) = 0$
$\Rightarrow \quad x^2 + y^2 + 2\left(\frac{\lambda - 3}{\lambda + 1}\right)x + 2\left(\frac{1 - 2\lambda}{\lambda + 1}\right)y + \left(\frac{2(2 - 3\lambda)}{\lambda + 1}\right) = 0$
So its centre is $\left(\frac{3 - \lambda}{\lambda + 1}, \frac{2\lambda - 1}{\lambda + 1}\right)$.
Since, the centre lies on the line $y = x$,
$\frac{2\lambda - 1}{\lambda + 1} = \frac{3 - \lambda}{\lambda + 1}$
$\Rightarrow \quad 3\lambda = 4$
$\Rightarrow \quad \lambda = 4/3$
Hence, the equation of the circle is $x^2 + y^2 - \frac{10}{7}x - \frac{10}{7}y - \frac{12}{7} = 0.$

83. The given circles are $x^2 + y^2 + 2x + 3y + 1 = 0$ and $x^2 + y^2 + 4x + 3y + 2 = 0$.
Hence, the equation of the common chord is $2x + 1 = 0$.
Therefore, the equation of the circle is
$(x^2 + y^2 + 2x + 3y + 1) + \lambda(x^2 + y^2 + 4x + 3y + 2) = 0 \quad \ldots(i)$
$\Rightarrow \quad (1 + \lambda)x^2 + (1 + \lambda)y^2 + 2(1 + 2\lambda)x + 3(1 + \lambda)y + (1 + 2\lambda) = 0$
$\Rightarrow \quad x^2 + y^2 + 2\left(\frac{1 + 2\lambda}{(1 + \lambda)}\right)x + 3y + \frac{(1 + 2\lambda)}{(1 + \lambda)} = 0$
So, its centre is $\left(-\frac{1 + 2\lambda}{\lambda + 1}, -\frac{3}{2}\right)$
Since $2x + 1 = 0$ is the diameter, so centre lies on it.
Therefore,
$2\left(-\frac{1 + 2\lambda}{\lambda + 1}\right) + 1 = 0$
$\Rightarrow \quad -2 - 4\lambda + \lambda + 1 = 0$
$\Rightarrow \quad \lambda = -\frac{1}{3}$
Hence, the equation of the circle (i) is
$2(x^2 + y^2) + 2x + 6y + 1 = 0$

Level III

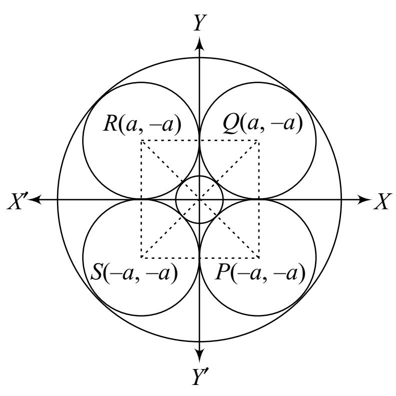
1. Given circle is $x^2 + y^2 = 16$.



Thus, the number of integral points inside the circle
$= 5 + 7 + 7 + 7 + 7 + 7 + 5$
$= 45$

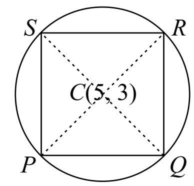
2. Let $S_1: x^2 + y^2 + 2gx + 2fy + c = 0$
$S_2: x^2 + y^2 = 4$
$S_3: x^2 + y^2 - 6x - 8y + 10 = 0$
and $S_4: x^2 + y^2 + 2x - 4y - 2 = 0$
Now, $S_1 - S_2 = 0$
$2gx + 2fy + c + 4 = 0$ …(i)
Centre of a circle $x^2 + y^2 = 4$ lies on (i)
So, $c + 4 = 0$
$c = -4$ …(ii)
Again, $S_1 - S_3 = 0$
$2(g + 3)x + 2(f + 4)y + (c - 10) = 0$ …(iii)
So, the centre of a circle $x^2 + y^2 - 6x - 8y + 10 = 0$ lies on (ii), so
$\Rightarrow$ $2(g + 3)3 + 2(f + 4)4 + (-4 - 10) = 0$
$\Rightarrow$ $2(g + 3)3 + 2(f + 4)4 + -14 = 0$
$\Rightarrow$ $6g + 8f + 18 + 32 - 14 = 0$
$\Rightarrow$ $6g + 8f + 36 = 0$
$\Rightarrow$ $3g + 4f + 18 = 0$ …(iv)
Also, $S_1 - S_4 = 0$
$\Rightarrow$ $2(g - 1)x + 2(f + 2)y + (c + 2) = 0$
So, centre $(-1, 2)$ lies on (iii).
Thus, $-2(g - 1) + 4(f + 2) + (-4 + 2) = 0$
$\Rightarrow$ $-2(g - 1) + 4(f + 2) - 2 = 0$
$\Rightarrow$ $(g - 1) - 2(f + 2) + 1 = 0$
$\Rightarrow$ $g - 2f = 4$ …(v)
On solving, we get
$g = -2$ and $f = -3$
Hence, the equation of the circle is
$x^2 + y^2 - 4x - 6y - 4 = 0$

3. Given circle is $(x \pm a)^2 + (y \pm a)^2 = a^2$



Hence, the radius of the smallest circle
$= (\sqrt{a^2 + a^2} - a), (\sqrt{a^2 + a^2} + a)$
$= a\sqrt{2} - a, a\sqrt{2} + a$
$= (\sqrt{2} - 1)a, (\sqrt{2} + 1)a$

4. Given circle is



$x^2 + y^2 - 10x - 6y + 30 = 0$
$\Rightarrow$ $(x - 5)^2 + (y - 3)^2$
$= 25 + 9 - 30$
$\Rightarrow$ $(x - 5)^2 + (y - 3)^2 = 4$
Clearly, CP = radius = 2
Let PQ is parallel to $y = x + 3$
Therefore, the co-ordinates of P and R are obtained by

$$\Rightarrow \quad \frac{x-5}{\cos(45°)} = \frac{y-3}{\sin(45°)} = \pm 2$$

$$\Rightarrow \quad \frac{x-5}{\frac{1}{\sqrt{2}}} = \frac{y-3}{\frac{1}{\sqrt{2}}} = \pm 2$$

$$\Rightarrow \quad x - 5 = y - 3 = \pm\frac{2}{\sqrt{2}}$$

$$\Rightarrow \quad x - 5 = y - 3 = \pm\sqrt{2}$$

$$\Rightarrow \quad x = 5 \pm \sqrt{2}, y = 3 \pm \sqrt{2}$$

Thus, $R = (5 + \sqrt{2}, 3 + \sqrt{2})$
and $Q = (5 - \sqrt{2}, 3 - \sqrt{2})$.

5. Clearly,
$(2\sqrt{2})^2 + (2\sqrt{2})^2 = a^2$
$\Rightarrow$ $a^2 = (2\sqrt{2})^2 + (2\sqrt{2})^2$
$\Rightarrow$ $a^2 = 8 + 8 = 16$
$\Rightarrow$ $|a| = 4$

6. Given circle is
$x^2 + y^2 - 8x + 2y + 12 = 0$
$\Rightarrow$ $(x - 4)^2 + (y + 1)^2 = 16 + 1 - 12$
$\Rightarrow$ $(x - 4)^2 + (y + 1)^2 = 5$
As we know that the line is a chord, tangent or does not meet the circle at all, if
$p < r, p = r$ or $p > 4$,
where p is the length of the perpendicular from the centre to the line.

$$\text{So,} \quad p = \left|\frac{4 + 2 - 1}{\sqrt{1 + 4}}\right| = \frac{5}{\sqrt{5}} = \sqrt{5} = r$$

Thus, the line be a tangent to the circle.

7. It is given that
$C_1C_2 = r_1 + r_2$

$$\Rightarrow \quad \sqrt{(a-b)^2 + (b-a)^2} = c + c$$

$$\Rightarrow \quad \sqrt{2(a-b)^2} = 2c$$

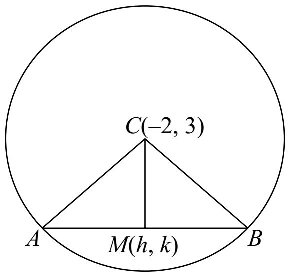
$\Rightarrow$ $2(a - b)^2 = 4c^2$
$\Rightarrow$ $(a - b)^2 = 2c^2$
$\Rightarrow$ $a - b = \pm c\sqrt{2}$
$\Rightarrow$ $a = b \pm c\sqrt{2}$

8. Given circle is

$$x^2 + y^2 + 4x - 6y - 12 = 0$$
$$(x + 2)^2 + (y - 3)^2 = 5^2$$

Thus, $C = (-2, 3)$ and $CA = 5 = CB$

and $\angle CAB = \frac{\pi}{3}$



Now, $\sin\left(\frac{\pi}{3}\right) = \frac{MC}{5}$

$\Rightarrow \quad \frac{\sqrt{3}}{2} = \frac{MC}{5}$

$\Rightarrow \quad MC = \frac{5\sqrt{3}}{2}$

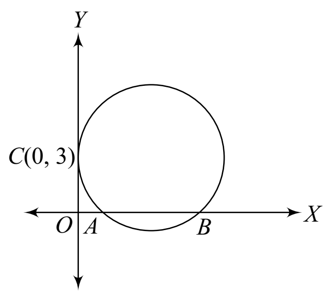
$\Rightarrow \quad MC^2 = \frac{75}{4}$

$\Rightarrow \quad (h+2)^2 + (k-3)^2 = \frac{75}{4}$

Hence, the locus of $M(h, k)$ is

$$(x+2)^2 + (y-3)^2 = \frac{75}{4}$$

9.



Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \ldots\text{(i)}$$

which touches the y-axis at C.

Put $x = 0$ in Eq. (i), we get

$$y^2 + 2fy + c = (y - 3)^2$$

$\Rightarrow \quad y^2 + 2fy + c = y^2 - 6y + 9$

$\Rightarrow \quad 2fy + c = -6y + 9$

Comparing the co-efficients, we get

$$2f = -6, c = 9$$

$\Rightarrow \quad f = -3, c = 9$

Intercepts on x-axis is

$\Rightarrow \quad 2\sqrt{g^2 - c} = 8$

$\Rightarrow \quad \sqrt{g^2 - c} = 4$

$\Rightarrow \quad \sqrt{g^2 - 9} = 4$

$\Rightarrow \quad g^2 - 9 = 16$

$\Rightarrow \quad g^2 = 25$

$\Rightarrow \quad g = \pm 5$

Hence, the equation of the circle is

$$x^2 + y^2 \pm 10x - 6y + 9 = 0$$

10. The equation of the chord of contact of tangents to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

from $(0, 0)$ is

$$x \cdot 0 + y \cdot 0 + g(x + 0) + f(y + 0) + c = 0$$
$$gx + fy + c = 0 \quad \ldots\text{(i)}$$

Hence, the required distance from the point (g, f) to the chord of contact (i)

$$= \left(\frac{g^2 + f^2 + c}{\sqrt{g^2 + f^2}}\right).$$

11. We have,

$$(1 + \alpha x)^n = 1 + 8x + 24x^2 + \cdots$$
$$1 + n(\alpha x) + \frac{n(n-1)}{2}(\alpha x)^2 + \cdots = 1 + 8x + 24x^2 + \cdots$$

Comparing the co-efficients, we get

$\Rightarrow \quad n\alpha = 8, \frac{n\alpha(n\alpha - \alpha)}{2} = 24$

$$\frac{8(8-\alpha)}{2} = 24$$

$\Rightarrow \quad (8 - \alpha) = 6$

$\Rightarrow \quad \alpha = 2, n = 4$

Thus, the point P is $(2, 4)$

Therefore, $PA \cdot PB$

$$= (PT)^2 = 4 + 16 - 4 = 16$$

12. Given circles are $x^2 + y^2 = 4$

and $\quad x^2 + y^2 + 2x + 3y - 5 = 0$

Hence, the common chord is

$$2x + 3y = 1 \quad \ldots\text{(i)}$$

Thus, the equation of the circle is

$$S + \lambda L = 0$$

$\Rightarrow \quad (x^2 + y^2 - 4) + \lambda(2x + 3\lambda - 1) = 0$

$\Rightarrow \quad x^2 + y^2 + 2\lambda x + 3\lambda y - (\lambda + 4) = 0$

So centre of the circle is

$$\left(-\lambda, -\frac{3\lambda}{2}\right)$$

Since the chord $2x + 3y = 1$ is a diameter of the circle, so

$$-2\lambda - \frac{9\lambda}{2} = 1$$

$\Rightarrow \quad -13\lambda = 2$

$\Rightarrow \quad \lambda = -\frac{2}{13}$

Hence, the equation of the circle is

$$x^2 + y^2 + 2\left(-\frac{2}{13}\right)x + 3\left(-\frac{2}{13}\right)y - \left(4 - \frac{2}{13}\right) = 0$$

$\Rightarrow \quad 13(x^2 + y^2) - 4x - 6y - 50 = 0$

Therefore, the centre of the circle is $\left(\frac{2}{13}, \frac{3}{13}\right)$.

13. Let (h, k) be a point lies on the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Thus, the length of the tangent from (h, k)

$$= \sqrt{h^2 + k^2 + 2gh + 2fk + d}$$
$$= \sqrt{(h^2 + k^2 + 2gh + 2fk) + \mathrm{d}}$$
$$= \sqrt{-c + d}$$
$$= \sqrt{d - \mathrm{c}}$$

14. Equation of the circle is

$$S + \lambda L = 0$$
$$\Rightarrow \quad (x^2 + y^2 - 4) + \lambda(x + y - 1) = 0$$
$$\Rightarrow \quad x^2 + y^2 + \lambda x + \lambda y - (\lambda + 4) = 0$$

So centre of the circle is $\left(-\frac{\lambda}{2}, -\frac{\lambda}{2}\right)$

Since the chord $x + y = 1$ is a diameter of the circle, so

$$-\frac{\lambda}{2} - \frac{\lambda}{2} = 1$$
$$\Rightarrow \quad -\lambda = 1$$
$$\Rightarrow \quad \lambda = -1$$

Hence, the equation of the circle is

$$x^2 + y^2 - x - y - 3 = 0$$

15. Now the image of the centre $(3, 2)$ to the line $x + y = 19$ is obtained by

$$\frac{\alpha - 3}{1} = \frac{\beta - 2}{1} = -2\left(\frac{3 + 2 - 19}{1^2 + 1^2}\right)$$
$$\frac{\alpha - 3}{1} = \frac{\beta - 2}{1} = 14$$
$$\alpha = 17, \beta = 16$$

Hence, the equation of the new circle is

$$(x - 17)^2 + (y - 16)^2 = 1$$

16. Given circles are

$$(x - 1)^2 + (y - 3)^2 = r^2 \quad \ldots\text{(i)}$$
and $\quad x^2 + y^2 - 8x + 2y + 8 = 0$
$$\Rightarrow \quad (x - 4)^2 + (y + 1)^2 = 16 + 1 - 8$$
$$\Rightarrow \quad (x - 4)^2 + (y + 1)^2 = 9 \quad \ldots\text{(ii)}$$

Two circles (i) and (ii) intersect, if

$$|r_1 - r_2| < C_1C_2 < r_1 + r_2$$
$$\Rightarrow \quad |r - 3| < \sqrt{(1 - 4)^2 + (3 + 1)^2} < r + 3$$
$$\Rightarrow \quad |r - 3| < 5 < r + 3$$
$$\Rightarrow \quad |r - 3| < 5 \text{ and } r + 3 > 5$$
$$\Rightarrow \quad -5 < (r - 3) < 5, r > 2$$
$$\Rightarrow \quad -2 < r < 5 + 3, r > 2$$
$$\Rightarrow \quad r < 8, r > 2$$
$$\Rightarrow \quad 2 < r < 8$$

17. Since there are two real tangents drawn, so the circles intersect.

Given circles are $(x - 1)^2 + (y - 1)^2 = 2$

and $\quad (x - 4)^2 + (y - 4)^2 = (32 - \lambda)$

It is given that $|r_1 - r_2| < C_1C_2 < r_2 + r_2$

$$|\sqrt{2} - \sqrt{32 - \lambda}| < 3\sqrt{2} < (\sqrt{2} + \sqrt{32 - \lambda})$$

Now, $3\sqrt{2} < (\sqrt{2} + \sqrt{32 - \lambda})$

$$\Rightarrow \quad \sqrt{32 - \lambda} > 2\sqrt{2}$$
$$\Rightarrow \quad 32 - \lambda > 8$$
$$\Rightarrow \quad \lambda < 24 \quad \ldots\text{(i)}$$
and $\quad |\sqrt{2} - \sqrt{32 - \lambda}| < 3\sqrt{2}$
$$\Rightarrow \quad -4\sqrt{2} < -\sqrt{32 - \lambda} < 2\sqrt{2}$$
$$\Rightarrow \quad -2\sqrt{2} < \sqrt{32 - \lambda} < 4\sqrt{2}$$
$$\Rightarrow \quad 8 < (32 - \lambda) < 32$$
$$\Rightarrow \quad -24 < -\lambda < 0$$
$$\Rightarrow \quad 0 < \lambda < 24 \quad \ldots\text{(ii)}$$

From Eqs (i) and (ii), we get

$$0 < \lambda < 24$$

18. Since two vertices of an equilateral triangle are $B(-1, 0)$ and $C(1, 0)$.

So, the third vertex must lie on the y-axis.

Let the third vertex be $A(0, b)$.

Now, $AB = BC = CA$

$$\Rightarrow \quad AB^2 = BC^2 = AC^2$$
$$\Rightarrow \quad 1 + b^2 = 4 = 1 + b^2$$
$$\Rightarrow \quad b^2 = 4 - 1$$
$$\Rightarrow \quad b = \sqrt{3}$$

Thus, the third vertex is $A = (0, \sqrt{3})$

As we know that in case of an equilateral triangle,

Circumcentre = Centroid = $\left(0, \frac{1}{\sqrt{3}}\right)$

Hence, the equation of the circumcircle is

$$\Rightarrow \quad (x - 0)^2 + \left(y - \frac{1}{\sqrt{3}}\right)^2 = (1 - 0)^2 + \left(0 - \frac{1}{\sqrt{3}}\right)^2$$
$$\Rightarrow \quad x^2 + \left(y - \frac{1}{\sqrt{3}}\right)^2 = \frac{4}{3}$$

19. The vertices of the triangle are

$$A = (0, 6), B = (2\sqrt{3}, 0), C = (0, 2\sqrt{3})$$

Let $P(h, k)$ be the circumcentre, then

$$PA = PB = PC$$
$$\Rightarrow \quad PA^2 = PB^2 = PC^2$$
$$\Rightarrow \quad h^2 + (k - 6)^2 = (h - 2\sqrt{3})^2 + k^2$$
$$= h^2 + (k - 2\sqrt{3})^2$$

On solving, we get

$$h = 0 \text{ and } k = 2$$

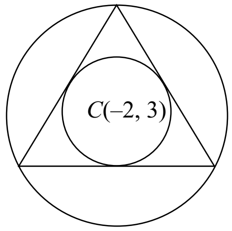
Thus, radius, $r = \sqrt{h^2 + (k - 6)^2} = 4$

Hence, the equation of the circle is

$$x^2 + (y - 2)^2 = 16$$
$$\Rightarrow \quad x^2 + y^2 - 4y - 12 = 0$$

20. Given circle is

$$x^2 + y^2 + 4x - 6y + 4 = 0$$
$$(x + 2)^2 + (y - 3)^2 = 3^2$$



So, $C = (-2, 3)$ and $r = 3$

As we know that the centroid divides the median in the ratio 2 : 1

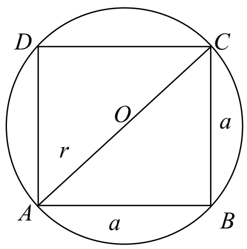
Here, circumradius = 2 and in-radius = 6

Hence, the equation of the circumcircle is

$$(x + 2)^2 + (y - 3)^2 = 6^2$$
$$\Rightarrow \quad x^2 + y^2 + 4x - 6y - 23 = 0$$

21. Given circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0$$



We have,

$$a^2 + a^2 = (r + r)^2$$
$$\Rightarrow \quad a^2 = 2r^2$$
$$\Rightarrow \quad a^2 = 2r^2$$
$$\Rightarrow \quad a = r\sqrt{2}$$

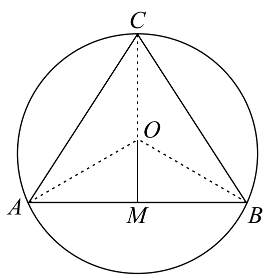
Hence, the result.

22. Given circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Here, $OA = r$, $\angle AOM = 60°$

Now,



$$\sin(60°) = \frac{AM}{OA} = \frac{AM}{r}$$
$$\Rightarrow \quad \frac{AM}{r} = \frac{\sqrt{3}}{2}$$
$$\Rightarrow \quad AM = \frac{\sqrt{3}}{2} r$$
$$\Rightarrow \quad AB = 2AM = r\sqrt{3}$$

Hence, the area of an equilateral triangle is

$$= \frac{\sqrt{3}}{4} \times (AB)^2$$
$$= \frac{\sqrt{3}}{4} \times 3r^2$$
$$= \frac{3\sqrt{3}}{4} r^2$$

23. Given centre of the circle is (0, 0).

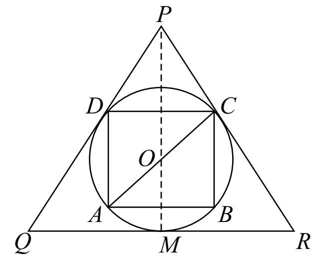
As we know that the centroid divides the median in the ratio 2 : 1

$\therefore$ Circumradius = $2a$

Hence, the equation of the circle is

$$x^2 + y^2 = (2a)^2$$
$$\Rightarrow \quad x^2 + y^2 = 4a^2$$

24. Let ABC be an equilateral triangle with $AB = a$.



Let $PM = p$

Now, $\sin(60°) = \dfrac{PM}{PQ} = \dfrac{p}{a}$

$$\Rightarrow \quad \frac{p}{a} = \frac{\sqrt{3}}{2}$$
$$\Rightarrow \quad p = \frac{\sqrt{3}}{2} a$$

Here O is the centroid.

So the centroid divides the median in the ratio 2 : 1

Thus, $OM = \dfrac{p}{3}$

$$\Rightarrow \quad r = \frac{p}{3}$$
$$\Rightarrow \quad 2r = \frac{2p}{3} = \frac{2}{3} \times \frac{\sqrt{3}}{2} a = \frac{a}{\sqrt{3}}$$

Let x be the side of a square.

Thus,

$$x^2 + x^2 = \left(\frac{a}{\sqrt{3}}\right)^2 = \frac{a^2}{3}$$
$$\Rightarrow \quad 2x^2 = \frac{a^2}{3}$$
$$\Rightarrow \quad x^2 = \frac{a^2}{6}$$

Area of the square = $\dfrac{a^2}{6}$

25. Given circle is

$$x^2 + y^2 - 12x + 4y + 30 = 0$$
$$\Rightarrow \quad (x - 6)^2 + (y + 2)^2 = 36 + 4 - 30$$
$$\Rightarrow \quad (x - 6)^2 + (y + 2)^2 = (\sqrt{10})^2$$

Any point on the circle is

$$P(6 + \sqrt{10}\cos\theta, -2 + \sqrt{10}\sin\theta)$$

Let d be the distance from the origin

Thus, $d = OP$

$$d^2 = (6 + \sqrt{10}\cos\theta)^2 + (-2 + \sqrt{10}\sin\theta)^2$$
$$= 50 + 4\sqrt{10}(3\cos\theta - \sin\theta)$$
$$= 40 + 4.10\left(\frac{3}{\sqrt{10}}\cos\theta - \frac{1}{\sqrt{10}}\sin\theta\right)$$
$$= 40 + 40(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$$
$$= 40 + 40(\cos(\theta + \alpha))$$
$$= 40 + 40, \text{ when } \cos(\theta + \alpha) = 1 = \cos 0°$$
$$= 80 \text{ and } \theta = -\alpha$$

Hence, the point is

$$= (6+\sqrt{10}\cos\theta, -2+\sqrt{10}\sin\theta)$$

$$= \left(6+\sqrt{10}\cdot\frac{3}{\sqrt{10}}, -2+\sqrt{10}\cdot\left(-\frac{1}{\sqrt{10}}\right)\right)$$

$$= (6+3, -2-1)$$

$$= (9, -3)$$

26.

27. The equation of any tangent through origin is

$$y = mx$$

If $y = mx$ be a tangent to the given circle, then

$$\left|\frac{7m+1}{\sqrt{m^2+1}}\right| = 5$$

$\Rightarrow \quad 25(m^2+1) = (7m+1)^2$

$\Rightarrow \quad 25m^2 + 25 = 46m^2 + 14m + 1$

$\Rightarrow \quad 24m^2 - 14m - 24 = 0$

Let its roots are m_1, m_2.

So, product of the roots = –1

$\Rightarrow \quad m_1 \cdot m_2 = -1$

$\Rightarrow \quad \theta = \dfrac{\pi}{2}$

28. Given circle is

$$x^2 + y^2 - 2x - 4y = 0$$

$\Rightarrow \quad (x-1)^2 + (y-2)^2 = 5 \qquad \text{...(i)}$

Equation of any tangent passing through (4, 3) is

$$y - 3 = m(x-4)$$

$\Rightarrow \quad mx - y + (3-4m) = 0 \qquad \text{...(ii)}$

If (ii) is a tangent of (i), then

$\Rightarrow \quad \left|\dfrac{m-2+3-4m}{\sqrt{m^2+1}}\right| = \sqrt{5}$

$$\left|\frac{1-3m}{\sqrt{m^2+1}}\right| = \sqrt{5}$$

$\Rightarrow \quad 5(m^2+1) = (1-3m)^2$

$\Rightarrow \quad 5m^2 + 5 = 1 - 6m + 9m^2$

$\Rightarrow \quad 4m^2 - 6m - 4 = 0$

$\Rightarrow \quad 2m^2 - 3m - 2 = 0$

Let its roots are m_1, m_2.

So, product of the roots = –1

$\Rightarrow \quad m_1 \cdot m_2 = -1$

$\Rightarrow \quad \theta = \dfrac{\pi}{2}$

29. Given circle is

$$x^2 + y^2 - 2x - 4y = 0$$

$\Rightarrow \quad (x-1)^2 + (y-2)^2 = 5 \qquad \text{...(i)}$

The equation of any tangent passing through (0, 1) is

$$y = m(x-1)$$

$\Rightarrow \quad mx - y - m = 0 \qquad \text{...(ii)}$

If (ii) is a tangent of (i), then

$\Rightarrow \quad \left|\dfrac{m-2-m}{\sqrt{m^2+1}}\right| = \sqrt{5}$

$\Rightarrow \quad \left|\dfrac{-2}{\sqrt{m^2+1}}\right| = \sqrt{5}$

$\Rightarrow \quad 5(m^2+1) = 2$

$\Rightarrow \quad (m^2+1) = \dfrac{2}{5}$

$\Rightarrow \quad m^2 = \dfrac{2}{5} - 1 = -\dfrac{3}{5}$

$\Rightarrow \quad m = \varphi$

So, no real tangents can be drawn.

Thus, number of tangents = 0.

30. The equation of the tangent to the circle $x^2 + y^2 = a^2$ at $(a\cos\theta, a\sin\theta)$ and $\left[a\cos\left(\frac{\pi}{3}+\theta\right), a\sin\left(\frac{\pi}{3}+\theta\right)\right]$ are $x\cos\theta + y\sin\theta = a$

and $\quad x\cos\left(\dfrac{\pi}{3}+\theta\right) + y\sin\left(\dfrac{\pi}{3}+\theta\right) = a$

Let (h, k) be the point of intersection.

On solving, we get

$$h = \frac{2a\left(\sin\left(\frac{\pi}{3}+\theta\right) - \sin\theta\right)}{\sqrt{3}}$$

and $\quad k = \dfrac{2a\left(\cos\left(\frac{\pi}{3}+\theta\right) - \cos\theta\right)}{\sqrt{3}}$

Now, squaring and adding, we get

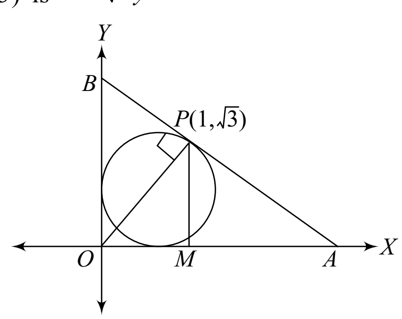
$$\frac{3h^2}{4a^2} + \frac{3k^2}{4a^2} = 1$$

$\Rightarrow \quad h^2 + k^2 = \dfrac{4a^2}{3}$

Hence, the locus of (h, k) is

$$x^2 + y^2 = \frac{4a^2}{3}$$

31. The equation of tangent to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$ is $x + \sqrt{3}y = 4$



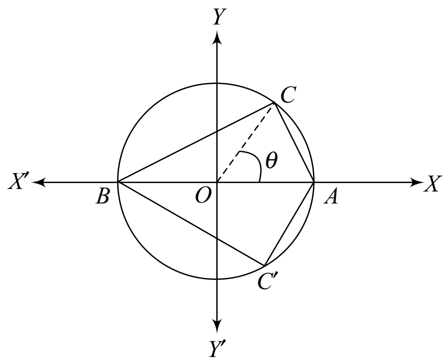
Clearly, $A = (4, 0), B = (0, 4/\sqrt{3})$

Thus, the area of the triangle OPA is

$$= \frac{1}{2} \times 4 \times \sqrt{3} = 2\sqrt{3} \text{ sq. u.}$$

Level IV

1. (i) Here, $AB = 13$ m and $AC = 5$ m



Let the co-ordinates of C be $\left(\frac{13}{2}\cos\theta, \frac{13}{2}\sin\theta\right)$

It is given that $AC = 5$

$\Rightarrow AC^2 = 25$

$$\Rightarrow \left(\frac{13}{2}\cos\theta - \frac{13}{2}\right)^2 + \left(\frac{13}{2}\sin\theta\right)^2 = 25$$

$$\Rightarrow \cos\theta = \frac{238}{338} = \frac{119}{159}$$

$$\Rightarrow \sin\theta = \sqrt{1 - \left(\frac{119}{159}\right)^2} = \frac{120}{169}$$

$$\Rightarrow \theta = \sin^{-1}\left(\frac{120}{169}\right)$$

The co-ordinates of C and C' will become

$$\left(\frac{119}{26}, \frac{60}{13}\right) \text{ and } \left(\frac{119}{26}, -\frac{60}{13}\right)$$

Thus, the co-ordinates of B are $\left(-\frac{13}{2}, 0\right)$

Hence, the equations of the pair of lines BC and BC' are

$$y - 0 = \frac{\pm(60/13)}{\left(\frac{119}{26} + \frac{13}{2}\right)}\left(x + \frac{13}{2}\right)$$

$$\Rightarrow y = \pm\frac{15}{36}\left(x + \frac{13}{2}\right)$$

$$\Rightarrow y = \pm\frac{5}{12}\left(x + \frac{13}{2}\right)$$

$$\Rightarrow 24y = \pm 5(2x + 13)$$

(ii) Area of the sector OAC

$$= \frac{1}{2}r^2\theta$$

$$= \frac{1}{2}\left(\frac{13}{2}\right)^2 \sin^{-1}\left(\frac{120}{169}\right)$$

$$= \frac{169}{8}\sin^{-1}\left(\frac{120}{169}\right) \text{ sq. m.}$$

Area of the triangle $OAC = \frac{1}{2}r^2 \sin\theta$

$$= \frac{1}{2}\left(\frac{13}{2}\right)^2\left(\frac{120}{169}\right)$$

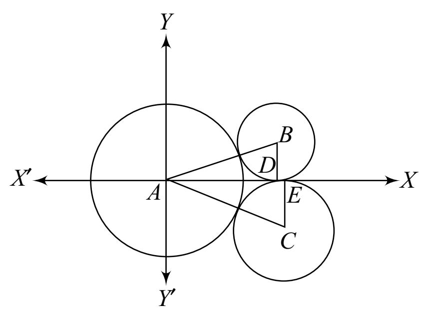
$$= 15 \text{ sq. m.}$$

Thus, the area of the smaller portion bounded by the circle and the chord AC.

= Area of sector OAC – Area of ΔAOC

$$= \left\{\frac{169}{8}\sin^{-1}\left(\frac{120}{169}\right) - 15\right\} \text{ sq. m.}$$

2. Here, the co-ordinates of the centres A, B and C are $(0, 0)$, $(\sqrt{55}, 3)$ and $(\sqrt{65}, 4)$.



3. The equation of the radical axis is

$$(x^2 + y^2 + 4x + 2y + 1) - \left(x^2 + y^2 - x + 3y - \frac{3}{2}\right) = 0$$

$$\Rightarrow 5x - y + \frac{5}{2} = 0$$

$$10x - 2y + 5 = 0 \quad \text{...(i)}$$

The equation of the co-axial circle is

$$\left(x^2 + y^2 - x + 3y - \frac{3}{2}\right) + \lambda(x^2 + y^2 + 4x + 2y + 10) = 0$$

$$x^2 + y^2 + \left(\frac{4\lambda - 1}{\lambda + 1}\right)x + \left(\frac{2\lambda + 3}{\lambda + 1}\right)y + \left(\frac{\lambda - 3/2}{\lambda + 1}\right) = 0 \quad \text{...(ii)}$$

Thus, $\left[\frac{1}{2}\left(\frac{4\lambda - 1}{\lambda + 1}\right), \frac{1}{2}\left(\frac{2\lambda + 3}{\lambda + 1}\right)\right]$

Clearly the centre lies on the radical axis.

So $\quad 10\left(\frac{1}{2}\left(\frac{4\lambda-1}{\lambda+1}\right)\right)+2\left(\frac{1}{2}\left(\frac{2\lambda+3}{\lambda+1}\right)\right)+5=0$

$\Rightarrow \quad \lambda = 1.$

Put the value of $\lambda = 1$ in Eq. (ii), we get

$4x^2 + 4y^2 + 6x + 10y - 1 = 0.$

4. A circle passing through the point of intersection of

$x^2 + y^2 + ax + by$ and $Ax + By + C = 0$ is

$x^2 + y^2 + ax + by + c + \lambda(Ax + By + C) = 0 \quad \ldots\text{(i)}$

and a circle passing through the point of intersection

$x^2 + y^2 + a'x + b'y + c' = 0$ and

$A'x + B'y + C' = 0$ is

$x^2 + y^2 + a'x + b'y + c' + \lambda' (a'x + B'y + C') = 0 \quad \ldots\text{(ii)}$

Since the point of intersection lie on the circle, so,

$a + \lambda A = a' + \lambda' A'$

$b + \lambda B = b' + \lambda' B'$

and $\quad c + \lambda C = c' + \lambda' C'$

Eliminating λ and λ', we get

$$\Rightarrow \quad \begin{vmatrix} A & A' & a-a' \\ B & B' & b-b' \\ C & C' & c-c' \end{vmatrix} = 0$$

$$\Rightarrow \quad A\begin{vmatrix} B' & b-b' \\ C' & c-c' \end{vmatrix} - B\begin{vmatrix} A' & a-a' \\ C' & c-c' \end{vmatrix} + C\begin{vmatrix} A' & a-a' \\ B' & b-b' \end{vmatrix} = 0$$

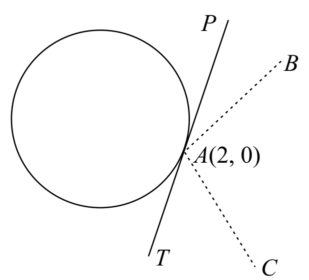
$$\Rightarrow \quad AB'(c-c') - AC'(b-b') - BA'(c-c') + BC'(a-a') + CA'(b-b') - CB'(a-a') = 0$$

$$\Rightarrow \quad (a-a')(BC' - CB') + (b-b')(CA' - AC') + (c-c')(AB' - BA') = 0$$

5. The equation of the tangent to the given circle

$(x + 2)^2 + (y + 3)^2 = 25$ at $A(2, 0)$ is

$4x - 3y - 8 = 0 \quad \ldots\text{(i)}$



Let m and m' be the slopes of the lines AB and AC.

Thus, $\tan(45°) = \dfrac{m - (4/3)}{1 + m \cdot (4/3)}$

and $\quad \tan(135°) = \dfrac{m' - (4/3)}{1 + m' \cdot (4/3)}$

On solving, we get

$m = -7, m' = \dfrac{1}{7}$

Therefore, the equations of the lines AB and AC are

$y - 0 = 7(x - 2)$ and $y - 0 = \dfrac{1}{7}(x - 2)$

$\Rightarrow \quad 7x + y = 14$ and $x - 7y = 2$

Now, the centres of the circles lie on the lines AB and AC at a distance of $5\sqrt{2}$ units.

Thus C_1: $\dfrac{x-2}{\frac{1}{5\sqrt{2}}} = \dfrac{y-0}{\frac{7}{5\sqrt{2}}} = 5\sqrt{2}$

$(x, y) = (1, 7)$

and $\quad C_2$: $\dfrac{x-2}{\frac{7}{5\sqrt{2}}} = \dfrac{y-0}{\frac{1}{5\sqrt{2}}} = 5\sqrt{2}$

$(x, y) = (9, 1)$

Hence, the equations of the circles are

$(x - 1)^2 + (y - 7)^2 = 9$

or $\quad (x - 7)^2 + (y - 1)^2 = 9$

6. Let the centre of the circle be (h, k) such that $k = h - 1$.

We have, $(h - 7)^2 + (k - 3)^2 = 3^2$

$\Rightarrow \quad (h - 7)^2 + (h - 4)^2 = 3^2$

$\Rightarrow \quad h^2 = 11h + 28 = 0$

$\Rightarrow \quad h = 7, 4$ and $k = 6, 3$

Hence, the equation of a circle can be

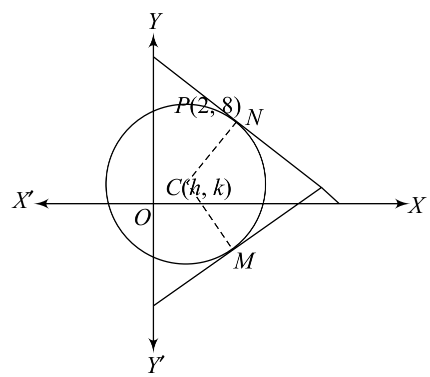
$(x - 7)^2 + (y - 6)^2 = 3^2$

and $\quad (x - 4)^2 + (y - 3)^2 = 3^2$

$\Rightarrow \quad x^2 + y^2 - 8x - 6y + 16 = 0$

and $\quad x^2 + y^2 - 8x - 6y + 16 = 0$

7. Here, $\quad CM = CN$



$$\Rightarrow \quad \left|\frac{4h - 3k - 24}{\sqrt{16+9}}\right| = \left|\frac{4h + 3k - 42}{\sqrt{16+9}}\right|$$

$\Rightarrow \quad (4h - 3k - 24) = \pm(4h + 3k - 42)$

$\Rightarrow \quad k = 3$

Also, $r = CM$

$\Rightarrow \quad r^2 = CM^2$

$$\Rightarrow \quad (h-2)^2 + (k-8)^2 = \left(\frac{4h+3k-42}{5}\right)^2$$

$$\Rightarrow \quad (h-2)^2 + (3-8)^2 = \left(\frac{4h+9-42}{5}\right)^2$$

$$\Rightarrow \quad (h-2)^2 + 25 = \left(\frac{4h-33}{5}\right)^2$$

On solving we get,

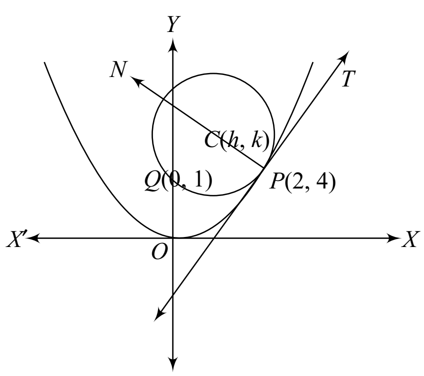
$$h = 2$$

Thus, the equation of the circle is

$$(x-2)^2 + (y-3)^2 = 25$$

$$\Rightarrow \quad x^2 + y^2 - 2x - 6y - 12 = 0$$

8. Let x_1, x_2 are the roots of $x^2 + 2x - a^2 = 0$.
   Then $x_1 + x_2 = -2$, $x_1x_2 = -a^2$
   Similarly, $y_1 + y_2 = -4$, $y_1y_2 = -b^2$
   Hence, the equation of the circle is
   $$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$
   $$x^2 - (x_1 + x_2)x + x_1x_2 + y^2 - (y_1 + y_2)y + y_1y_2 = 0$$
   $$x^2 + 2x - a^2 + y^2 + 4y - b^2 = 0$$
   $$x^2 + y^2 + 2x + 4y - (a^2 + b^2) = 0$$
   Therefore, the centre of the circle = (–1, –2) and the radius = $\sqrt{a^2 + b^2 + 5}$.
9. Equation of the tangent to the circle $x^2 + y^2 = 5$ at (1, –2) is $x - 2y = 5$.
   Given circle is
   $$x^2 + y^2 - 8x + 6y + 20 = 0$$
   Centre = (4, –3) and radius = $\sqrt{5}$.
   Let the point of contact be (h, k).
   Thus, $h - 2k = 5$ …(i)
   and $(h-4)^2 + (k+3)^2 = 5$
   $\Rightarrow \quad h^2 + k^2 - 8h + 6k = 20 = 0$ …(ii)
   On solving Eqs (i) and (ii), we get
   $$h = 3, k = -1$$
   Hence, the point of contact is (3, –1).
10. Given curve is $y = x^2$



$$\Rightarrow \quad \frac{dy}{dx} = 2x$$

Thus, $m = \left(\frac{dy}{dx}\right)_{/(2,4)} = 4$

The equation of the normal is

$$(y-4) = -\frac{1}{4}(x-2)$$

$\Rightarrow \quad 4(y-4) = -(x-2)$

$\Rightarrow \quad x + 4y = 18$

Let (h, k) be the centre of the circle.

Thus, $h + 4k = 18$ …(i)

Also, $CP = CQ$

$\Rightarrow \quad h^2 + (k-1)^2 = (h-2)^2 + (k-4)^2$

$\Rightarrow \quad -2k + 1 = -4h + 4 - 8k + 16$

$\Rightarrow \quad 4h + 6k = 19$ …(ii)

On solving Eqs (i) and (ii), we get

$$h = -\frac{16}{5}, k = \frac{53}{10}$$

Hence, the required centre is $\left(\frac{16}{5}, \frac{53}{10}\right)$.

11. Let the mid point be $M(h, k)$.
    Equation of the chord bisected at M is
    $$T = S_1$$
    $$hx + ky - (x + h) - 3(y + k) = h^2 + k^2 - 2h - 6k$$
    which is passing through the origin. So,
    $$-h - 3k = h^2 + k^2 - 2h - 6k$$
    $\Rightarrow \quad h^2 + k^2 - h - 3k = 0$
    Hence, the locus of (h, k) is
    $$x^2 + y^2 - x - 3y = 0$$
12. The equation of a circle passing through the points of intersection of $x^2 + y^2 = 1$
    $x^2 + y^2 - 2x - 4y + 1 = 0$ is
    $$(x^2 + y^2 - 2x - 4y + 1) + \lambda(x^2 + y^2 - 1) = 0$$
    $\Rightarrow \quad (1+\lambda)x^2 + (1+\lambda)y^2 - 2x - 4y + (1-\lambda) = 0$
    $\Rightarrow \quad x^2 + y^2 - \frac{2}{(1+\lambda)}x - \frac{4}{(1+\lambda)}y + \frac{(1-\lambda)}{(1+\lambda)} = 0$ …(i)
    $$\text{Centre} = \left(\frac{1}{(1+\lambda)}, \frac{2}{(1+\lambda)}\right)$$
    $$\text{Radius} = \sqrt{\left(\frac{1}{(1+\lambda)}\right)^2 + \left(\frac{2}{(1+\lambda)}\right)^2 + \left(\frac{(1-\lambda)}{(1+\lambda)}\right)^2}$$
    Equation (i) touching the straight line $x + 2y = 0$. So, the length of perpendicular from the centre is equal to the radius of a circle.
    Thus, $\frac{\sqrt{5}}{(1+\lambda)} = \sqrt{\left(\frac{1}{(1+\lambda)}\right)^2 + \left(\frac{2}{(1+\lambda)}\right)^2 + \left(\frac{(1-\lambda)}{(1+\lambda)}\right)^2}$
    $\Rightarrow \quad (1-\lambda)^2 + 5 = 5$
    $\Rightarrow \quad \lambda = 1$
    Hence, the equation of the circle is
    $$x^2 + y^2 - x - 2y = 0$$
13. As we know that the normal always passes through the centre of the circle.
    So, the equation of the circle is
    $$(x-3)^2 + y^2 = 6$$

14 Let the equations of the two given circles are

$$x^2 + y^2 + 2gx + g^2 = 0$$

and $x^2 + y^2 + 2fy + f^2 = 0$

whose centres lie on x and y axes.

Thus, the equation of the radical axis is

$$2gx + g^2 - 2fy - f^2 = 0$$

$\Rightarrow \quad 2gx - 2fy + g^2 - f^2 = 0$

15. Equations of the normals of the circle are

$$x + 2xy + 3x + 6y = 0$$

$\Rightarrow \quad (x + 2y)(x + 3) = 0$

$\Rightarrow \quad (x + 2y) = 0, (x + 3) = 0$

Thus, the centre of the circle is C_1: $\left(-3, \dfrac{3}{2}\right)$.

Given circle is

$$x(x - 4) + y(y - 3) = 0$$

$\Rightarrow \quad x^2 + y^2 - 4x - 3y = 0$

Centre is C_2: $\left(2, \dfrac{3}{2}\right)$ and the radius is $\dfrac{5}{2}$.

According to the questions, we get

$$C_1C_2 = r_1 - r_2$$

$\Rightarrow \quad \sqrt{(-3-2)^2 + 0} = r - \dfrac{5}{2}$

$\Rightarrow \quad r = 5 + \dfrac{5}{2} = \dfrac{15}{2}$

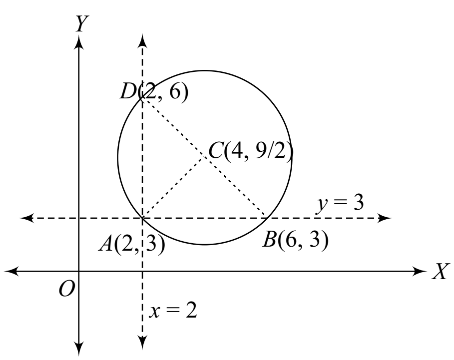
Hence, the equation of the circle is

$$(x+3)^2 + \left(y - \frac{3}{2}\right)^2 = \left(\frac{15}{2}\right)^2$$

$\Rightarrow \quad x^2 + y^2 + 6x - 3y - 45 = 0$

16. Here $A = (2, 3)$, $B = (6, 3)$ and $D = (2, 6)$

Thus, the centre of the circle is (4, 9/2) and the radius = 5/2.



Hence, the equation of the circle is

$$(x-4)^2 + \left(y - \frac{9}{2}\right)^2 = \frac{25}{4}$$

$\Rightarrow \quad x^2 + y^2 - 8x - 9x + 30 = 0$

17. Let the centre be (h, k)

Thus, $(h-1)^2 + (k-2)^2 = (h-5)^2 + (k-2)^2$

$= (h-5)^2 + (k+2)^2$

Now, $(h-5)^2 + (k-2)^2 = (h-5)^2 + (k+2)^2$

$\Rightarrow \quad (k-2)^2 = (k+2)^2$

$\Rightarrow \quad k^2 - 4k + 4 = k^2 + 4k + 4$

$\Rightarrow \quad 8k = 0$

$\Rightarrow \quad k = 0$

Also, $(h-1)^2 + (k-2)^2 = (h-5)^2 + (k-2)$

$\Rightarrow \quad (h-1)^2 = (h-5)^2$

$\Rightarrow \quad h^2 - 2h + 1 = h^2 - 10h + 25$

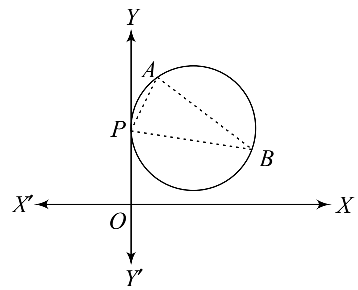
$\Rightarrow \quad 8h = 24$

$\Rightarrow \quad h = 3.$

Thus, the centre is (3, 0)

Hence, the radius of the circle is $2\sqrt{2}$.

18. Let the co-ordinates of P be $(0, k)$ and the centre of the circle be $C(h, k)$.



Now, $CA = CB = CP$

$\Rightarrow \quad CA^2 = CB^2 = CP^2$

$\Rightarrow \quad (h-4)^2 + (k-3)^2 = (h-2)^2 + (k-5)^2 = h^2$

Now,

$$(h-4)^2 + (k-3)^2 = (h-2)^2 + (k-5)^2$$

$\Rightarrow \quad -8h - 6k = -4h + 4 - 10k$

$\Rightarrow \quad 4h - 4k + 4 = 0$

$\Rightarrow \quad h - k + 1 = 0 \qquad \ldots$(i)

and $\quad (h-2)^2 + (k-1)^2 = h^2$

$\Rightarrow \quad (h-2)^2 + (h-4)^2 = h^2$

$\Rightarrow \quad h^2 - 4h + 4 + h^2 - 8h + 16 = h^2$

$\Rightarrow \quad -4h + 4 + h^2 - 8h + 16 = 0$

$\Rightarrow \quad h^2 - 12h + 20 = 0$

$\Rightarrow \quad (h-2) = 0, (h-10) = 0$

$\Rightarrow \quad (h-2) = 0, (h-10) = 0$

$\Rightarrow \quad h = 2, 10$

Thus, $k = h + 1 = 3, 11$

Hence, the point on y-axis is P (0, 3).

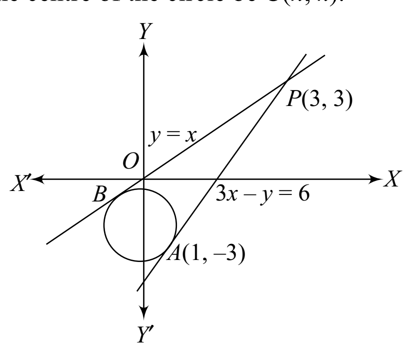
Thus, the equation of the circle is

$$(x-2)^2 + (y-3)^2 = 9$$

$\Rightarrow \quad x^2 - 4x + 4 + y^2 - 6y + 9 = 9$

$\Rightarrow \quad x^2 + y^2 - 4x - 6y + 4 = 0$

19. Let the centre of the circle be $C(h, k)$.



Here, $PA = 2\sqrt{10}$

Let $\quad B$ be (a, a).

Thus, $PB = 2\sqrt{10}$

$\Rightarrow \quad (a-3)^2 + (a-3)^2 = 40$

$\Rightarrow \quad (a-3)^2 = 20$

$\Rightarrow \quad (a-3) = \pm 2\sqrt{5}$

$\Rightarrow \quad a = 3 \pm 2\sqrt{5}$

$\Rightarrow \quad a = 3 - 2\sqrt{5}$

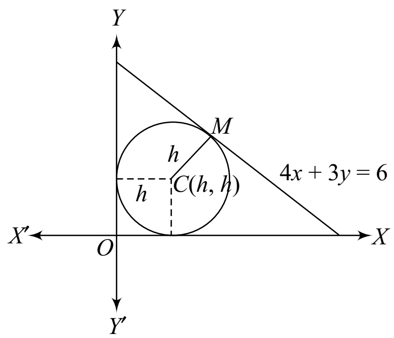
Hence, $B = (3-2\sqrt{5}, 3-2\sqrt{5})$.

20. The length of the line = the length of the tangent

$$= \sqrt{4+9+4+30+1}$$

$$= \sqrt{48} = 4\sqrt{3}$$

21. Let the centre be $C(h, h)$



Now, $CM = h$

$\Rightarrow \quad \left|\frac{4h+3h}{5}\right| = 6$

$\Rightarrow \quad 7h = 30$

$\Rightarrow \quad h = \frac{30}{7}$

Hence, the equation of the circle is

$\Rightarrow \quad \left(x - \frac{30}{7}\right)^2 + \left(y - \frac{30}{7}\right)^2 = \left(\frac{30}{7}\right)^2$

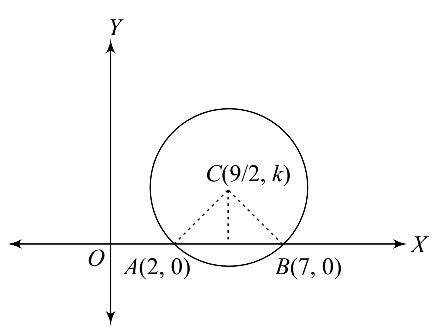
$\Rightarrow \quad x^2 - \frac{60}{7}x + \frac{900}{49} + y^2 - \frac{60}{7}y = 0$

$\Rightarrow \quad x^2 + y^2 - \frac{60}{7}x - \frac{60}{7}y + \frac{900}{49} = 0$

$\Rightarrow \quad 49(x^2 + y^2) - 420(x + y) + 900 = 0$

22. Hence, the equation of the circle is

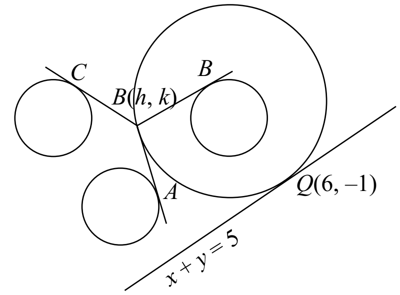
$$\left(x - \frac{9}{2}\right)^2 + (y-k)^2 = \left(\frac{9}{2} - 2\right)^2 + k^2$$



$\Rightarrow \quad x^2 - 9x + \frac{81}{4} + y^2 - 2ky = \frac{25}{4}$

$\Rightarrow \quad x^2 + y^2 - 9x - 2ky + 14 = 0$

23. Let P be (h, k)



Here, $PA = PB = PC$

$\Rightarrow \quad PA^2 = PB^2 = PC^2$

$\Rightarrow \quad h^2 + k^2 + h - 3 = 3(h^2 + k^2) - 5h + 3k$

$$= 4(h^2 + k^2) + 8h + 7k + 9$$

On solving, we get

$h = 0, k = -3.$

So, the point P is $(0, -3)$.

Let the centre of the new circle be (a, b).

We have,

$$\left|\frac{a+b-5}{\sqrt{2}}\right| = \sqrt{(a-6)^2 + (b+1)^2}$$

$$= \sqrt{a^2 + (b+3)^2}$$

On solving, we get

$$a = \frac{7}{2}, b = -\frac{7}{2}$$

and radius is $r = \frac{5\sqrt{2}}{2}$

Hence, the equation of the circle is

$$\left(x - \frac{7}{2}\right)^2 + \left(y + \frac{7}{2}\right)^2 = \left(\frac{5\sqrt{2}}{2}\right)^2$$

$\Rightarrow \quad x^2 + y^2 - 7x + 7y + 12 = 0$

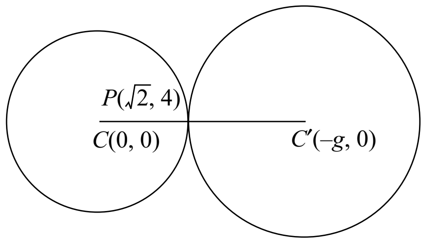
24. Clearly, the locus of the mutually perpendicular tangents to the circle is the director circle. So its equation is $x^2 + y^2 = 18$.

Thus, its centre is $(0, 0)$ and the radius $= 3\sqrt{2}$.

Let the equation of the co-axial system be

$$x^2 + y^2 + 2gx + c = 0$$

whose centre is $(-g, 0)$ and the radius $= \sqrt{g^2 - c}$.



Clearly, $\dfrac{-g+0}{2} = \sqrt{2}$

$\Rightarrow \quad g = -2\sqrt{2}$

Also, $CC' = r_1 + r_2$

$\Rightarrow \quad g = 3\sqrt{2} + \sqrt{g^2 - c}$

$\Rightarrow \quad -2\sqrt{2} = 3\sqrt{2} + \sqrt{8-c}$

$\Rightarrow \quad -5\sqrt{2} = \sqrt{8-c}$

$\Rightarrow \quad 8 - c = 50$

$\Rightarrow \quad c = -42$

Hence, the equation of the co-axial circle is $x^2 + y^2 - 4\sqrt{2}x - 42 = 0$.

25. Equation of the tangent to the circle $x^2 + y^2 = 5$ at $(1, -2)$ is $x - 2y = 5$.

Given circle is

$$x^2 + y^2 - 8x + 6y + 20 = 0$$

Centre = $(4, -3)$ and the radius = $\sqrt{5}$.

Let the point of contact be (h, k).

Thus, $h - 2k = 5$ …(i)

and $(h-4)^2 + (k+3)^2 = 5$

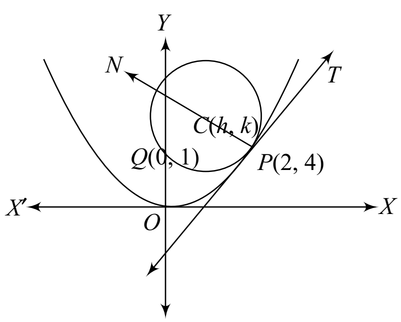
$\Rightarrow \quad h^2 + k^2 - 8h + 6k + 20 = 0$ …(ii)

On solving Eqs (i) and (ii), we get

$$h = 3, k = -1$$

Hence, the point of contact is $(3, -1)$.

26. Given curve is



$$y = x^2$$

$\Rightarrow \quad \dfrac{dy}{dx} = 2x$

Thus, $m = \left(\dfrac{dy}{dx}\right)_{/(2,4)} = 4$

The equation of the normal is

$$(y-4) = -\frac{1}{4}(x-2)$$

$\Rightarrow \quad 4(y-4) = -(x-2)$

$\Rightarrow \quad x + 4y = 18$

Let (h, k) be the centre of the circle.

Thus, $h + 4k = 18$ …(i)

Also, $CP = CQ$

$\Rightarrow \quad h^2 + (k-1)^2 = (h-2)^2 + (k-4)^2$

$\Rightarrow \quad -2k + 1 = -4h + 4 - 8k + 16$

$\Rightarrow \quad 4h + 6k = 19$ …(ii)

On solving (i) and (ii), we get

$$h = -\frac{16}{5}, k = \frac{53}{10}$$

Hence, the required centre is $\left(-\dfrac{16}{5}, \dfrac{53}{10}\right)$.

27. Given equation of normals are

$$x^2 - 3xy - 3x + 9y = 0$$

$\Rightarrow \quad (x - 3y)(x - 3) = 0$

$\Rightarrow \quad (x - 3y) = 0, (x - 3) = 0$

As we know that the normal always passes through the centre of a circle.

Thus, centre, $C_1 = (3, 1)$

Also, given circle is

$$x^2 + y^2 - 6x + 6y + 17 = 0$$

$\Rightarrow \quad (x-3)^2 + (y+3)^2 = 1$

So the centre $C_2 = (3, -3)$ and $r_2 = 1$

Since the circle touches externally, so

$$C_1C_2 = r_1 + r_2$$

$\Rightarrow \quad 4 = r_1 + 1$

$\Rightarrow \quad r_1 = 3$

Hence, the equation of the circle is

$$(x-3)^2 + (y-1)^2 = 9$$

$\Rightarrow \quad x^2 + y^2 - 6x - 2y + 1 = 0$

28. Given lines are $2x - 4y = 9$

and $\quad 2x - 4y + \dfrac{7}{3} = 0$

Hence, the radius = $\dfrac{1}{2}\left|\dfrac{9 + \frac{7}{3}}{\sqrt{4+16}}\right|$

$$= \frac{1}{2}\left(\frac{34}{3\sqrt{20}}\right)$$

$$= \frac{1}{2}\left(\frac{34}{6\sqrt{5}}\right) = \frac{17}{6\sqrt{5}}$$

29. Let the co-ordinates of the other end be (a, b).

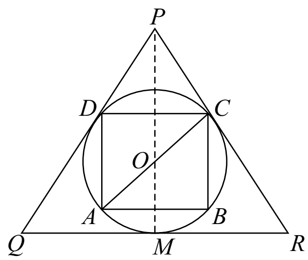
Given that the centre of the given circle = $(4, 2)$.

Thus, $\dfrac{a-3}{2} = 4, \dfrac{b+2}{2} = 2$

$$a = 11, b = 2$$

Hence, the other end be $(11, 2)$.

30.



Let $PM = p$

Now, $\sin(60°) = \frac{PM}{PQ} = \frac{p}{a}$

$\Rightarrow \quad \frac{p}{a} = \frac{\sqrt{3}}{2}$

$\Rightarrow \quad p = \frac{\sqrt{3}}{2}a$

Here O is the centroid.
So the centroid divides the median in the ratio 2 : 1.

Thus, $OM = \frac{p}{3}$

$\Rightarrow \quad r = \frac{p}{3}$

$\Rightarrow \quad 2r = \frac{2p}{3} = \frac{2}{3} \times \frac{\sqrt{3}}{2}a = \frac{a}{\sqrt{3}}$

Let x be the side of a square.

Thus, $x^2 + x^2 = \left(\frac{a}{\sqrt{3}}\right)^2 = \frac{a^2}{3}$

$\Rightarrow \quad 2x^2 = \frac{a^2}{3}$

$\Rightarrow \quad x^2 = \frac{a^2}{6}$

Area of a square $= \frac{a^2}{6}$

31. Given circle is $x^2 + y^2 = 3$
and the line is $x + y = 2$
Now,

$x^2 + (2 - x)^2 = 3$

$\Rightarrow \quad x^2 + x^2 - 4x + 4 = 3$

$\Rightarrow \quad 2x^2 - 4x + 1 = 0$

$\Rightarrow \quad x = \frac{4 \pm \sqrt{8}}{4} = 1 \pm \frac{1}{\sqrt{2}}$

$\Rightarrow \quad x = 1 + \frac{1}{\sqrt{2}}$

Also, $y = 2 - x = 2 - 1 - \frac{1}{\sqrt{2}} = 1 - \frac{1}{\sqrt{2}}$

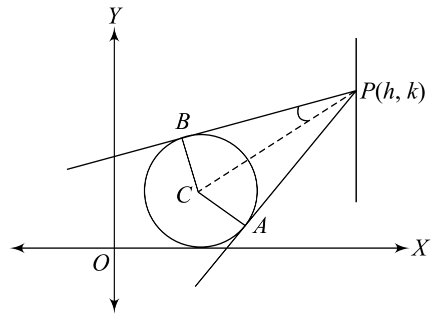
Thus, the point of intersection is

$$\left(1 + \frac{1}{\sqrt{2}}, 1 - \frac{1}{\sqrt{2}}\right)$$

Hence, the equation of the line is

$$y = \left(\frac{\sqrt{2} - 1}{\sqrt{2} + 1}\right)x = (3 - 2\sqrt{2})x$$

32. Given circle is $x^2 + y^2 - 6x - 4y + 4 = 0$



So, the centre is (3, 2) and the radius is 3.

Let P be (h, k)

Thus, $4h - 3k = 6$ …(i)

Let $\angle CPB = \theta$

Then $\sin\theta = \frac{BC}{PC}$

$= \frac{3}{\sqrt{(h-3)^2 + (k-2)^2}}$ …(ii)

From Eqs (i) and (ii), we get

$$\sin\theta = \frac{3}{\sqrt{(h-3)^2 + \left(\frac{4h-6}{3} - 2\right)^2}} \quad \text{…(iii)}$$

It is given that,

$2\theta = \tan^{-1}\left(\frac{24}{7}\right)$

$\Rightarrow \quad \tan(2\theta) = \frac{24}{7}$

$\Rightarrow \quad \cos(2\theta) = \frac{7}{25}$

$\Rightarrow \quad 1 - 2\sin^2(\theta) = \frac{7}{25}$

$\Rightarrow \quad 2\sin^2(\theta) = 1 - \frac{7}{25} = \frac{18}{25}$

$\Rightarrow \quad \sin^2(\theta) = \frac{9}{25}$

$\Rightarrow \quad \sin(\theta) = \frac{3}{5}$

From Eq. (iii), we get

$$\frac{3}{5} = \frac{3}{\sqrt{(h-3)^2 + \left(\frac{4h-6}{3} - 2\right)^2}}$$

$\Rightarrow \quad (h-3)^2 + \left(\frac{4h-6}{3} - 2\right)^2 = 25$

$\Rightarrow \quad (h-3)^2 + \left(\frac{4h-12}{3}\right)^2 = 25$

$\Rightarrow \quad 9(h-3)^2 + (4h-12)^2 = 225$

$\Rightarrow \quad 25h^2 - 150h = 0$

$\Rightarrow \quad h = 0, 6$

If $h = 0$, then $k = -2$.

If $h = 6$, then $k = 6$.

Hence, the points are $(0, -2)$, $(6, 6)$.

33. Here, $C_1 = (5, 0)$, $r_1 = 3$, $C_2 = (0, 0)$, $r_2 = r$

It is given that two circles intersect.

So, $\quad |r_1 - r_2| < C_1C_2 < r_1 + r_2$

$\Rightarrow \quad |3 - r| < 5 < 3 + r$

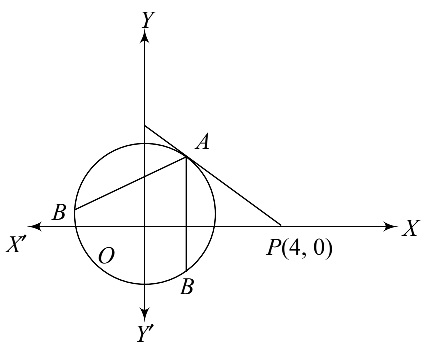
$\Rightarrow \quad |3 - r| < 5, 5 < 3 + r$

$\Rightarrow \quad -5 < (r - 3) < 5, r > 2$

$\Rightarrow \quad -2 < r < 8, r > 2$

$\Rightarrow \quad 2 < r < 8$

34. Given circle is $x^2 + y^2 = 8$



Let the point A be (h, k).

The equation of the tangent at A is $hx + ky = 8$ which is passing through $P(4, 0)$.

So, $4h = 8 \Rightarrow h = 2$

Now PA = length of the tangent = $2\sqrt{2}$

$\Rightarrow \quad PA^2 = 8$

$\Rightarrow \quad (h-4)^2 + k^2 = 8$

$\Rightarrow \quad (2-4)^2 + k^2 = 8$

$\Rightarrow \quad k^2 = 8 - 4$

$\Rightarrow \quad k = 2$

Hence, the point A is $(2, 2)$.

Now, for the point B,

$$\frac{x-2}{\cos\theta} = \frac{y-2}{\sin\theta} = 4$$

$\Rightarrow \quad x = 2 + 4\cos\theta, y = 2 + 4\sin\theta$

Thus, the point B is $(2 + 4\cos\theta, 2 + 4\sin\theta)$.

Since the point B lies on the circle $x^2 + y^2 = 8$, so

$(2 + 4\cos\theta)^2 + (2 + 4\sin\theta)^2 = 8$

$\Rightarrow \quad 16\cos\theta + 16\sin\theta + 16 = 0$

$\Rightarrow \quad \cos\theta + \sin\theta + 1 = 0$

$\Rightarrow \quad \cos\theta + \sin\theta = -1$

$\Rightarrow \quad \frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta = -\frac{1}{\sqrt{2}}$

$\Rightarrow \quad \cos\left(\theta - \frac{\pi}{4}\right) = \cos\left(\frac{3\pi}{4}\right)$

$\Rightarrow \quad \theta = \pi \text{ or } -\frac{\pi}{2}$

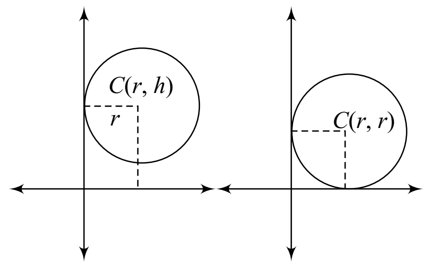
Hence, the point B be $(-2, 2)$ or $(2, -2)$.

35. Given circle is $x^2 + y^2 - 2rx - 2hy + h^2 = 0$

$\Rightarrow \quad (x - r)^2 + (y - h)^2 = r^2$.

So, the centre is (r, h) and radius is r.

It is possible only when $r = h$.



36. Let the equation of the circle C be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \ldots\text{(i)}$$

Given circle is $x^2 + y^2 = 1$

$\Rightarrow \quad x^2 + y^2 - 1 = 0 \qquad \ldots\text{(ii)}$

Since two circles are orthogonal, so

$2(g \cdot 0 + f \cdot 0) = c - 1$

$\Rightarrow \quad c = 1$

Also, the equation of radical axis is

$2gx + 2fy + c + 1$

$\Rightarrow \quad 2gx + 2fy + 1 + 1 = 0$

$\Rightarrow \quad gx + fy + 1 = 0$

It is given that the radical axis parallel to y-axis, so,

$f = 0, g \in R - \{0\}$

Thus, $x = -\frac{1}{g}$

Hence, the equation of the circle is

$\Rightarrow \quad x^2 + y^2 + 2gx + 1 = 0$

$\Rightarrow \quad x^2 + y^2 + x + 1 = 0$

or $\quad x^2 + y^2 - x + 1 = 0$

37. Ans.

38. Given circle is $(x + r)^2 + (y - h)^2 = r^2$.

The equation of any tangent passing through origin is

$y = mx$. i.e. $mx - y = 0$

As we know that the length of the perpendicular from the centre of the circle to the tangent is equal to the radius of a circle.

Thus, $\left|\frac{-rm - h}{\sqrt{m^2+1}}\right| = r$

$\Rightarrow \quad (-rm - h)^2 = (r\sqrt{m^2+1})^2$

$\Rightarrow \quad (rm + h)^2 = r^2(m^2 + 1)$

$\Rightarrow \quad r^2m^2 + 2rmh + h^2 = r^2m^2 + r^2$

$\Rightarrow \quad 2rmh + h^2 = r^2$

$$\Rightarrow \quad m = \frac{r^2 - h^2}{2rh}$$

Hence, the equation of the circle is

$$y = \left(\frac{r^2 - h^2}{2rh}\right)x$$

$$\Rightarrow \quad (r^2 - h^2)x - 2rhy = 0$$

$$\Rightarrow \quad (h^2 - r^2)x + 2rhy = 0$$

39. Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \ldots\text{(i)}$$

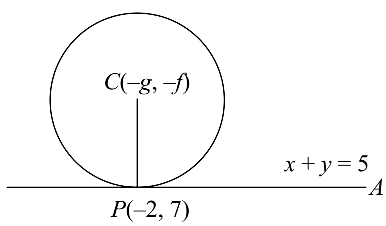
Also (i) is orthogonal to

$$x^2 + y^2 + 4x - 6y + 9 = 0$$

Thus, $2(g_1g_2 + f_1f_2) = c_1 + c_2$

$$\Rightarrow \quad 2(g \cdot 2 + f \cdot (-3)) = c + 9$$

$$\Rightarrow \quad 4g - 6f = c + 9 \quad \ldots\text{(ii)}$$



Also, $CP \perp PA$

Thus, $\dfrac{7+f}{g-2} \times -1 = -1$

$$\Rightarrow \quad \frac{7+f}{g-2} = 1$$

$$\Rightarrow \quad 7 + f = g - 2$$

$$\Rightarrow \quad f = g - 9 \quad \ldots\text{(ii)}$$

From Eqs (i) and (ii), we get

$$4g - 6f = c + 9$$

$$\Rightarrow \quad 4(f + 9) - f = c + 9$$

$$\Rightarrow \quad 36 - 2f = c + 9$$

$$\Rightarrow \quad c = 27 - 2f \quad \ldots\text{(iii)}$$

Again, CP = Radius

$$\left|\frac{-g - f - 5}{\sqrt{1^2 + 1^2}}\right| = \sqrt{g^2 + f^2 - c}$$

$$\Rightarrow \quad (g + f + 5)^2 = 2(g^2 + f^2 - c)$$

$$\Rightarrow \quad (2f + 14)^2 = 2((9 + f)^2 + f^2 - (27 - 2f))$$

$$\Rightarrow \quad 2(f + 7)^2 = (2f^2 + 18f + 81 - (27 - 2f))$$

$$\Rightarrow \quad 2(f + 7)^2 = (2f^2 + 20f + 54)$$

$$\Rightarrow \quad (f + 7)^2 = (f^2 + 10f + 27)$$

$$\Rightarrow \quad f^2 + 14f + 49 = (f^2 + 10f + 27)$$

$$\Rightarrow \quad 4f = 27 - 49$$

$$\Rightarrow \quad f = -\frac{22}{4} = -\frac{11}{2}$$

Thus, $g = f + 9 = 9 - \dfrac{11}{2} = \dfrac{7}{2}$

and $c = 27 - 2f = 27 + 11 = 38$

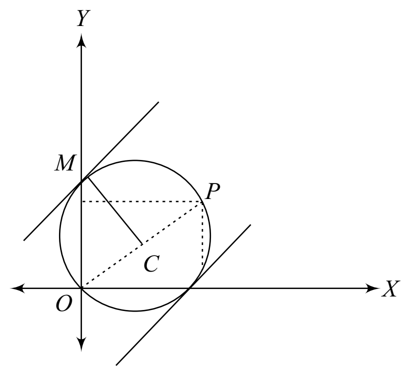
Hence, the equation of the circle is

$$x^2 + y^2 + 7x - 11y + 38 = 0.$$

40. Here, the centre = $\left(2, \dfrac{3}{2}\right)$ and the radius = $\dfrac{5}{2}$

Therefore, the equation of the circle is

$$(x - 2)^2 + \left(y - \frac{3}{2}\right)^2 = \frac{25}{4}$$



The equation of the tangent parallel to the diagonal is

$$4y - 3x + k = 0$$

Now, $CM = \dfrac{5}{2}$

$$\Rightarrow \quad \left|\frac{4\left(\frac{3}{2}\right) - 3.2 + k}{5}\right| = \frac{5}{2}$$

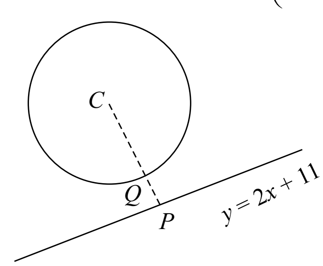
$$\Rightarrow \quad k = \pm\frac{25}{2}$$

Hence, the equations of the tangents are

$$4y - 3x \pm \frac{25}{2} = 0$$

$$\Rightarrow \quad 8y - 6x \pm 25 = 0$$

41. Clearly, the centre of the circle is $\left(-1, -\dfrac{1}{4}\right)$



Given line is $y = 2x + 11$

$$\Rightarrow \quad 2x - y + 11 = 0 \quad \ldots\text{(i)}$$

The equation of CP is $-x - 2y + k = 0$

$$\Rightarrow \quad x + 2y - k = 0$$

which is passing through the centre $C\left(-1, -\dfrac{1}{4}\right)$. so,

$$-1 - \frac{1}{2} = k$$

$\Rightarrow \quad k = -\frac{3}{2}$

Hence, the line CP is $x + 2y + \frac{3}{2} = 0$

$\Rightarrow \quad 2x + 4y + 3 = 0 \qquad \ldots\text{(ii)}$

On solving Eqs (i) and (ii), we get

$$x = -\frac{9}{2}, y = 2$$

Hence, the required point is $\left(-\frac{9}{2}, 2\right)$.

42. Given circle is $x^2 + y^2 + 4x = 0$

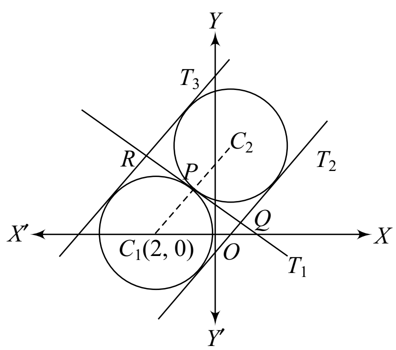
$$(x + 2)^2 + y^2 = 4$$

So, the centre is (–2, 0) and the radius is = 2.

The equation of the line joining the centres of the circles is

$$y + 2 = m(x - 0)$$

$$y + 2 = \sqrt{3}x, m = \tan(60°) = \sqrt{3}$$



Here, let the centre C_2 be (h, k).

Thus, $\dfrac{h+2}{\frac{1}{2}} = \dfrac{k-0}{\frac{\sqrt{3}}{2}} = 4$

$\Rightarrow \quad h = 0, k = 2\sqrt{3}$

Therefore, $C_2 = (h, k) = (0, 2\sqrt{3})$

Hence, the locus of the centre of the outer circle is

$$(x - 0)^2 + (y - 2\sqrt{3})^2 = 4$$

$$x^2 + y^2 - 4\sqrt{3}y - 8 = 0$$

The equation of the common tangent T_1 is

$$-x - \sqrt{3}y + k = 0$$

which is passing through $P(-1, \sqrt{3})$.

So, $\quad k = 2$

Hence, the equation of the common tangent T_1 is $x + \sqrt{3}y = 2$.

Clearly, the co-ordinates of Q and R are

$$(\sqrt{3} - 1, \sqrt{3} - 1) \text{ and } (-\sqrt{3} - 1, \sqrt{3} + 1)$$

Hence, the equation of the tangents T_2 and T_3 are

$$y - (\sqrt{3} - 1) = \sqrt{3}[x - (\sqrt{3} - 1)]$$

and $\quad y - (\sqrt{3} - 1) = \sqrt{3}[x + (\sqrt{3} + 1)]$

43. If R be the radius of the circumcircle

so, $\quad R = \dfrac{ph + qk - r}{\sqrt{p^2 + q^2}} \qquad \ldots\text{(i)}$

Since TP and TQ are tangents to the circle $x^2 + y^2 = a^2$, so the circumcircle through T will pass through its centre.

Thus, $R = \sqrt{h^2 + k^2} \qquad \ldots\text{(ii)}$

From Eqs (i) and (ii), we get

$$\frac{ph + qk - r}{\sqrt{p^2 + q^2}} = \sqrt{h^2 + k^2}$$

$\Rightarrow \quad (ph + qk - r)^2 = (p^2 + q^2)(h^2 + k^2)$

Hence, the locus of (h, k) is

$$(px + qy - r)^2 = (p^2 + q^2)(x^2 + y^2)$$

44. Let $\quad S_1: x^2 + y^2 - 2x - 6y + 6 = 0$

$\quad S_2: x^2 + y^2 + 2x - 6y + 6 = 0$

and $\quad S_3: x^2 + y^2 + 4x + 6y + 6 = 0$

Let the equation of the circle passing through the point of intersection S_1 and S_2 is

$$S_4: S_1 + \lambda S_2 = 0$$

$\Rightarrow (x^2 + y^2 - 2x - 6y + 6) + \lambda(x^2 + y^2 + 2x - 6y + 6) = 0$

$\Rightarrow x^2 + y^2 + \left(\dfrac{2\lambda - 2}{\lambda + 1}\right)x + \left(\dfrac{-6\lambda - 6}{\lambda + 1}\right)yz + \left(\dfrac{6\lambda + 6}{\lambda + 1}\right) = 0 \qquad \ldots\text{(i)}$

Also, S_4 is orthogonal to S_3.

Thus, $2(g_1g_2 + f_1f_3) = c_1 + c_2$

$\Rightarrow \quad 2.2\left(\dfrac{2\lambda - 2}{\lambda + 1}\right) + 2.3\left(\dfrac{-6\lambda - 6}{\lambda + 1}\right) = 6 + \left(\dfrac{6\lambda + 6}{\lambda + 1}\right)$

On solving, we get

$$\lambda^2 + 6\lambda + 8 = 0$$

$\Rightarrow \quad (\lambda + 2)(\lambda + 4) = 0$

$\Rightarrow \quad \lambda = -2, -4$

Put the values of $\lambda = -2, -4$, we get

$$x^2 + y^2 + 6x - 6y + 6 = 0$$

or $\quad x^2 + y^2 + \left(\dfrac{10}{3}\right)x - 6y + 6 = 0$

Integer Type Questions

1. Clearly, $C_1C_2 = 5$

Here, $r_1 = 10, r_2 = 5$

Thus, $C_1C_2 = 5 = r_1 - r_2$

So, the number of common tangents = 1

2. We have, $2(g_1g_2 + f_1f_2) = c_1 + c_2$

$\Rightarrow \quad 2(1.0 + k \cdot k) = k + 6$

$\Rightarrow \quad 2k^2 - k - 6 = 0$

$\Rightarrow \quad (k-2)(2k+3) = 0$

$\Rightarrow \quad k = 2, -3/2$

3. Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

and $\left(m, \frac{1}{m}\right)$ be a variable point lies on the circle

Thus, $m^2 + \frac{1}{m^2} + 2gm + \frac{2f}{m} + c = 0$

$\Rightarrow \quad m^4 + 1 + 2gm^3 + 2fm + cm^2 = 0$

$\Rightarrow \quad m^4 + 2gm^3 + cm^2 + 2fm + 1 = 0$

It has four roots, say m_1, m_2, m_3 and m_4.

Thus, $m_1m_2m_3m_4 = 1$

$\Rightarrow \quad m_1m_2m_3m_4 + 4 = 5$

4. The equation of the common chord is

$$6x + 14y + p + q = 0$$

Here, the centre of the 1st circle (1, –4) lies on the common chord.

So, $\quad 6 - 56 + p + q = 0$

$p + q = 50$

Hence, the value of $\left(\frac{p+q}{10} + 2\right) = 7$

5. We have, $|r_1 - r_2| < C_1C_2 < r_1 + r_2$

$\Rightarrow \quad |r - 3| < C_1C_2 < r + 3$

$\Rightarrow \quad |r - 3| < 5 < r + 3$

$\Rightarrow \quad -5 < (r-3) < 5, r + 3 > 5$

$\Rightarrow \quad -2 < r < 8, r > 2$

$\Rightarrow \quad 2 < r < 8$

Clearly, $n = 2$, $m = 8$

Hence, the value of $(m - n)$ is 6.

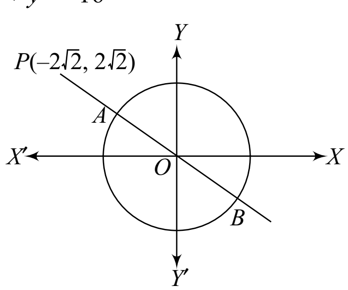
6. The equation of any line passing through P is

$$y - 2\sqrt{2} = -(x + 2\sqrt{2})$$

$\Rightarrow \quad y = -x$

and the equation of the circle is

$$x^2 + y^2 = 16$$



Clearly, $AB = 8$

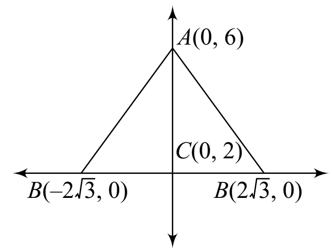
7. As we know that if the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ cut the x and y axes in four concyclic points, then

$$a_1a_2 = b_1b_2$$

$\Rightarrow \quad \lambda \cdot 1 = (-1)(-2)$

$\Rightarrow \quad \lambda = 2$

8.



Clearly, radius = 4.

9. Given circle is

$$x^2 + y^2 - 4x - 6y + \lambda = 0$$

$\Rightarrow \quad (x-2)^2 + (y-3)^2 = (\sqrt{13-\lambda})^2$

Clearly, $\sqrt{13-\lambda} = 3$

$\Rightarrow \quad 13 - \lambda = 9$

$\Rightarrow \quad \lambda = 4$

10. Clearly, $m_1m_2 = 1$

$\Rightarrow \quad m_1m_2 + 4 = 5$

Previous Years' JEE-Advanced Examinations

1. Given lines are $3x + 5y - 1 = 0$,

$$(2 + c)x + 5c^2y - 1 = 0$$

On solving, we get

$$\Rightarrow \quad \frac{x}{-5+5c^2} = \frac{y}{-(2+c)+3} = \frac{1}{15c^2 - 5(2+c)}$$

$$\frac{x}{5(c^2-1)} = \frac{y}{-(c-1)} = \frac{1}{5(3c^2-c-2)}$$

$$\Rightarrow \quad x = \frac{(c^2-1)}{(3c^2-c-2)}, y = \frac{-(c-1)}{5(3c^2-c-2)}$$

$$\Rightarrow \quad x = \frac{(c-1)(c+1)}{(c-1)(3c+2)}, y = \frac{-(c-1)}{5(c-1)(3c+2)}$$

$$\Rightarrow \quad x = \frac{(c+1)}{(3c+2)}, y = -\frac{1}{5(3c+2)}$$

when c tends to 1, then

$$x = \frac{2}{5}, y = -\frac{1}{25}$$

Now, radius,

$$r = \sqrt{\left(2 - \frac{2}{5}\right)^2 + \left(0 + \frac{1}{25}\right)^2}$$

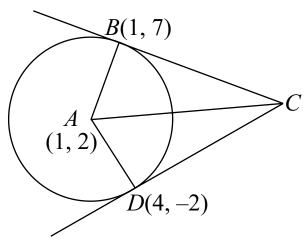
$$= \sqrt{\frac{64}{25} + \frac{1}{625}} = \sqrt{\frac{1601}{625}}$$

Hence, the equation of the circle is

$$\left(x - \frac{2}{5}\right)^2 + \left(y + \frac{1}{25}\right)^2 = \frac{1601}{625}.$$

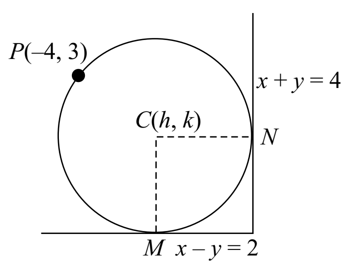
2. The equation of a circle passing through the point of intersection of $x^2 + y^2 = 6$ and $x^2 + y^2 - 6x + 8 = 0$ is
$\Rightarrow \quad (x^2 + y^2 - 6) + \lambda(x^2 + y^2 - 6x + 8) = 0).$
which is passing through (1, 1).
So, $\quad (2 - 6) + \lambda(10 - 6) = 0$
$\Rightarrow \quad \lambda = 1$
Hence, the equation of the circle is
$(x^2 + y^2 - 6) + 1 \cdot (x^2 + y^2 - 6x + 8) = 0$
$\Rightarrow \quad 2(x^2 + y^2) - 6x + 2 = 0$
$\Rightarrow \quad (x^2 + y^2) - 3x + 1 = 0$

3. The equations of the tangents at B and D are
$y = 7$ and $3x - 4y = 20$



Thus, the co-ordinates of C are (16, 7).
Hence, the area of the quadrilateral $ABCD = 2(\Delta ABC)$
$= 2\left(\frac{1}{2} \times AB \times BC\right)$
$= AB \cdot BC$
$= (5 \times BC)$
$= 5 \times 15$
$= 75$ sq. u.

4.



Clearly, $CM = CN$
$\Rightarrow \quad \left|\frac{h + k - 4}{\sqrt{2}}\right| = \left|\frac{h - k - 2}{\sqrt{2}}\right|$
$\Rightarrow \quad (h + k - 4) = \pm(h - k - 2)$
$\Rightarrow \quad h = 3, k = 1$
Now, $r = \sqrt{(3+4)^2 + (1-3)^2} = \sqrt{53}$
Hence, the equation of the circle is
$(x - 3)^2 + (y - 1)^2 = 53$
$\Rightarrow \quad x^2 + y^2 - 6x - 2y - 43 = 0$

5. Ans. $k \neq 1$

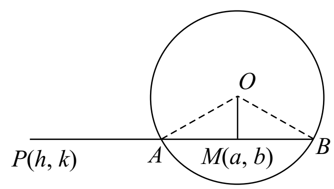
6. Given circle is
$x^2 + y^2 - 2x + 4y - 20 = 0$
$\Rightarrow \quad \left(\frac{3y + 10}{4}\right)^2 + y^2 - 2\left(\frac{3y + 10}{4}\right) + 4y - 20 = 0$
$\Rightarrow \quad (3y + 10)^2 + 16y^2 - 8(3y + 10) + 64y - 320 = 0$
$\Rightarrow \quad 9y^2 + 60y + 100 + 16y^2 - 24y - 80 + 64y - 320 = 0$
$\Rightarrow \quad 25y^2 + 100y - 300 = 0$
$\Rightarrow \quad y^2 + 4y - 12 = 0$
$\Rightarrow \quad (y + 6)(y - 2) = 0$
$\Rightarrow \quad y = 2, -6$
when $y = 2, -6$, then $x = 4, -2$
Hence, the points of intersection are (4, 2), (–2, –6)

7. Equation of any circle passing through the point of intersection of the given circles is
$(2x^2 + 2y^2 + 4x - 7y - 25) + \lambda(x^2 + y^2 + 13x - 3y) = 0$
which is passing through the point (1, 1).
$\Rightarrow \quad -24 + 12\lambda = 0$
$\Rightarrow \quad \lambda = 2$
Hence, the equation of the circle is
$(2x^2 + 2y^2 + 4x - 7y - 25)$
$+ 2(x^2 + y^2 + 13x - 3y) = 0$
$\Rightarrow \quad 4x^2 + 4y^2 + 30x - 13y - 25 = 0$

8. Ans. (b)

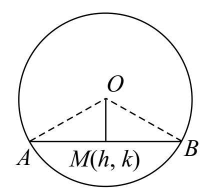
9. Do yourself

10. Equation of the chord bisected at M is
$ax + by = a^2 + b^2$



which is passing through $P(h, k)$.
So, $\quad ah + bk = a^2 + b^2$
Hence, the locus of $M(a, b)$
$hx + ky = x^2 + y^2$

11. Let AB be a chord of the circle.
Draw a perpendicular from the centre O to the chord AB at M. Then $AM = BM$.
Now, $\angle AOM = 45°$



$\Rightarrow \quad \cos 45° = \frac{OM}{OA} = \frac{OM}{2}$
$\Rightarrow \quad \frac{OM}{2} = \frac{1}{\sqrt{2}}$
$\Rightarrow \quad OM^2 = 2$
$\Rightarrow \quad h^2 + k^2 = 2$
Hence, the locus of $M(h, k)$ is
$x^2 + y^2 = 2$

12. Given equation is $x^2 = 2ax - b^2 = 0$
Let its roots are x_1 and x_2.
Thus, $x_1 + x_2 = 2a, x_1 \cdot x_2 = -b^2$
Similarly, $y_1 + y_2 = -2p, y_1 \cdot y_2 = -q^2$
Hence, the equation of the circle is
$\Rightarrow \quad (x - x_1)x - x_2) + (y - y_1)(y - y_2) = 0$
$\Rightarrow \quad x^2 - (x_1 + x_2)x + x_1x_2 + y^2 - (y_1 + y_2)y + y_1y_2 = 0$
$\Rightarrow \quad x^2 + 2ax - b^2 + y^2 + 2py - q^2 = 0$
$\Rightarrow \quad x^2 + y^2 + 2ax + 2py - (b^2 + q^2) = 0$

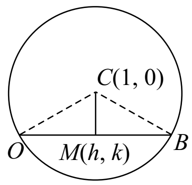
13. Diameter = Distance between two parallel lines

$$= \left| \frac{\left(4 - \left(-\frac{7}{2}\right)\right)}{\sqrt{3^2 + 4^2}} \right|$$

$$= \frac{3}{2}$$

Hence, the radius is $\frac{3}{4}$.

14. Equation of the chord bisected at $M(h, k)$ is $T = S_1$



$\Rightarrow \quad xx_1 + yy_1 - (x + x_1) = x_1^2 + y_1^2 - 2x_1$

$\Rightarrow \quad hx + ky - (x + h) = h^2 + k^2 - 2h$

which is passing through origin.

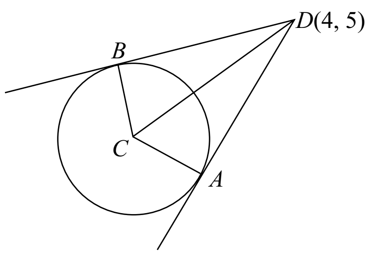
So, $\quad 0 + 0 - 0 - h = h^2 + k^2 - 2h$

$\Rightarrow \quad h^2 + k^2 - h = 0$

Hence, the locus of $M(h, k)$ is

$$x^2 + y^2 - x = 0.$$

15.



Here, $C = (2, 1)$ and $r = 4$

$\therefore$ Area of quadrilateral $ACBD = 2(\Delta ACD)$

$$= 2 \times \frac{1}{2} \times 4 \times 2$$

$$= 8 \text{ sq. u.}$$

16. The equation of the line of intersection is

$$\left(x^2 + y^2 - \frac{2}{3}x + 4y - 3\right)$$

$\Rightarrow \quad -(x^2 + y^2 + 6x + 2y - 15) = 0$

$\Rightarrow \quad -\left(\frac{2}{3} + 6\right)x + (4 - 2)y + 12 = 0$

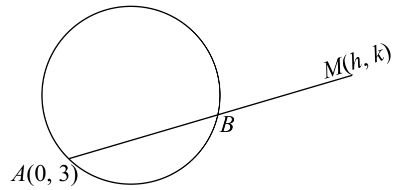
$\Rightarrow \quad -\frac{20}{3}x + 2y + 12 = 0$

$\Rightarrow \quad -\frac{10}{3}x + y + 6 = 0$

$\Rightarrow \quad -10x + 3y + 18 = 0$

$\Rightarrow \quad 10x - 3y - 18 = 0.$

17. Let the co-ordinates of M be (h, k).



Clearly, B is the mid-point of AM.

Thus, $\left(\frac{h}{2}, \frac{k+3}{2}\right)$

Since B lies on $x^2 + 4x + (y - 3)^2 = 0$. So

$$\left(\frac{h}{2}\right)^2 + 4\left(\frac{h}{2}\right) + \left(\frac{k+3}{2} - 2\right)^2 = 0$$

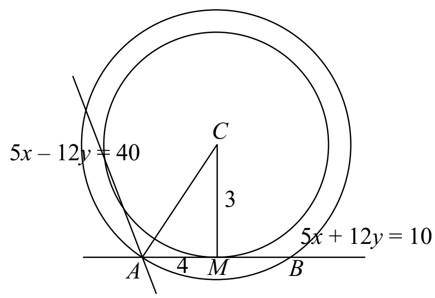
$\Rightarrow \quad h^2 + 8h + (k - 3)^2 = 0$

Hence, the locus of $M(h, k)$ is

$$x^2 + 8x + (y - 3)^2 = 0$$

18. Since the lines $5x + 12y = 10$ and $5x - 12y = 40$ touch the circle C_1, its centre lies on one of the angle bisectors of the given lines.

$$\left|\frac{5x + 12y - 10}{\sqrt{25 + 144}}\right| = \left|\frac{5x - 12y - 40}{\sqrt{25 + 144}}\right|$$



$\Rightarrow \quad |5x + 12y - 10| = |5x - 12y - 40|$

$\Rightarrow \quad (5x + 12y - 10) = \pm(5x - 12y - 40)$

$\Rightarrow \quad x = 5$ and $y = -5/4$

Since the centre lies on the first quadrant, let its co-ordinates be $(5, k)$.

We have $CM = 3$

$\Rightarrow \quad \left|\frac{5.5 + 12k - 10}{\sqrt{25 + 144}}\right| = 3$

$\Rightarrow \quad \left|\frac{15 + 12k}{13}\right| = 3$

$\Rightarrow \quad (15 + 12k) = \pm 39$

$\Rightarrow \quad k = 2, -\frac{9}{2}$

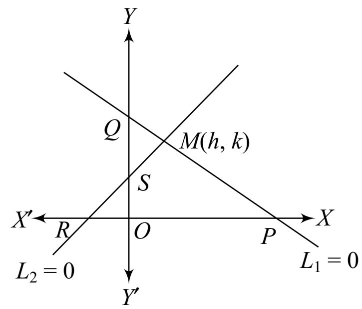
Thus, $k = 2$ and $r = \sqrt{3^2 + 4^2} = 5$

Hence, the equation of the circle is

$$(x - 5)^2 + (y - 2)^2 = 25$$

$\Rightarrow \quad x^2 + y^2 - 10x - 4y + 4 = 0$

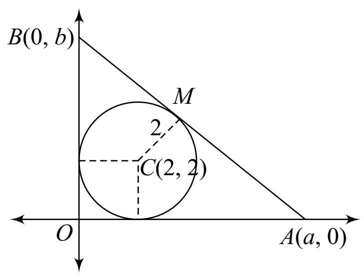
19.



20. Given circle is

$$x^2 + y^2 - 4x - 4y + 4 = 0$$

$$\Rightarrow \quad (x-2)^2 + (y-2)^2 = 4$$



Let AB: $\dfrac{x}{a} + \dfrac{y}{b} = 1$

Now, $CM = 2$

$$\Rightarrow \quad \left| \frac{\frac{2}{a} + \frac{2}{b} - 1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} \right| = 2$$

$$\Rightarrow \quad \frac{\frac{2}{a} + \frac{2}{b} - 1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} = \pm 2$$

$$\Rightarrow \quad \frac{\frac{2}{a} + \frac{2}{b} - 1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} = -2$$

[since the origin and (2, 2) lie on the same side of the line AB]

Here, circumcentre = $M = \left(\dfrac{a}{2}, \dfrac{b}{2}\right)$

Therefore the locus of the circumcentre is

$$\Rightarrow \quad \frac{1}{x} + \frac{1}{y} - 1 = -2\sqrt{\frac{1}{4x^2} + \frac{1}{4y^2}}$$

$$\Rightarrow \quad \frac{1}{x} + \frac{1}{y} - 1 = -\sqrt{\frac{1}{x^2} + \frac{1}{y^2}}$$

$$\Rightarrow \quad x + y - xy + \sqrt{x^2 + y^2} = 0$$

Thus, the value of k is 1.

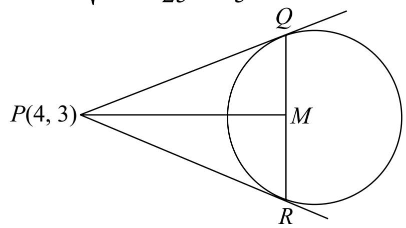
21. The equation of the chord of contact is

$$4x + 3y = 9$$

$\therefore$ Length of the tangent $PQ = PR$

$$= \sqrt{16 + 9 - 9} = 4$$

Now, $QM = \sqrt{16 - \dfrac{256}{25}} = \dfrac{12}{5}$



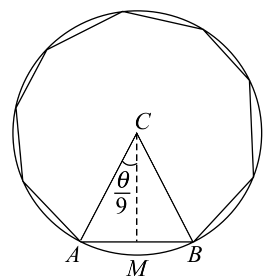
Hence, the area of the triangle PQR

$$= 2(\Delta PQM)$$

$$= 2 \times \frac{1}{2} \times \frac{12}{5} \times \frac{16}{5}$$

$$= \frac{192}{25}$$

22.

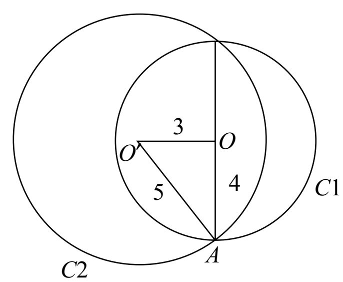


Here, $AB = 2$. So, $AM = 1$

In ΔACM, $\sin\left(\dfrac{\pi}{9}\right) = \dfrac{AM}{AC} = \dfrac{1}{r}$

$$\Rightarrow \quad r = \frac{1}{\sin\left(\frac{\pi}{9}\right)} = \operatorname{cosec}\left(\frac{\pi}{9}\right)$$

23.



Clearly, $OO' = \sqrt{25 - 16} = 3$

It is given that, $\tan\theta = \dfrac{3}{4}$

$$\Rightarrow \quad \frac{\sin\theta}{3} = \frac{\cos\theta}{4} = \frac{1}{5}$$

Let the co-ordinates of the centre $O' = (x, y)$ of the circle C_2.

Therefore, $\dfrac{x-0}{\frac{4}{5}} = \dfrac{y-0}{\frac{3}{5}} = \pm 3$

$\Rightarrow \quad x = \pm\dfrac{12}{5}, y = \pm\dfrac{9}{5}$

Thus, $O' = \left(\pm\dfrac{12}{5}, \pm\dfrac{9}{5}\right)$.

24. The equation of a circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \ldots\text{(i)}$$

which passes through (a, b). So

$$a^2 + b^2 + 2ga + 2fb + c = 0 \quad \ldots\text{(ii)}$$

Also (i) is orthogonal to $x^2 + y^2 = k^2$. So

$$2(g\cdot 0 + f\cdot 0) = c - k^2$$

$$\Rightarrow \quad c = k^2 \quad \ldots\text{(iii)}$$

From Eqs (ii) and (iii), we get

$$a^2 + b^2 + 2ga + 2fb + k^2 = 0$$

$$\Rightarrow \quad a^2 + b^2 - 2(-g)a - 2(-f)b + k^2 = 0$$

Hence, the locus of the centre $(-g, -f)$ is

$$a^2 + b^2 - 2xa - 2yb + k^2 = 0$$

$$\Rightarrow \quad 2ax - 2by - (a^2 + b^2 + k^2) = 0$$

25. Given circle is $(x + r)^2 + (y - h)^2 = r^2$

Equation of any tangent passing through origin is $y = mx \Rightarrow mx - y = 0$

As we know that the length of the perpendicular from the centre of the circle to the tangent is equal to the radius of a circle.

Thus, $\left|\dfrac{-rm - h}{\sqrt{m^2+1}}\right| = r$

$$\Rightarrow \quad (-rm - h)^2 = (r\sqrt{m^2+1})^2$$

$$\Rightarrow \quad (rm + h)^2 = r^2(m^2 + 1)$$

$$\Rightarrow \quad r^2m^2 + 2rmh + h^2 = r^2m^2 + r^2$$

$$\Rightarrow \quad 2rmh + h^2 = r^2$$

$$\Rightarrow \quad m = \frac{r^2 - h^2}{2rh}$$

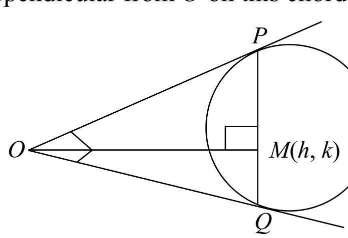
Hence, the equation of the circle is

$$y = \left(\frac{r^2 - h^2}{2rh}\right)x$$

$$\Rightarrow \quad (r^2 - h^2)x - 2rhy = 0$$

$$\Rightarrow \quad (h^2 - x^2)x + 2rhy = 0$$

26. Let PQ be the chord of the given circle which subtends a right angle at the origin O and $M(h, k)$ be the foot of the perpendicular from O on this chord PQ.



Equation of PQ is

$$y - k = -\frac{h}{k}(x - h)$$

$$\Rightarrow \quad hx + ky = h^2 + k^2$$

$$\Rightarrow \quad \frac{hx + ky}{h^2 + k^2} = 1 \quad \ldots\text{(i)}$$

The equation of the pair of lines joining the point of intersection of (i) with $S = 0$ is

$$x^2 + y^2 + (2gx + 2fy)\left(\frac{hx + ky}{h^2 + k^2}\right) + c\left(\frac{hx + ky}{h^2 + k^2}\right)^2 = 0 \quad \ldots\text{(ii)}$$

As the line (ii) are at right angles, so

$$1 + 1 + \left(\frac{2gh + 2fk}{h^2 + k^2}\right) + c\frac{(h^2 + k^2)}{(h^2 + k^2)^2} = 0$$

$$\Rightarrow \quad (h^2 + k^2) + gh + fk + \frac{c}{2} = 0$$

Hence, the locus of $M(h, k)$ is

$$\Rightarrow \quad (x^2 + y^2) + gx + fy + \frac{c}{2} = 0$$

27. Given two circles intersect. We have

$$|r_1 - r_2| < C_1C_2 < r_1 + r_2$$

$$\Rightarrow \quad |r - 3| < 5 < r + 3$$

$$\Rightarrow \quad |r - 3| < 5,\ 5 < r + 3$$

$$\Rightarrow \quad -5 < (r - 3) < 5,\ r > 2$$

$$\Rightarrow \quad -2 < r < 8,\ r > 2$$

$$\Rightarrow \quad 2 < r < 8$$

28. Let r be the radius of the circle.

So, $\pi r^2 = 154$

$$\Rightarrow \quad \frac{22}{7} \times r^2 = 154$$

$$\Rightarrow \quad r^2 = 7 \times 7$$

$$\Rightarrow \quad r = 7$$

Now, the centre of the circle is the point of intersection of $2x - 3y = 5$ and $3x - 4y = 7$. So

$$C = (1, -1)$$

Hence, the equation of the circle is

$$(x - 1)^2 + (y + 1)^2 = 7^2$$

$$\Rightarrow \quad x^2 + y^2 - 2x + 2y - 47 = 0$$

29 Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Let $\left(m, \dfrac{1}{m}\right)$ be an arbitrary point on the circle.

$$\Rightarrow \quad m^2 + \frac{1}{m^2} + 2gm + \frac{2f}{m} + c = 0$$

$$\Rightarrow \quad m^4 + 1 + 2gm^3 + 2fm + cm^2 = 0$$

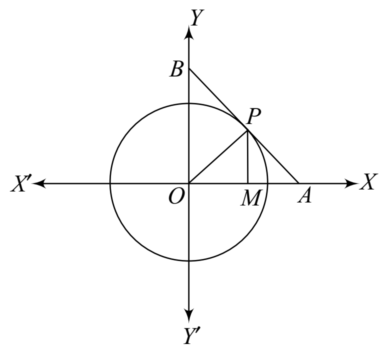
$$\Rightarrow \quad m^2 + 2gm^3 + cm^2 + 2fm + 1 = 0$$

Let m_1, m_2, m_3, m_4 be the roots.

Thus, $m_1 \cdot m_2 \cdot m_3 \cdot m_4 = \dfrac{1}{1} = 1$.

30. Given circle is $x^2 + y^2 - 6x + 2y = 0$
Centre is $(3, -1)$.
Clearly, the centre $(3, -1)$ satisfies the equation of the diameter $x + 3y = 0$.

31.



The equation of tangent to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$ is

$$x + \sqrt{3}y = 4$$

The equation of normal to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$ is

$$y = \sqrt{3}x$$

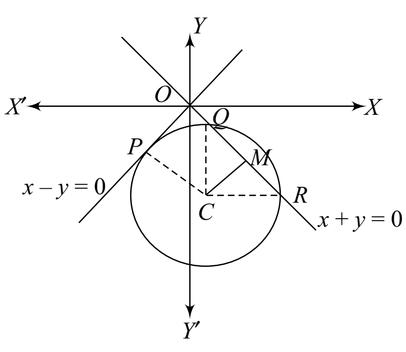
Thus, area of the triangle OPA

$$= \frac{1}{2} \times OA \times PM$$

$$= \frac{1}{2} \times 4 \times \sqrt{3}$$

$$= 2\sqrt{3} \text{ sq. u.}$$

32. Let $C(h, k)$ be the centre and r be the radius of the given circle.



Thus, $\left|\frac{h-k}{\sqrt{2}}\right| = r$ …(i)

Given $OP = 4\sqrt{2}, QR = 6\sqrt{2}$

So, $QM = MR = 3\sqrt{2}$

Clearly, $CM = OP = 4\sqrt{2}$

$$\left|\frac{h+k}{\sqrt{2}}\right| = 4\sqrt{2} \quad \ldots(ii)$$

Also, $r = \sqrt{CM^2 + (3\sqrt{2})^2}$

$\Rightarrow \quad r = \sqrt{CM^2 + 18}$

$\Rightarrow \quad r^2 = CM^2 + 18$

$\Rightarrow \quad r^2 = \frac{(h+k)^2}{2} + 18$

$\Rightarrow \quad \frac{(h-k)^2}{2} = \frac{(h+k)^2}{2} + 18$

$\Rightarrow \quad \frac{(h-k)^2}{2} - \frac{(h+k)^2}{2} = 18$

$\Rightarrow \quad -2hk = 18$

$\Rightarrow \quad hk = -9$ …(iii)

From Eq. (ii), we get

$h + k = 8$

$\Rightarrow \quad h - \frac{9}{h} = 8$

$\Rightarrow \quad h^2 - 8h - 9 = 0$

$\Rightarrow \quad (h-9)(h+1) = 0$

$\Rightarrow \quad h = -1, 9$

and $\quad k = 9, -1$

Hence, the equation of the circle is

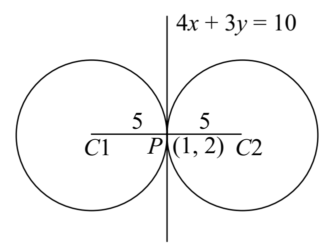
$(x+1)^2 + (y-9)^2 = 50$

or

$(x-9)^2 + (y+1)^2 = 50$

33. Ans. $\frac{3\sqrt{3}}{4} \times r^2$ sq. units

34.



The equation of the line where both the centres lie is $3x - 4y + k = 0$ which is passing through the point.
Thus, $k = 5$
Hence, the required line is $3x - 4y + 5 = 0$

Clearly, $\tan\theta = \frac{3}{4}$

$\Rightarrow \quad \frac{\sin\theta}{3} = \frac{\cos\theta}{4} = \frac{1}{5}$

Therefore, the co-ordinates of C_1 and C_2 can be obtained from

$\Rightarrow \quad \frac{x-1}{\frac{4}{5}} = \frac{y-2}{\frac{3}{5}} = \pm 5$

$\Rightarrow \quad x = 1 \pm 4, y = 2 \pm 3$

$\Rightarrow \quad x = 5, -3; y = 5, -1$

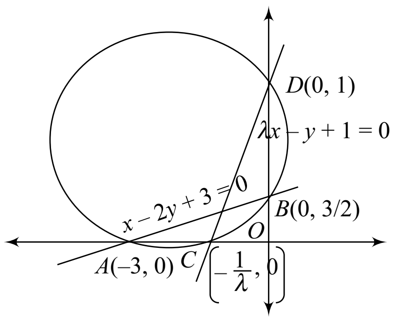
Therefore, the equations of the required circles are

$$(x-5)^2+(y-5)^2=25$$

or

$$(x+3)^2+(y+1)^2=25$$

35. Here, A, B, C and D are concyclics.

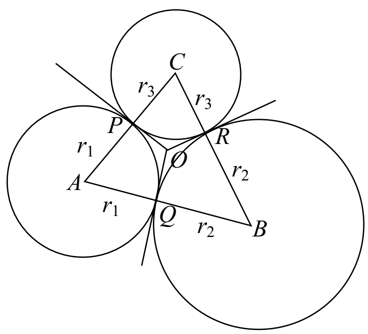


Thus, $OA.OC = OB.OD$

$\Rightarrow \quad -3.-\dfrac{1}{\lambda}=\dfrac{3}{2}\cdot 1$

$\Rightarrow \quad \lambda = 2$

36. Consider three circles with centres at A, B and C with radii r_1, r_2, r_3 respectively, which touch each other externally at P, Q, R.



Let the common tangents at P, Q, R meet each other at O.

Then $OP = OQ = QR = 4$

Also, $OP \perp AB$, $OQ \perp AC$, $OR \perp BC$

Here, O is the incentre of the triangle ABC.

For ΔABC,

$$s=\frac{(r_1+r_2)+(r_3+r_2)+(r_1+r_3)}{2}=r_1+r_2+r_3$$

and $\quad \Delta=\sqrt{(r_1+r_2+r_3)r_1r_2r_3}$

Now, from the relation $r=\dfrac{\Delta}{s}$, we get

$$\frac{\sqrt{(r_1+r_2+r_3)r_1r_2r_3}}{r_1+r_2+r_3}=4$$

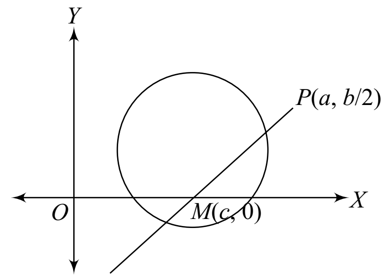
$\Rightarrow \quad \sqrt{\dfrac{r_1r_2r_3}{r_1+r_2+r_3}}=4$

$\Rightarrow \quad \dfrac{r_1r_2r_3}{r_1+r_2+r_3}=16=\dfrac{16}{1}$

$\Rightarrow \quad (r_1r_2r_3):(r_1+r_2+r_3)=16:1$

37. Given circle is

$$2x(x-a)+y(2y-b)=0$$



$\Rightarrow \quad 2x^2-2ax+2y^2-by=0$

$\Rightarrow \quad 2x^2+2y^2-2ax-by=0$

The equation of the chord bisected at M is $T = S_1$

$\Rightarrow \quad 2xx_1+2yy_1-a(x+x_1)-b\left(\dfrac{y+y_1}{2}\right)$

$$=2x_1^2+2y_1^2-2ax_1-by_1$$

$\Rightarrow \quad 2x\cdot c+0-a(x+c)-b\left(\dfrac{y+0}{2}\right)$

$$=2c^2+0-2ac-0$$

$\Rightarrow \quad 4cx+2ax-2ac-by=4c^2-4ac$

which is passing through $P(a, b/2)$. So

$$4ca-2a^2-2ac-\frac{b^2}{2}=4c^2-4ac$$

$\Rightarrow \quad 8ca-4a^2-4ac-b^2=18c^2-8ac$

$\Rightarrow \quad 8c^2-12ac+(4a^2+b^2)=0$

Now, $D > 0$

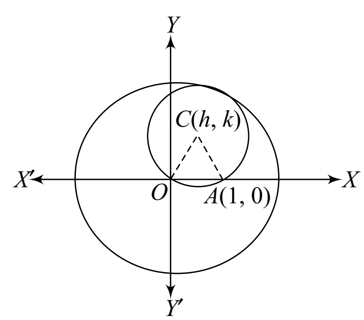
$\Rightarrow \quad 144a^2-32(4a^2+b^2)>0$

$\Rightarrow \quad 9a^2-2(4a^2+b^2)>0$

$\Rightarrow \quad 9a^2+8a^2-2b^2>0$

$\Rightarrow \quad a^2>2b^2$

38.



Clearly, $OC = OA$

$\Rightarrow \quad OC^2 = CA^2$

$\Rightarrow \quad (h^2+k^2)=(h-1)^2+k^2$

$\Rightarrow \quad (h^2+k^2)=h^2-2h+1+k^2$

$\Rightarrow \quad -2h+1=0$

$\Rightarrow \quad h=\dfrac{1}{2}$

Also, $C_1C_2 = |r_1 - r_2|$

$\Rightarrow \quad OC = 3 - \sqrt{h^2 + k^2}$

$\Rightarrow \quad \sqrt{h^2 + k^2} = 3 - \sqrt{h^2 + k^2}$

$\Rightarrow \quad 2\sqrt{h^2 + k^2} = 3$

$\Rightarrow \quad h^2 + k^2 = \frac{9}{4}$

$\Rightarrow \quad k^2 = \frac{9}{4} - h^2 = \frac{9}{4} - \frac{1}{4} = 2$

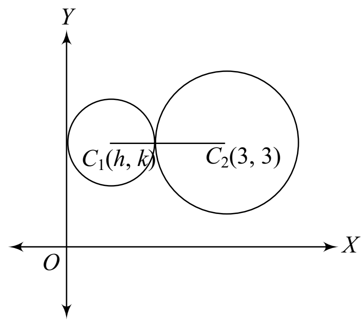
$\Rightarrow \quad k = \sqrt{2}$

Hence, the centre is $\left(\frac{1}{2}, \sqrt{2}\right)$.

39. Given circle is

$x^2 + y^2 - 6x - 6y + 14 = 0$

$\Rightarrow \quad (x - 3)^2 + (y - 3)^2 = 4$



We have,

$C_1C_2 = r_1 + r_2$

$\Rightarrow \quad \sqrt{(h-3)^2 + (k-3)^2} = h + 2$

$\Rightarrow \quad (h - 3)^2 + (k - 3)^2 = (h + 2)^2$

$\Rightarrow \quad h^2 - 6h + 9 + k^2 - 6k + 9 = h^2 + 4h + 4$

$\Rightarrow \quad k^2 - 10h - 6k + 14 = 0$

Hence, the locus of (h, k) is

$y^2 - 10x - 6y + 14 = 0$

40. The equation of any circle passing through $A(3, 7)$ and $B(6, 5)$ is

$$(x-3)(x-6) + (y-7)(y-5) + \lambda \begin{vmatrix} x & y & 1 \\ 3 & 7 & 1 \\ 6 & 5 & 1 \end{vmatrix} = 0$$

$\Rightarrow S_1: (x - 3)(x - 6) + (y - 7)(y - 5) + \lambda(2x + 3y - 27) = 0$

$\Rightarrow S_1: x^2 + y^2 - 9x - 12y + 53 + \lambda(2x + 3y - 27) = 0$ …(i)

The equation of the common chord of (i)

and $S_2: x^2 + y^2 - 4x - 6y - 3 = 0$ …(ii)

is $S_1 - S_2 = 0$.

$\Rightarrow \quad -5x = 6y + 56 + k(2x + 3y - 27) = 0$

$\Rightarrow \quad -5x - 6y + 56 = 0,\ 2x + 3y - 27 = 0$

$\Rightarrow \quad x = 2,\ y = \frac{23}{3}$

Hence, the co-ordinates of the point is $\left(2, \frac{23}{3}\right)$.

41. Given circles are

$x^2 + y^2 - 4x - 2y + 4 = 0$

$\Rightarrow \quad (x - 2)^2 + (y - 1)^2 = 1$

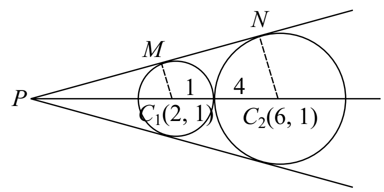
and $(x - 6)^2 = (y - 4)^2 = 4^2$

$\Rightarrow \quad (x - 6)^2 + (y - 4)^2 = 4^2$

Here, $C_1 = (2, 1),\ r_1 = 3;\ C_2 = (6, 4),\ r_2 = 4$

Now, $C_1C_2 = \sqrt{(6-2)^2 + (4-1)^2} = 5$

$= r_1 + r_2$



Therefore, $D = \left(\frac{4.2 + 1.6}{4 + 1}, \frac{4.1 + 1.4}{4 + 1}\right) = \left(\frac{14}{5}, \frac{8}{5}\right)$

and $P = \left(\frac{4.2 - 1.6}{4 - 1}, \frac{4.1 - 1.4}{4 - 1}\right) = \left(\frac{2}{3}, 0\right)$

The equations of the tangent through P is

$\Rightarrow \quad y - 0 = m\left(x - \frac{2}{3}\right)$

$\Rightarrow \quad 3y = m(3x - 2)$

$\Rightarrow \quad 3mx - 3y - 2m = 0$

Now, $C_1M = 1$

$\Rightarrow \quad \left|\frac{3m.2 - 3.1 - 2m}{\sqrt{9m^2 + 9}}\right| = 1$

$\Rightarrow \quad \left|\frac{4m - 3}{3\sqrt{m^2 + 1}}\right| = 1$

$\Rightarrow \quad (4m - 3)^2 = 9(m^2 + 1)$

$\Rightarrow \quad 16m^2 - 24m + 9 = 9m^2 + 9$

$\Rightarrow \quad 7m^2 - 24m = 0$

$\Rightarrow \quad m(7m - 24) = 0$

$\Rightarrow \quad m = 0,\ (7m - 24) = 0$

$\Rightarrow \quad m = 0,\ m = \frac{24}{7}$

Hence, the equations of tangents are

$y = 0$ and $y = \frac{24}{7}(x - 3)$

$\Rightarrow \quad y = 0$ and $24x - 7y - 72 = 0$

42. Given circle is

$4x^2 + 4y^2 - 12x + 4y + 1 = 0$

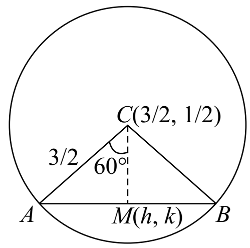
$x^2 + y^2 - 3x + y + (1/4) = 0$

$x^2 + y^2 - 3x + y + (1/4) = 0$

Thus, $C = \left(\frac{3}{2}, -\frac{1}{2}\right), r = \sqrt{\frac{9}{4} + \frac{1}{4} - \frac{1}{4}} = \frac{3}{2}$

In ΔACM,

$$\cos(60°) = \frac{CM}{AC}$$



$\Rightarrow \quad \frac{CM}{3/2} = \frac{1}{2}$

$\Rightarrow \quad CM = \frac{3}{4}$

$\Rightarrow \quad CM^2 = \frac{9}{16}$

$\Rightarrow \quad \left(h - \frac{3}{2}\right)^2 + \left(k - \frac{1}{2}\right)^2 = \frac{9}{16}$

Hence, the locus of $M(h, k)$ is

$$\left(x - \frac{3}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{9}{16}$$

43. Here, $C_1 = (5, 0), r_1 = 3, C_2 = (0, 0), r_2 = r$

It is given that two circles intersect.

So, $|r_1 - r_2| < C_1C_2 < r_1 + r_2$

$\Rightarrow \quad |3 - r| < 5 < 3 + r$

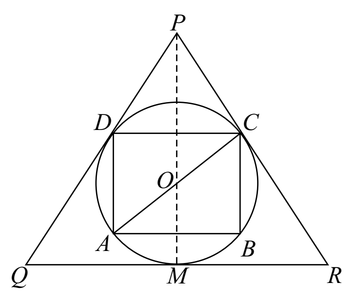
$\Rightarrow \quad |3 - r| < 5, 5 < 3 + r$

$\Rightarrow \quad -5 < (r - 3) < 5, r > 2$

$\Rightarrow \quad -2 < r < 8, r > 2$

$\Rightarrow \quad 2 < r < 8$

44. Let $PM = p$



Now, $\sin(60°) = \frac{PM}{PQ} = \frac{p}{a}$

$\Rightarrow \quad \frac{p}{a} = \frac{\sqrt{3}}{2}$

$\Rightarrow \quad p = \frac{\sqrt{3}}{2}a$

Here O is the centroid.

So the centroid divides the median in the ratio 2 : 1.

Thus, $OM = \frac{p}{3}$

$\Rightarrow \quad r = \frac{p}{3}$

$\Rightarrow \quad 2r = \frac{2p}{3} = \frac{2}{3} \times \frac{\sqrt{3}}{2}a = \frac{a}{\sqrt{3}}$

Let x be the side of a square.

Thus, $x^2 + x^2 = \left(\frac{a}{\sqrt{3}}\right)^2 = \frac{a^2}{3}$

$\Rightarrow \quad 2x^2 = \frac{a^2}{3}$

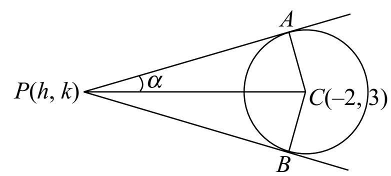
$\Rightarrow \quad x^2 = \frac{a^2}{6}$

Area of a square $= \frac{a^2}{6}$

45. Given circle is

$$x^2 + y^2 + 4x - 6y + 9\sin^2\alpha + 13\cos^2\alpha = 0$$
$$(x + 2)^2 + (y - 3)^2 = 4 - 4\cos^2\alpha$$
$$(x + 2)^2 + (y - 3)^2 = (2\sin\alpha)^2$$
$$(x + 2)^2 + (y - 3)^2 = (2\sin\alpha)^2$$

Let the P be (h, k).



We have

$$\sin\alpha = \frac{AC}{PC}$$

$\Rightarrow \quad \sin\alpha = \frac{2\sin\alpha}{\sqrt{(h + 2)^2 + (k - 3)^2}}$

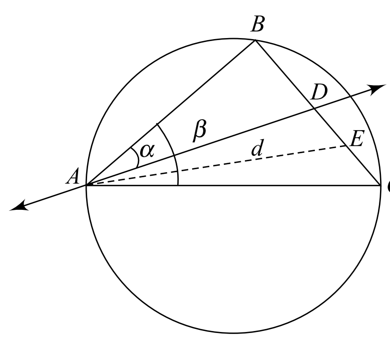
$\Rightarrow \quad (h + 2)^2 + (k - 3)^2 = 4$

$\Rightarrow \quad h^2 + k^2 + 4h - 6k + 9 = 0$

Hence, the locus of (h, k) is

$$x^2 + y^2 + 4x - 6y + 9 = 0$$

46. Let r be the radius of a circle, then $AC = 2r$



Since, AC is the diameter $\angle ABC = 90°$

In ΔABC, $BC = 2r\sin\beta$, $AB = 2r\cos\beta$

$BD = AB\tan\alpha = 2r\cos\beta\tan\alpha$

$AD = AB\sec\alpha = 2r\cos\beta\sec\alpha$

$DC = BC - BD = 2r\sin\beta - 2r\cos\beta\tan\alpha$

Since E is the mid-point of DC, so

$$DE = \frac{DC}{2} = r\sin\beta - r\cos\beta\tan\alpha$$

Now, in ΔADC, AE is the median

$$2(AE^2 + DE^2) = AD^2 + AC^2$$

$$2(d^2 + r^2(\sin\beta - \cos\beta\tan\alpha)^2)$$

$$= 4r^2\cos^2\beta\sec^2\alpha + 4r^2$$

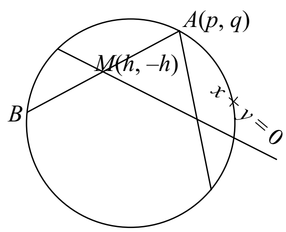
$$r^2 = \frac{d^2\cos^2\alpha}{\cos^2\alpha + \cos^2\beta + 2\cos\alpha\cos\beta\cos(\beta-\alpha)}$$

Hence, the area of a circle = pr^2

$$\frac{\pi d^2\cos^2\alpha}{\cos^2\alpha + \cos^2\beta + 2\cos\alpha\cos\beta\cos(\beta-\alpha)}$$

47. Let $(p, q) = \left(\frac{1+a\sqrt{2}}{2}, \frac{1-a\sqrt{2}}{2}\right)$

Given circle is $2(x^2 + y^2) - 2px - 2qy = 0$



The equation of the chord bisected at M is

$$2(hx - hy) - p(x + h) - q(y - h)$$

$$= 2(h^2 + h^2) - 2ph + 2qh$$

which is passing through $A(p, q)$.

$\Rightarrow \quad 2(hp - hq) - p(p + h) - q(q - h) = 4h^2 - 2ph + 2qh$

$\Rightarrow \quad 3hp - 3hq - p^2 - q^2 - 4h^2 = 0$

$\Rightarrow \quad 4h^2 + 3(q - p)h + (p^2 + q^2) = 0$

Since chords are distinct, so

$D > 0$

$\Rightarrow \quad 9(q - p)^2 - 16(p^2 + q^2) > 0$

$\Rightarrow \quad 9(-a\sqrt{2})^2 - 8(1 + 2a^2) > 0$

$\Rightarrow \quad 18a^2 - 16a^2 - 8 > 0$

$\Rightarrow \quad 2a^2 - 8 > 0$

$\Rightarrow \quad a^2 - 4 > 0$

$\Rightarrow \quad (a + 2)(a - 2) > 0$

$\Rightarrow \quad a < -2, a > 2$

Thus, $a \in (-\infty, 2) \cup (2, \infty)$

48. The equation of any circle passing through the point of intersection of $x^2 + y^2 - 2x = 0$ and $y = x$ is

$x^2 + y^2 - 2x + \lambda(x - y) = 0$

$\Rightarrow \quad x^2 + y^2 + (\lambda - 2)x - \lambda y = 0$

Its centre is $\left(\frac{2-\lambda}{2}, \frac{\lambda}{2}\right)$

The centre lies on $y = x$

So, $\quad \frac{2-\lambda}{2} = \frac{\lambda}{2}$

$\Rightarrow \quad \lambda = 1$

Hence, the required equation of the circle is

$x^2 + y^2 - 2x + (x - y) = 0$

$\Rightarrow \quad x^2 + y^2 - x - y = 0$

49. Given C is the circle with centre at $(0, \sqrt{2})$ and radius r (say)

Then $x^2 + (y - \sqrt{2})^2 = r^2$

$(y - \sqrt{2})^2 = r^2 - x^2$

$(y - \sqrt{2}) = \pm\sqrt{r^2 - x^2}$

$y = \sqrt{2} \pm \sqrt{r^2 - x^2}$

The only rational value of y is 0

Suppose the possible value of x for which y is 0 is x_1. Certainly, $y - x_1$ will also give the value of y as 0. Thus, atmost there are two rational points which satisfy the equation of the circle.

50. Let the point P be (h, k).

Thus, $\frac{x-p}{\cos\theta} = \frac{y-q}{\sin\theta} = r$ …(i)

$\Rightarrow \quad x = p + r\cos\theta, y = q + r\sin\theta$

The point P lies on the curve.

So, $\quad a(p + r\cos\theta)^2 + b(q + r\sin\theta)^2$

$+ 2h(p + r\cos q)(q + r\sin q) = 1$

$\Rightarrow \quad (a^2\cos^2\theta + 2h\cos\theta\sin\theta + b^2\sin^2\theta)r^2$

$+2[p(a\cos\theta + h\sin\theta)r + q(h\cos q + b\sin\theta)]r$

$+ ap^2 + 2hpq + bq^2 - 1 = 0$ …(ii)

Let $PQ = r_1$ and $PR = r_2$.

Also, let r_1 and r_2 are the roots of Eq. (ii)

Thus, $r_1r_2 = \frac{2(ap^2 + 2hpq + bq^2 - 1)}{(a+b) + 2h\sin 2\theta + (a-b)\cos 2\theta}$

Since the product of PQ and PR is the independent of θ so, $h = 0$, $a = b$ and $a \neq 0$.

So the given product becomes

$$x^2 + y^2 = \frac{1}{a}$$

which represents a circle.

51.

52. Any point on the line $2x + y = 4$ can be considered as $P(a, 4 - 2a)$.

The equation of the chord of contact of the tangent to the circle $x^2 + y^2 = 1$ from P is

$xx_1 + yy_1 - 1 = 0$

$\Rightarrow \quad ax + (4 - 2a)y - 1 = 0$

$\Rightarrow \quad a(x - 2y) + (4y - 1) = 0$

$\Rightarrow \quad (x - 2y) + \frac{1}{a}(4y - 1) = 0$

$\Rightarrow \quad (x - 2y) + \lambda(4y - 1) = 0, \lambda = \frac{1}{a}$

Thus, $x - 2y = 0$, $4y - 1 = 0$

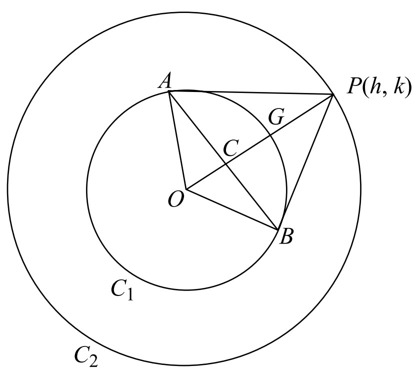
$\Rightarrow \quad x = \frac{1}{2}, y = \frac{1}{4}$.

Hence, the required point is $\left(\frac{1}{2}, \frac{1}{4}\right)$.

53. Since two vertices of an equilateral triangle are $B(-1, 0)$ and $C(1, 0)$.
So, the third vertex must lie on the y-axis.
Let the third vertex be $A(0, b)$
Now, $AB = BC = CA$
$\Rightarrow \quad AB^2 = BC^2 = AC^2$
$\Rightarrow \quad 1 + b^2 = 4 = 1 + b^2$
$\Rightarrow \quad b^2 = 4 - 1$
$\Rightarrow \quad b = \sqrt{3}$
Thus, the third vertex is $A = (0, \sqrt{3})$.
As we know that in case of an equilateral triangle,
Circumcentre = Centroid = $\left(0, \frac{1}{\sqrt{3}}\right)$
Hence, the equation of the circumcircle is
$$(x-0)^2 + \left(y - \frac{1}{\sqrt{3}}\right)^2 = (1-0)^2 + \left(0 - \frac{1}{\sqrt{3}}\right)^2$$
$$\Rightarrow \quad x^2 + \left(y - \frac{1}{\sqrt{3}}\right)^2 = \frac{4}{3}.$$

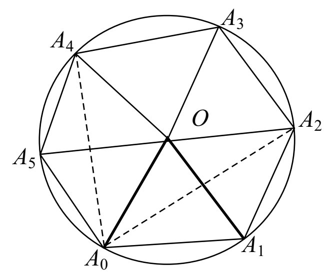
54. Given circles are $x^2 + y^2 = 4$
and $\quad x^2 + y^2 - 6x - 8y - 24 = 0$
Here $C_1 = (0, 0), r_1 = 2$; $C_2 = (3, 4), r_2 = 7$
Now, $C_1C_2 = 5 = r_2 - r_1$
So, two circles touch each other internally.
Thus, the number of common tangents = 1

55. Let $P(h, k)$ be on C_2
So, $\quad h^2 + k^2 = 4r^2 \qquad$ ...(i)

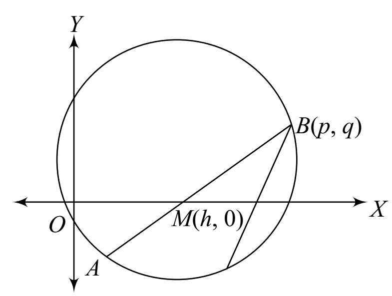


Chord of contact of P w.r.t C_1 is $hx + ky = r^2$
It intersects C_1 $x^2 + y^2 = a^2$ in A and B.
Eliminating y, we get,
$$x^2 + \left(\frac{r^2 - hx}{k}\right)^2 = r^2$$
$$(h^2 + k^2)x^2 - 2r^2hx + r^2(r^2 - k^2) = 0$$
$$4r^2x^2 - 2r^2hx + r^2(r^2 - k^2) = 0$$
Thus, $x_1 + x_2 = \frac{h}{2}, y_1 + y_2 = \frac{k}{2}$
Let (x, y) be the centroid of ΔPAB
Thus, $3x = x_1 + x_2 + h = \frac{h}{2} + h = \frac{3h}{2}$
$\quad h = 2x$
Similarly, $k = 2y$
Putting in (i), we get,
$\quad 4x^2 + 4y^2 = 4r^2$
$\quad x^2 + y^2 = r^2$
Hence, the locus is $x^2 + y^2 = r^2$

56. Here $A_0A_1 = 1$
$$\Rightarrow \quad \cos(120°) = \frac{1^2 + 1^2 - A_0A_2^2}{2.1.1}$$
$$\Rightarrow \quad -\frac{1}{2} = \frac{1^2 + 1^2 - A_0A_2^2}{2}$$
$\Rightarrow \quad A_0A_2^2 = 3$
$\Rightarrow \quad A_0A_2 = \sqrt{3}$
Similarly, $A_0A_4 = \sqrt{3}$
Hence, the value of
$\quad A_0A_1 A_0A_2 A_0A_4$
$\quad = 1 \cdot \sqrt{3} \cdot \sqrt{3}$
$\quad = 3.$



57.



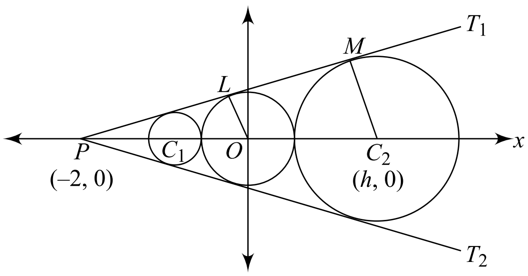
The equation of the chord bisected at $M(h, 0)$ is
$$xx_1 + yy_1 - p\left(\frac{x + x_1}{2}\right) - q\left(\frac{y + y_1}{2}\right)$$
$$= x_1^2 + y_1^2 - px_1 - qy_1$$
$$\Rightarrow \quad hx + 0 - p\left(\frac{x + h}{2}\right) - q\left(\frac{y + 0}{2}\right)$$
$$= h^2 + 0 - ph - 0$$
$\Rightarrow \quad 2hx - px - ph - qy = 2h^2 - 2ph$
$\Rightarrow \quad 2hx - px - qy = 2h2 - ph$
$\Rightarrow \quad 2h^2 - 2hx - ph + (px + qy) = 0$
which is passing through (p, q)
So, $\quad 2h^2 - 2ph - ph + (p^2 + q^2) = 0$
$\Rightarrow \quad 2h^2 - 3ph + (p^2 + q^2) = 0$
Clearly, $D > 0$
$\Rightarrow \quad 9p^2 - 8(p^2 + q^2) > 0$
$\Rightarrow \quad p^2 - 8q^2 > 0$
$\Rightarrow \quad p^2 > 8q^2$

58. The given circle is $x^2 + y^2 = r^2 \qquad$ ...(i)
Centre is $(0, 0)$ and radius = 1
Let T_1 and T_2 be the tangents drawn from $(-2, 0)$ to the circle (i)

Let m be the slope of the tangent, then the equations of tangents are

$y - 0 = m(x + 2)$

$mx - y + 2m = 0$ ...(ii)



Clearly, $\left|\dfrac{2m}{\sqrt{m^2+1}}\right| = 1$

$$m = \pm\frac{1}{\sqrt{3}}$$

Thus, the two tangents are

$T_1: x + \sqrt{3}\, y = 2$

$T_2: x - \sqrt{3}\, y = 2$

Now any other circle touching (i) and T_1, T_2 is such that its centre lies on x-axis

Let $(h, 0)$ be the centre of such circle

Thus, $OC_1 = OA + AC_1$

$|h| = 1 + |AC_1|$

But AC_1 = Perpendicular distance from $(h, 0)$ to the tangents

$$|h| = 1 + \left|\frac{h+2}{2}\right|$$

$$|h| - 1 = \left|\frac{h+2}{2}\right|$$

$$h^2 - 2|h| + 1 = \frac{h^2 + 4h + 4}{4}$$

$$h = -\frac{4}{3}, 4$$

Hence, the centres of the circles are

$\left(-\dfrac{4}{3}, 0\right), (4, 0)$

Radius of the circle with centre $(4, 0)$ is $4 - 1 = 3$ and the radius of the circle with centre $\left(-\dfrac{4}{3}, 0\right)$ is $\dfrac{4}{3} - 1 = \dfrac{1}{3}$

Thus, two possible circles are

$(x - 4)^2 + y^2 = 9$ ...(iii)

and $\left(x + \dfrac{4}{3}\right)^2 + y^2 = \dfrac{1}{9}$ ...(iv)

Since (i) and (iii) are two touching circles, so they have three common tangents T_1, T_2 and $x = 1$

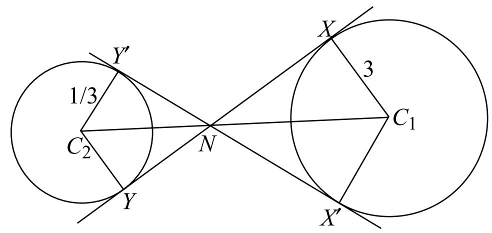
Similarly, common tangents of (i) and (iv) are T_1, T_2 and $x = -1$

For the circles (iii) and (iv), there will be four common tangents of which 2 are direct and another two are transverse common tangents.

In two triangles, ΔC_1XN, ΔC_2YN

$$\frac{C_1N}{C_2N} = \frac{3}{1/3} = 9$$

Thus, N divides C_1C_2 in the ratio 9:1



Clearly, N lies on x-axis

$$N = \left(-\frac{4}{5}, 0\right)$$

Any line through N is $y = m\left(x + \dfrac{4}{5}\right)$

$5mx - 5y + 4m = 0$

If is is tangent to the circle (iii), then

$$\left|\frac{20m + 4m}{\sqrt{2m^2 + 25m}}\right| = 3$$

$$m = \pm\frac{5}{\sqrt{39}}$$

Hence, the required tangents are

$$y = \pm\frac{5}{\sqrt{39}}\left(x + \frac{4}{5}\right)$$

59. Let $z_1 = Q = 3 + 4i$

Then $z_2 = R = iz = i(3 + 4i) = 3i - 4$

$= -4 + 3i$

$= (-4, 3)$

Thus, $\angle QOR = \dfrac{\pi}{2}$.

Clearly, $\angle QPR = \dfrac{\pi}{4}$.

60. The given circles are orthogonal. So,

$2(g_1g_2 + f_1f_2) = c_1 + c_2$

$\Rightarrow$ $2(1.0 + k \cdot k) = k + 6$

$\Rightarrow$ $2k^2 = k + 6$

$\Rightarrow$ $2k^2 - k - 6 = 0$

$\Rightarrow$ $2k^2 - 4k + 3k - 6 = 0$

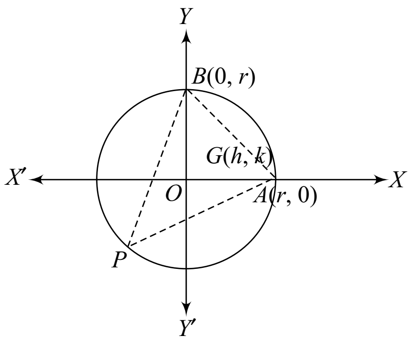
$\Rightarrow$ $2k(k - 2) + 3(k - 2) = 0$

$\Rightarrow$ $(k - 2)(2k + 3) = 0$

$\Rightarrow$ $(k - 2) = 0, (2k + 3) = 0$

$\Rightarrow$ $k = 2, -\dfrac{3}{2}$

61.



Let $A = (r, 0)$, $B = (0, r)$
and $P = (r\cos\theta, r\sin\theta)$.

We have $h = \frac{1}{3}(r + r\cos\theta)$, $k = \frac{1}{3}(r + r\sin\theta)$

$$\therefore \quad \left(h - \frac{r}{3}\right)^2 + \left(k - \frac{r}{3}\right)^2 = \frac{r^2}{9}(\cos^2\theta + \sin^2\theta)$$

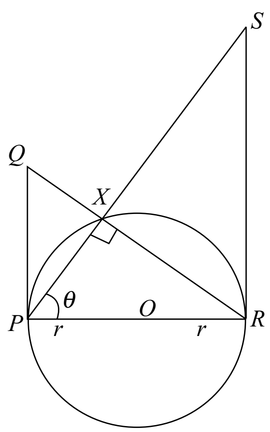
Hence, the locus of $G(h, k)$ is

$$\left(x - \frac{r}{3}\right)^2 + \left(y - \frac{r}{3}\right)^2 = \frac{r^2}{9}$$

which represents a circle.

62. Ans. $OA = 9 + 3\sqrt{10}$

63. Let $\angle SPR = \theta$



Then $\angle QRP = \left(\frac{p}{2} - \theta\right)$, $\angle PQR = \theta$

In ΔPQR, $\tan\left(\frac{\pi}{2} - \theta\right) = \frac{PQ}{PR}$

$$\Rightarrow \quad \cot\theta = \frac{PQ}{2r} \quad \ldots\text{(i)}$$

Also, in ΔPRS, $\tan\theta = \frac{RS}{PR} = \frac{RS}{2r}$ …(ii)

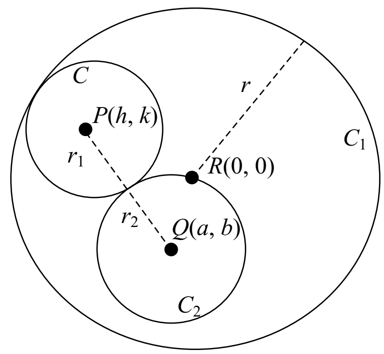
From Eqs (i) and (ii), we get

$$\frac{PQ}{2r} \cdot \frac{RS}{2r} = 1$$

$$\Rightarrow \quad 4r^2 = PQ \cdot RS$$

$$\Rightarrow \quad 2r = \sqrt{PQ \cdot RS}$$

64.



Let C_1 be the circle with centre $R(0, 0)$ and radius r.
Thus, its equation is $x^2 + y^2 = r^2$.
Let C_2 be $(x - a)^2 + (y - b)^2 = r_1^{\,2}$
and C be $(x - h)^2 + (y - k)^2 = r_2^{\,2}$.
It is given that $PR = r - r_1$ and $QR = r + r_2$

$$\sqrt{h^2 + k^2} = r - r_1 \text{ and } \sqrt{(h - a)^2 + (k - b)^2} = r_1 + r_2$$

Adding, we get

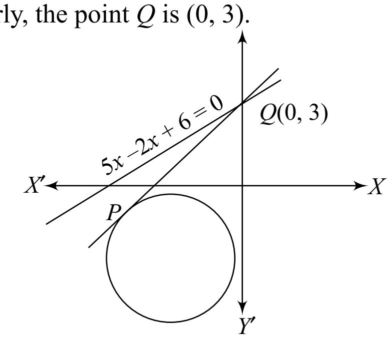
$$\sqrt{h^2 + k^2} + \sqrt{(h - a)^2 + (k - b)^2} = r + r_2$$

Thus, the locus of $P(h, k)$ is

$$\sqrt{x^2 + y^2} + \sqrt{(x - a)^2 + (y - b)^2} = r + r_2$$

which represents an ellipse with foci at $R(0, 0)$ and $Q(a, b)$ and the length of the major axis is $r + r_2$.

65. Clearly, the point Q is $(0, 3)$.



Now, the length of the tangent PQ from Q to the circle $x^2 + y^2 + 6x + 6y = 2$ is

$$PQ = \sqrt{0 + 9 + 0 + 18 - 2} = 5$$

66. Given common tangent is

$$y = mx - b\sqrt{1 + m^2}$$

$$\Rightarrow \quad mx - y - b\sqrt{1 + m^2} = 0$$

Now, the length of the perpendicular from the 2nd circle is equal to the radius of the circle.

$$\Rightarrow \quad \left|\frac{am - b\sqrt{1 + m^2}}{\sqrt{m^2 + 1}}\right| = b$$

$$\Rightarrow \quad am - b\sqrt{1 + m^2} = b\sqrt{m^2 + 1}$$

$$\Rightarrow \quad am = 2b\sqrt{1 + m^2}$$

$\Rightarrow \quad a^2m^2 = 4b^2(1 + m^2)$

$\Rightarrow \quad a^2m^2 = 4b^2 + 4b^2m^2$

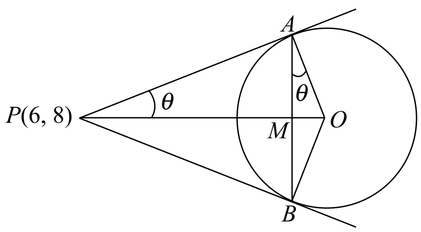
$\Rightarrow \quad (a^2 - 4b^2)\, m^2 = 4b^2$

$$\Rightarrow \quad m^2 = \frac{4b^2}{(a^2 - 4b^2)}$$

$$\Rightarrow \quad m = \sqrt{\frac{4b^2}{(a^2 - 4b^2)}}$$

$$\Rightarrow \quad m = \frac{2b}{\sqrt{(a^2 - 4b^2)}}$$

67. Given circle is $x^2 + y^2 = r^2$



We have $OP = \sqrt{6^2 + 8^2} = 10$

$BM = r \cos \theta$, $OM = r \sin \theta$

where $0 < \theta < \dfrac{\pi}{2}$

Also, $\sin \theta = \dfrac{r}{10}$

If A denotes the area of the triangle PAB, then

$$A = 2ar\,(\Delta PBM)$$
$$= 2 \times \frac{1}{2} \times PM \times BM$$
$$= PM \times BM$$
$$= (OP - OM) \cdot BM$$
$$= (10 - r \sin \theta) \cdot r \cos \theta$$
$$= (10 - 10 \sin \theta \cdot \sin \theta)(10 \sin \theta \cdot \cos \theta)$$
$$= 100 \sin \theta \cos^3 \theta$$

$$\Rightarrow \quad \frac{dA}{d\theta} = 100 \cos \theta \cos^3\theta - 300 \sin^2\theta \cos^2\theta$$

$$= 300 \cos^4\theta \left(\frac{1}{\sqrt{3}} - \tan \theta\right)\left(\frac{1}{\sqrt{3}} + \tan \theta\right)$$

For maxima and minima, $\dfrac{dA}{d\theta} = 0$ gives

$$\frac{1}{\sqrt{3}} - \tan \theta = 0$$

$$\Rightarrow \quad \theta = \frac{\pi}{6}$$

$$\text{Thus, } \frac{dA}{d\theta} = \begin{cases} > 0 : 0 < \theta < \dfrac{\pi}{6} \\ < 0 : \dfrac{\pi}{6} < \theta < \dfrac{\pi}{2} \end{cases}$$

Therefore A is maximum, when $\theta = \dfrac{\pi}{6}$ and $r = 10.\sin\left(\dfrac{\pi}{6}\right) = 5$

68. From Figure (i),

$$I_n = n \cdot \frac{1}{2} \cdot (OA_1) \cdot (OA_1) \sin\left(\frac{2\pi}{n}\right)$$
$$= \frac{\pi}{2} \sin\left(\frac{2\pi}{n}\right)$$

From figure (ii)

$$B_1B_2 = 2(B_1L)$$
$$= 2(OL) \tan\left(\frac{\pi}{n}\right)$$
$$= 2 \cdot 1 \cdot \tan\left(\frac{\pi}{n}\right)$$
$$= 2 \tan\left(\frac{\pi}{n}\right)$$

Thus, $O_n = n\left(\dfrac{1}{2}(B_1B_2)(OL)\right) = n \tan\left(\dfrac{\pi}{n}\right)$

Now, $\dfrac{I_n}{O_n} = \dfrac{(n/2)\sin(2\theta)}{n \tan \theta}$,

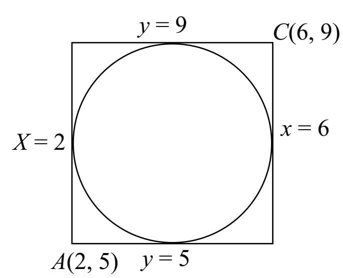
where $\theta = \dfrac{\pi}{n} = \dfrac{2 \tan \theta}{(1 + \tan^2\theta)} \cdot \dfrac{1}{2 \tan \theta} = \cos^2 \theta$

$$= \frac{1}{2}(2 \cos^2\theta)$$
$$= \frac{1}{2}(1 + \cos 2\theta)$$
$$= \frac{1}{2}\left(1 + \sqrt{1 - \left(\frac{2I_n}{n}\right)^2}\right)$$
$$I_n = \frac{O_n}{2}\left(1 + \sqrt{1 - \left(\frac{2I_n}{n}\right)^2}\right)$$

69. Given $x^2 - 8x + 12 = 0$ and $y^2 + 14y + 45 = 0$

$\Rightarrow \quad (x - 2)(x - 6) = 0$ and $(y - 5)(y - 9) = 0$

$\Rightarrow \quad x = 2, x = 6$ and $y = 5, y = 9$



Hence, the centre is $\left(\dfrac{2 + 6}{2}, \dfrac{5 + 9}{2}\right) = (4, 7)$

70. The equation of a circle C is

$$(x - 2)^2 + (y - 1)^2 = r^2 \quad \ldots\text{(i)}$$

Given circle is

$$x^2 + y^2 - 2x - 6y + 6 = 0 \quad \ldots\text{(ii)}$$

The equation of the common chord is

$$-2x + 4y + 5 - r^2 - 6 = 0$$

which is a diameter of the circle (ii).

Thus, $-2.1 + 4.3 + 5 - r^2 - 6 = 0$

$\Rightarrow \quad 9 - r^2 = 0$

$\Rightarrow \quad r = 3$

71. Given $\left|\frac{z-\alpha}{z-\beta}\right| = k$

$\Rightarrow \quad \frac{|z-\alpha|^2}{|z-\beta|^2} = k^2$

$\Rightarrow \quad \frac{(z-\alpha)(\bar{z}-\bar{\alpha})}{(z-\beta)(\bar{z}-\bar{\beta})} = k^2$

$\Rightarrow \quad (z-\alpha)(\bar{z}-\bar{\alpha}) = k^2(z-\beta)(\bar{z}-\bar{\beta})$

$\Rightarrow \quad |z|^2 - \alpha\bar{z} - \bar{\alpha}z + |\alpha|^2 = k^2(|z|^2 - \beta\bar{z} - \bar{\beta}z + |\beta|^2)$

$\Rightarrow \quad (1-k^2)|z|^2 - (\alpha - k^2\beta)\bar{z} - (\bar{\alpha} - \bar{\beta}k^2)z + (|\alpha|^2 - k|\beta|^2) = 0$

$\Rightarrow \quad |z|^2 - \frac{(\alpha - k^2\beta)}{(1-k^2)}\bar{z} - \frac{(\bar{\alpha} - \bar{\beta}k^2)}{(1-k^2)}z + \frac{(|\alpha|^2 - k|\beta|^2)}{(1-k^2)} = 0$

Thus, the centre of a circle is $= \frac{(\alpha - k^2\beta)}{(1-k^2)}$.

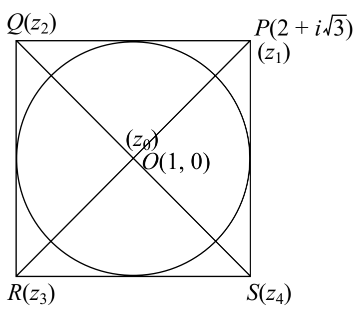
and its radius

$$= \sqrt{\left|\frac{(\alpha - k^2\beta)}{(1-k^2)}\right|^2 - \left(\frac{|\alpha|^2 - k^2\beta\bar{\beta}}{(1-k^2)}\right)}$$

$$= \sqrt{\left(\frac{(\alpha - k^2\beta)}{(1-k^2)}\right)\left(\frac{(\bar{\alpha} - k^2\bar{\beta})}{(1-k^2)}\right) - \left(\frac{|\alpha|^2 - k^2\beta\bar{\beta}}{(1-k^2)}\right)}$$

$$= \left|\frac{k(\alpha-\beta)}{1-k^2}\right|$$

72.



Since the centre of a square coincides with the centre of a circle, so

$\frac{z_1 + z_3}{2} = 1$

$\Rightarrow \quad z_1 + z_3 = 2$

$\Rightarrow \quad z_3 = 2 - z_1 = 2 - (2 + i\sqrt{3}) = -i\sqrt{3}$

Here, $\angle z_1 z_0 z_2 = \frac{\pi}{2}$

By the rotation theorem,

$$\left(\frac{z_2 - z_0}{z_1 - z_0}\right) = \left|\left(\frac{z_2 - z_0}{z_1 - z_0}\right)\right| e^{i\pi/2} = i$$

$\Rightarrow \quad (z_2 - z_0) = i(z_1 - z_0)$

$\Rightarrow \quad z_2 = z_0 + i(z_1 - z_0)$

$\Rightarrow \quad z_2 = 1 + i(2 + i\sqrt{3} - 1) = 1 + i - \sqrt{3}$

$\Rightarrow \quad z_2 = (1 - \sqrt{3}) + i$

Also, $z_4 = 2 - z_2 = 2 - (1 - \sqrt{3}) + i$

$\Rightarrow \quad z_4 = (1 + \sqrt{3}) - i$

73. The equation of the family of circle is

$(x-1)^2 + (y+1)^2 + \lambda(2x + 3y + 1) = 0$

$x^2 + y^2 - 2x + 2y + 2 + \lambda(2x + 3y + 1) = 0$

$x^2 + y^2 + 2(\lambda - 1)x + (3\lambda + 2)y + (\lambda + 2) = 0 \quad$ ...(i)

The equation (i) is orthogonal to the circle

$x(x+2) + (y+1)(y-3) = 0$

$x^2 + y^2 + 2x - 2y - 3 = 0$

Therefore,

$$2\left[(\lambda - 1)\cdot 1 + \frac{(3\lambda + 2)}{2}\cdot(-1)\right] = \lambda - 1$$

$\Rightarrow \quad 2\lambda - 2 - 3\lambda - 2 = \lambda - 1$

$\Rightarrow \quad \lambda = -\frac{3}{2}$

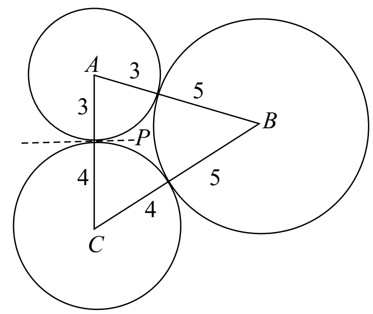
Hence, the equation of the circle is

$$(x-1)^2 + (y+1)^2 - \frac{3}{2}(2x + 3y + 1) = 0$$

$\Rightarrow \quad 2(x^2 + y^2) - 4x + 4y + 4 - 6x - 9y - 3 = 0$

$\Rightarrow \quad 2(x^2 + y^2) - 10x - 5y + 1 = 0$

74. Let A, B, C be the centres of the 3- given circles.



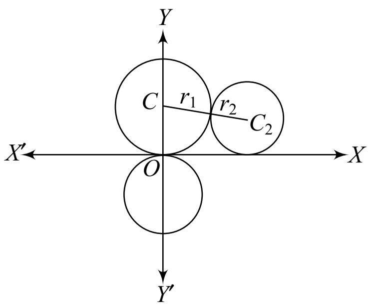
Clearly P is the incentre of the ΔABC.

Thus, $r = \frac{\Delta}{s} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}$

$\Rightarrow \quad r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$

$\Rightarrow \quad r = \sqrt{\frac{5.4.3}{12}} = \sqrt{5}$, since $s = 12$

75. Given $C_1 = (0, 1)$



Let $C_2 = (x, y)$

Then $r_1 + r_2 = C_1C_2$

$\Rightarrow \quad 1 + |y| = \sqrt{x^2 + (y-1)^2}$

$\Rightarrow \quad 1 + y^2 + 2|y| = x^2 + y^2 - 2y + 1$

$\Rightarrow \quad x^2 = 2|y| + 2y$

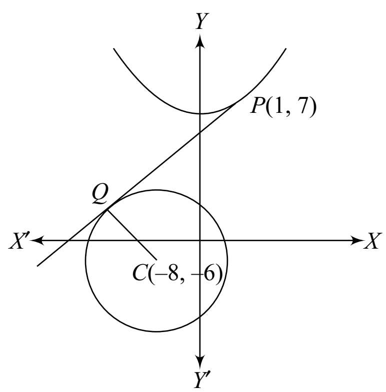
$\Rightarrow \quad x^2 = 4y$ if $y \geq 0$

Thus, the locus of its centre is

$\{(x, y) : x^2 = 4y\} \cup \{(0, y) : y \leq 0\}$

76. The equation of the tangent to the curve $y = x^2 + 6$ at $P(1, 7)$ is

$2x - y + 5 = 0 \qquad \ldots\text{(i)}$



Here CQ is perpendicular to PQ.

The equation of CQ is $-x - 2y + k = 0$

which is passing through the centre $(-8, -6)$. So,

$8 + 12 + k = 0$

$k = -20$

The equation of CQ is $-x - 2y - 20 = 0$

$x + 2y + 20 = 0 \qquad \ldots\text{(ii)}$

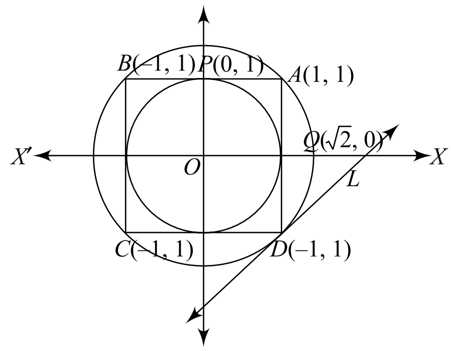
On solving Eqs. (i) and (ii), we get

$x = -6$ and $y = -7$.

Therefore, the co-ordinates of Q are $(-6, -7)$

77. Without loss of genrality, we can assume the square $ABCD$ with its vertices $A(1, 1)$, $B(-1, 1)$, $C(-1, 1)$, $D(1, -1)$

Let P be $(0, 1)$ and Q at $(\sqrt{2}, 0)$



Then $\dfrac{PA^2 + PB^2 + PC^2 + PD^2}{QA^2 + QB^2 + QC^2 + QD^2}$

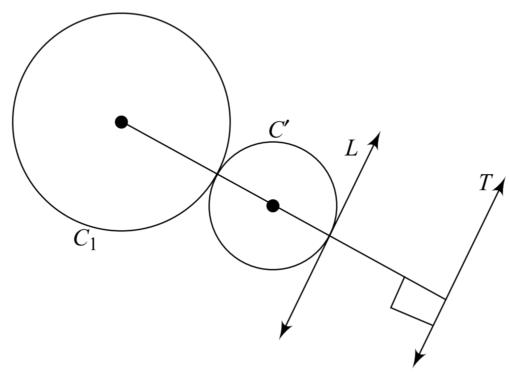
$= \dfrac{1 + 1 + 5 + 5}{2[(\sqrt{2} - 1)^2 + 1] + 2[(\sqrt{2} + 1)^2 + 1]}$

$= \dfrac{12}{16} = \dfrac{3}{4} = 0.75$

78. Let C' be the said circle touching C_1 and L, so that C_1 and C' are on the same side of L.

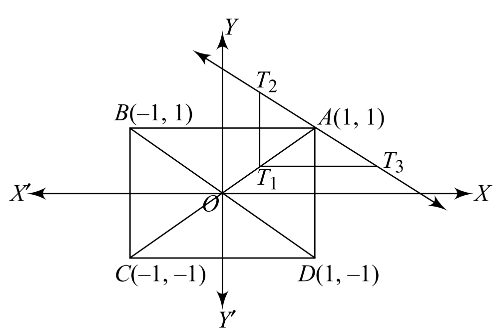
Let us draw a line T parallel to L at a distance equal to the radius of the circle C_1, on opposite side of L

Then the centre of C' is equivalent from the centre of C_1 and from line T



Locus of centre of C' is a parabola

79. Since S is equidistant from A and line BD, it traces a parabola.



Clearly, AC is the axis, $A(1, 1)$ is the focus and $T_1\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$ is the vertex of the parabola.

$AT_1 = \frac{1}{\sqrt{2}}$ and T_2T_3 = latus rectum of parabola

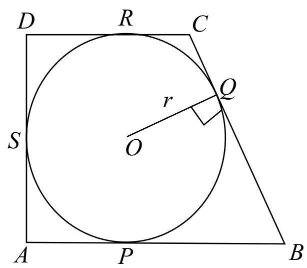
$= 4 \times \frac{1}{\sqrt{2}} = 2\sqrt{2}$

Thus, the area of the $\Delta T_1T_2T_3$

$= \frac{1}{2} \times \frac{1}{\sqrt{2}} \times 2\sqrt{2} = 1$ Sq unit.

Note. No questions asked in 2015

80. Given $AB \parallel CD \cdot CD = 2AB$.



Let $AB = 2a$, $CD = a$

and the radius of the circle be r.

Let the circle touches AB at P, BC at Q, AD at R and CD at S.

Then $AR = AP = r$, $BP = BQ = a - r$,

$DR = DS = r$ and $CQ = CS = 2a - r$

In ΔBEC,

$BC^2 = BE^2 + EC^2$

$\Rightarrow \quad (a - r + 2a - r)^2 = (2r)^2 + a^2$

$\Rightarrow \quad (3a - 2r)^2 = (2r)^2 + a^2$

$\Rightarrow \quad 9a^2 + 4r^2 - 12ar = 4r^2 + a^2$

$\Rightarrow \quad a = \frac{3}{2}r$

Also, ar(Quad. $ABCD$) = 18

$\Rightarrow \quad ar$(Quad. $ABED$) + $ar(\Delta BCE)$ = 18

$\Rightarrow \quad a \cdot 2r + \frac{1}{2} \cdot a \cdot 2r = 18$

$\Rightarrow \quad 3ar = 18$

$\Rightarrow \quad 3 \times \frac{3}{2} \times r^2 = 18$

$\Rightarrow \quad r^2 = 4$

$\Rightarrow \quad r = 2$

Thus, the radius is $r = 2$.

81. The equation od any tangent to the given circle is

$$y = mx + a\sqrt{1 + m^2}$$

which is passing through (17, 7).

Thus, $7 = 17m + 13\sqrt{1 + m^2}$

$\Rightarrow \quad (7 - 17m)^2 = 169(1 + m^2)$

$\Rightarrow \quad 49 + 289\, m^2 - 238m = 169(1 + m^2)$

$\Rightarrow \quad 120m^2 - 238m - 120 = 0$

Let its roots are m_1, m_2.

Therefore, $m_1m_2 = -1$

$\Rightarrow$ the tangents are mutually perpendicular.

As we know that the point of intersection of two mutually perpendicular tangents is the director circle.

So, the equation of the director circle is

$$x^2 + y^2 = 338$$

Therefore, the Statement II is the correct explanation of the Statement I.

82. Given circle is $(x + 3)^2 + (y - 5)^2 = 4$

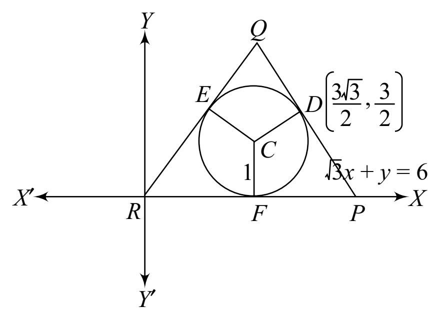
So, the radius = 2

Distance between the parallel lines L_1 and L_2 is

$$\left|\frac{(p+3)-(p-3)}{\sqrt{4+9}}\right| = \frac{6}{\sqrt{13}} < \text{radius } (2)$$

So, the Statement II is false, but the Statement I is true.

Comprehension



83. Co-ordinates of C are

$$\Rightarrow \quad \frac{x - \frac{3\sqrt{3}}{2}}{\cos\left(\frac{\pi}{6}\right)} = \frac{y - \frac{3}{2}}{\sin\left(\frac{\pi}{6}\right)} = -1$$

$$\Rightarrow \quad x - \frac{3\sqrt{3}}{2} = -\frac{\sqrt{3}}{2},\ y - \frac{3}{2} = -\frac{1}{2}$$

$$\Rightarrow \quad x = \sqrt{3},\ y = 1$$

Thus, $C = (\sqrt{3}, 1)$

Hence, the equation of the circle is

$$(x - \sqrt{3})^2 + (y - 1)^2 = 1$$

84. Clearly, the point F is $(\sqrt{3}, 0)$.

Now the co-ordinates of E are

$$\Rightarrow \quad \frac{x - \sqrt{3}}{\cos(150°)} = \frac{y - 1}{\sin(150°)} = 1$$

$$\Rightarrow \quad \frac{x - \sqrt{3}}{-\frac{\sqrt{3}}{2}} = \frac{y - 1}{\frac{1}{2}} = 1$$

$$\Rightarrow \quad x - \sqrt{3} = -\frac{\sqrt{3}}{2},\ y = 1 - \frac{1}{2}$$

$\Rightarrow \quad x = \sqrt{3} - \frac{\sqrt{3}}{2}, y = \frac{1}{2}$

$\Rightarrow \quad x = \frac{\sqrt{3}}{2}, y = \frac{1}{2}$

Therefore, $E = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), F = (\sqrt{3}, 0)$

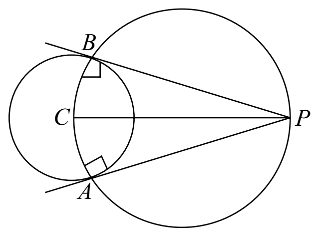
85. Co-ordinates of Q are $(\sqrt{3}, 3)$.

The equation of QR is $y - 0 = \frac{3}{\sqrt{3}}(x - 0)$

$\Rightarrow \quad y = \sqrt{3}x$

and the equation of RP is $y = 0$.

86. Given circle is $x^2 + y^2 - 6x - 4y - 11 = 0$



Therefore, the centre is (3, 2)

Since CA and CB are perpendicular to PA and PB.

So, CP is the diameter of the circumcircle PAB.

Thus, the equation of the circumcircle triangle PAB is

$(x - 3)(x - 1) + (y - 2)(y - 8) = 0$

$\Rightarrow \quad x^2 + y^2 - 4x - 10y + 19 = 0$

87. Clearly, the length of the perpendicular from the centre of the circle is equal to the radius of the circle.

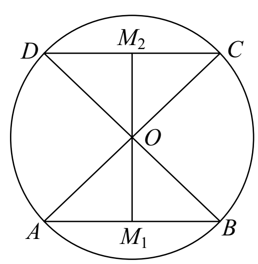
Thus, $\left|\frac{h \cdot 0 + k \cdot 0 - 1}{\sqrt{h^2 + k^2}}\right| = 2$

$\Rightarrow \quad (h^2 + k^2) = \frac{1}{4}$

Hence, the locus of the (h, k) is

$x^2 + y^2 = \frac{1}{4}$

88.



In ΔOM_1A,

$OA = 2, \angle AOM_1 = \frac{\pi}{2k}$

$\cos\left(\frac{\pi}{2k}\right) = \frac{OM_1}{2} \Rightarrow OM_1 = 2\cos\left(\frac{\pi}{2k}\right)$

Similarly, $OM_2 = 2\cos\left(\frac{\pi}{k}\right)$

It is given that,

$OM_1 + OM_2 = 2$

$\Rightarrow \quad 2\cos\left(\frac{\pi}{2k}\right) + 2\cos\left(\frac{\pi}{k}\right) = (\sqrt{3} + 1)$

$\Rightarrow \quad \cos\left(\frac{\pi}{2k}\right) + \cos\left(\frac{\pi}{k}\right) = \left(\frac{\sqrt{3} + 1}{2}\right)$

$\Rightarrow \quad \cos\left(\frac{\theta}{2}\right) + \cos(\theta) = \left(\frac{\sqrt{3} + 1}{2}\right)$ where $\left(\frac{\pi}{k}\right) = \theta$

$\Rightarrow \quad \cos\left(\frac{\theta}{2}\right) + 2\cos^2\left(\frac{\theta}{2}\right) - 1 = \left(\frac{\sqrt{3} + 1}{2}\right)$

$\Rightarrow \quad 2\cos^2\left(\frac{\theta}{2}\right) + \cos\left(\frac{\theta}{2}\right) - \left(\frac{\sqrt{3} + 3}{2}\right) = 0$

$\Rightarrow \quad 2b^2 + b - \left(\frac{\sqrt{3} + 3}{2}\right) = 0, b = \cos\left(\frac{\theta}{2}\right)$

$\Rightarrow \quad 4b^2 + 2b - (\sqrt{3} + 3) = 0$

$\Rightarrow \quad b = \frac{-2 \pm \sqrt{4 + 16(3 + \sqrt{3})}}{8}$

$\Rightarrow \quad b = \frac{-1 \pm \sqrt{1 + 4(3 + \sqrt{3})}}{2}$

$\Rightarrow \quad b = \frac{-1 \pm \sqrt{1 + 4\sqrt{3} + 12}}{2}$

$\Rightarrow \quad b = \frac{-1 \pm \sqrt{(1 + 2\sqrt{3})^2}}{2} = \frac{-1 \pm (1 + 2\sqrt{3})}{2}$

$\Rightarrow \quad b = \frac{\sqrt{3}}{2}, \frac{-\sqrt{3} - 1}{2}$

$\Rightarrow \quad b = \frac{\sqrt{3}}{2}$

$\Rightarrow \quad \cos\left(\frac{\theta}{2}\right) = \frac{\sqrt{3}}{2} = \cos\left(\frac{\pi}{6}\right)$

$\Rightarrow \quad \cos\left(\frac{\theta}{2}\right) = \cos\left(\frac{\pi}{6}\right)$

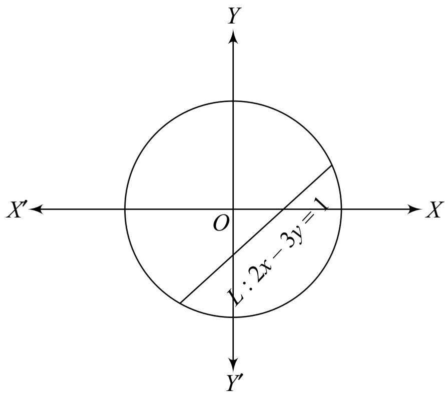
$\Rightarrow \quad \theta = \frac{\pi}{3}$

$\Rightarrow \quad \frac{\pi}{k} = \frac{\pi}{3}$

$\Rightarrow \quad k = 3$

$\Rightarrow \quad [k] = 3$

89. Let $L: 2x - 3y = 1$



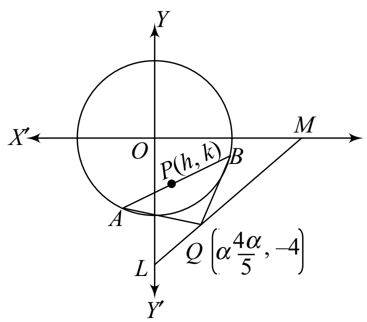
and $S: x^2 + y^2 \le 6$

If $L > 0$ and $S < 0$, the points lie on the smaller part.

Thus, the points $\left(2, \frac{3}{4}\right)$ and $\left(\frac{1}{4}, -\frac{1}{4}\right)$ lie inside the triangle.

90. The equation of the chord bisected at $P(h, k)$ is

$$hx + ky = h^2 + k^2 \quad \ldots(i)$$



Let any point on line LM be $\left(\alpha, \frac{4\alpha}{5} - 4\right)$.

The equation of the chord of contact is

$$\alpha x + \left(\frac{4\alpha}{5} - 4\right)y = 9 \quad \ldots(ii)$$

Comparing Eqs (i) and (ii), we get

$$\frac{h}{\alpha} = \frac{k}{\frac{4}{5}\alpha - 4} = \frac{h^2 + k^2}{9}$$

$$\Rightarrow \quad \alpha = \frac{20h}{4h - 5k}$$

Therefore, $\dfrac{h}{\dfrac{20h}{4h - 5k}} = \dfrac{h^2 + k^2}{9}$

$$\Rightarrow \quad \frac{4h - 5k}{20} = \frac{h^2 + k^2}{9}$$

$$\Rightarrow \quad 20(h^2 + k^2) = 9(4h - 5)$$

Hence, the locus of $P(h, k)$ is

$$20(x^2 + y^2) = 9(4x - 5y)$$

91. (i) Equation of the tangent to the circle $x^2 + y^2 = 4$

at $P(\sqrt{3}, 1)$ is $\sqrt{3}x + y = 4$

i.e $PT: \sqrt{3}x + y = 4$

Now, $m(PT) = -\sqrt{3}$

So, $m(L) = \frac{1}{\sqrt{3}}$.

The line L is $y = m(x - 3) \pm a\sqrt{1 + m^2}$

$$\Rightarrow \quad y = \frac{1}{\sqrt{3}}(x - 3) \pm 1\sqrt{1 + \frac{1}{3}}$$

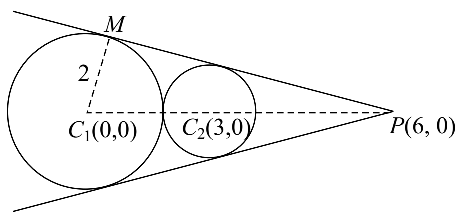
$$\Rightarrow \quad y = \frac{1}{\sqrt{3}}(x - 3) \pm \frac{2}{\sqrt{3}}$$

$$\Rightarrow \quad y = \frac{x - 5}{\sqrt{3}}, \frac{x - 1}{\sqrt{3}}$$

$$\Rightarrow \quad x - \sqrt{3}y - 5 = 0, x - \sqrt{3}y - 1 = 0$$

(ii) Given circles are

$x^2 + y^2 = 4, (x - 3)2 + y^2 = 1$



Clearly, the point of intersection is (6, 0).

The equation of the direct common tangent is

$$y - 0 = m(x - 6)$$

$$\Rightarrow \quad mx - y + 6m = 0$$

Now, $C_1M = 2$

$$\Rightarrow \quad \left|\frac{6m}{\sqrt{m^2 + 1}}\right| = 2$$

$$\Rightarrow \quad m^2 + 1 = 9m^2$$

$$\Rightarrow \quad 8m^2 = 1$$

$$\Rightarrow \quad m = \pm\frac{1}{2\sqrt{2}}.$$

Hence, the equation of the common tangents are

$$y = \pm\frac{1}{2\sqrt{2}}(x - 6) \text{ and } x = 2$$

92. Given circles are $x^2 + y^2 - 2x - 15 = 0$

and $x^2 + y^2 = 1$

Let the equation of the circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

It passes through (0, 1), so

$$1 + 2f + c = 0$$

Applying condition of orthogonality, we have

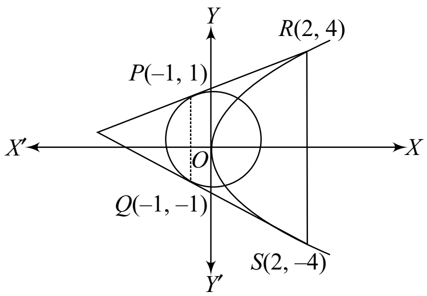
$$-2g = c - 15, 0 = c - 1$$

$$\Rightarrow \quad c = 1, g = 7, f = -1$$

Thus, $r = \sqrt{49 + 1 - 1} = 7$

and centre = (–7, 1)

93.



The equation of any tangent to the parabola can be considered as $y = mx + \frac{a}{m} = mx + \frac{2}{m}$.

i.e. $m^2x - my + 2 = 0$

As we know that the length of the perpendicular from the centre to the tangent to the circle is equal to the radius of a circle.

Thus, $\frac{2}{\sqrt{m^4 + m^2}} = \sqrt{2}$

$m^4 + m^2 = 2$

$m^4 + m^2 - 2 = 0$

$(m^2 + 2)(m^2 - 1) = 0$

$m = \pm 1$

Hence, the equation of the tangents are

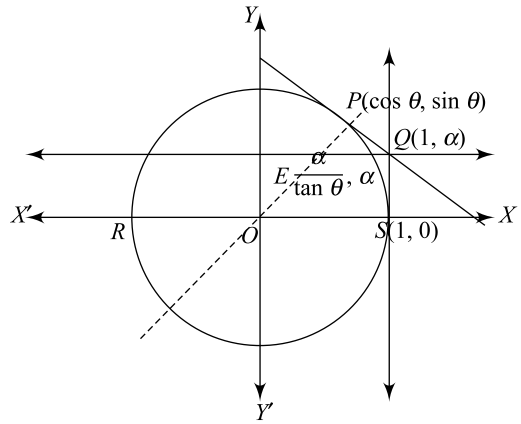
$y = x + 2, y = -x - 2$

Therefore, the points P, Q are (–1, 1), (–1, –1) and R, S are (2, 4) and (2, –4) respectively.

Thus, the area of the equadrilateral $PQRS$

$= \frac{1}{2} \times (2 + 8) \times 3 = 15$

94. Ans. (a, c)



$E = \left(\frac{\alpha}{\tan\theta}, \alpha\right)$

$\cos\theta + \alpha \sin\theta = 1$

$\alpha = \tan\left(\frac{\theta}{2}\right)$

Hence, the locus is $y^2 = 1 - 2x$.

Thus, the required points are $\left(\frac{1}{3}, \frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{3}, -\frac{1}{\sqrt{3}}\right)$.

CHAPTER

4

Parabola

Concept Booster

1. Introduction

It is believed that the first definition of a conic section is due to Menaechmus (died 320 BC). His work did not survive and is only known through secondary accounts. The definition used at that time differs from the one commonly used today. It requires the plane cutting the cone to be perpendicular to one of the lines (a generatrix), that generates the cone as a surface of revolution. Thus the shape of the conic is determined by the angle formed at the vertex of the cone (between two opposite generatrices). If the angle is acute, the conic is an ellipse; if the angle is right, the conic is a parabola; and if the angle is obtuse, the conic is a hyperbola.

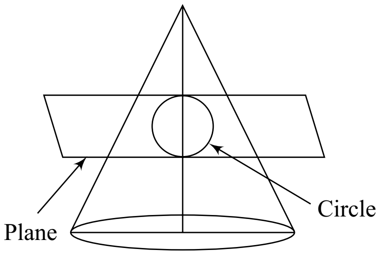
Note: The circle cannot be defined in this way and was not considered as a conic at this time.

Euclid (300 BC) is said to have written four books on conics but these were lost as well. Archimedes (died 212 BC) is known to have studied conics, having determined the area bounded by a parabola and an ellipse. The only part of this work to survive is a book on the solids of revolution of conics.

2. Basic definitions

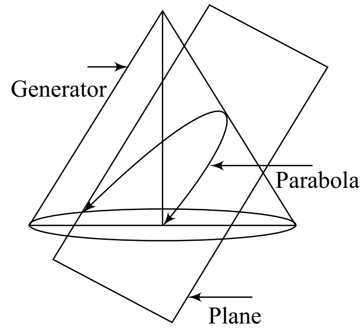
(i) Circle

The section of a right circular cone by a plane which is parallel to its base is called a circle.



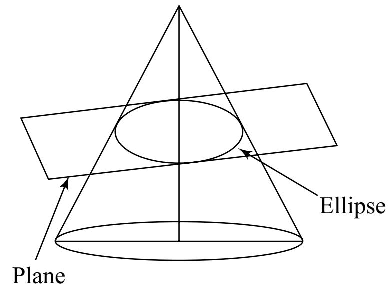
(ii) Parabola

The section of a right circular cone by a plane which is parallel to a generator of a cone is called a parabola.



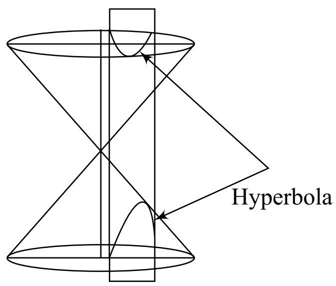
(iii) Ellipse

The section of a right circular cone by a plane which is neither parallel to a generator of a cone nor parallel or perpendicular to the axis of a cone is called an ellipse.



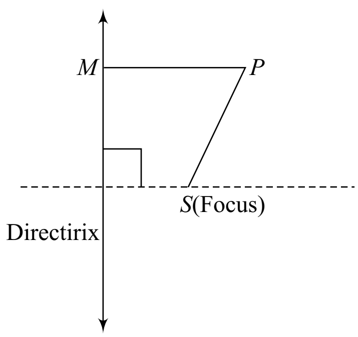
(iv) Hyperbola

The section of a double right circular cone by a plane which is parallel to the axis of a cone is called a hyperbola.



3. Conic Section

It is the locus of a point which moves in a plane in such a way that its distance from a fixed point to its perpendicular distance from a fixed straight line is always constant. The fixed point is called the *focus* of the conic and this fixed straight line is called the *directrix* of the conic and this constant ratio is known as the *eccentricity* of the conic. It is denoted as e.



Conic Section with Respect to Eccentricity

(i) If $e = 0$, the conic section is called a circle
(ii) If $e = 1$, the conic section is called a parabola.
(iii) If $e < 1$, the conic section is called an ellipse.
(iv) If $e > 1$, the conic section is called a hyperbola.
(v) If $e = \sqrt{2}$, the conic section is called a rectangular hyperbola.

Some Important Definitions to Remember

Axis: The straight line passing through the focus and perpendicular to the directrix is called the axis of the conic section.

Vertex: The point of intersection of the conic section and the axis is called the vertex of the conic section.

Double ordinate
Any chord, which is perpendicular to the axis of the conic section, is called a double ordinate of the conic section.

Focal chord
Any chord passing through the focus is called the focal chord of the conic section.

Focal distance
The distance between the focus and any point on the conic is known as the focal distance of the conic section.

Latus rectum
Any chord passing through the focus and perpendicular to the axis is known as latus rectum of the conic section.

Centre
The point which bisects every chord of the conic passing through it, is called the centre of the conic section.

4. Recognition of Conics

A general equation of 2nd degree is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

where $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

$$= abc + 2fgh - af^2 - bg^2 - ch^2$$

and $H = \begin{vmatrix} a & h \\ h & b \end{vmatrix} = ab - h^2$

There are two types of conics.
(i) Degenerate conic, and
(ii) Non-degenerate conic.

We use the term degenerate conic sections to describe the single point, single line and pair of lines and the term non-degenerate conic sections to describe those conic sections that are circles, parabolas, ellipses or hyperbolas.

A non-degenerate conic represents
(i) a circle if, $\Delta \neq 0$, $h = 0$, $a = b$
(ii) a parabola if $\Delta \neq 0$, $H = 0$
(iii) an ellipse, if $\Delta \neq 0$, $H < 0$
(iv) a hyperbola, if $\Delta \neq 0$, $H > 0$
(v) a rectangular hyperbola, if $\Delta \neq 0$, $H > 0$ and $a + b = 0$.

Now, the centre of the conics is obtained by

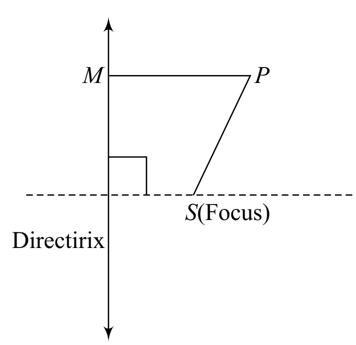
$$\frac{\delta f}{\delta x} = 2ax + 2hy + 2g = 0$$

and $$\frac{\delta f}{\delta y} = 2hx + 2by + 2f = 0.$$

$\Rightarrow \quad ax + hy + g = 0, hx + by + f = 0$

Solving the above equations, we get the required centre of the given conic.

5. Equation of Conic Section



Let the focus be (h, k), directrix be $ax + by + c = 0$ and the eccentricity is e.

Then the equation of the conic section is $\dfrac{SP}{PM} = e$

$$\Rightarrow \qquad \sqrt{(x-h)^2 + (y-k)^2} = e\left|\frac{ax+by+c}{\sqrt{a^2+b^2}}\right|$$

$$\Rightarrow \qquad (x-h)^2 + (y-k)^2 = e^2\left(\frac{(ax+by+c)^2}{a^2+b^2}\right)$$

which is the general equation of the conic section.

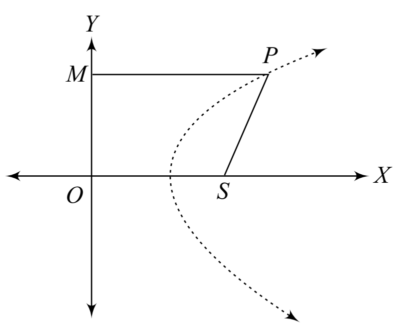
6. Parabola

The term parabola comes from Greek word, *para* 'alongside, nearby, right up to,' and *bola*, from the verb *ballein* means 'to cast, to throw.' Understandably, parallel and many of its derivatives start with the same root. The word parabola may thus mean 'thrown parallel' in accordance with the definition.

7. Mathematical Definitions

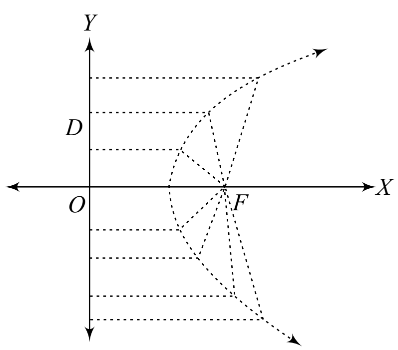
Definition 1

It is the locus of a point which moves in a plane in such a way that its distance from a fixed point is equal to its distance from a fixed straight line. The fixed point S is called the focus and the fixed straight line OM is called the directrix.



Definition 2

Let D be a line in the plane and F a fixed point not on D. A parabola is the collection of points in the plane that are equidistant from F and D. The point F is called the focus and the line D is called the directrix.



Definition 3

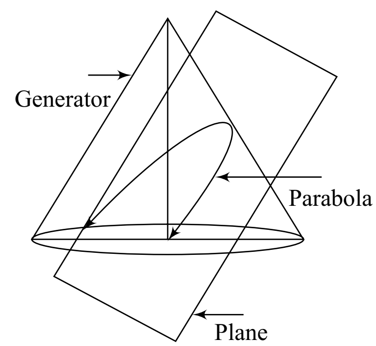
In algebra, the parabolas are frequently encountered as graphs of quadratic functions, such as $y = ax^2 + bx + c$ or $x = ay^2 + by + c$.

Definition 4

It is a section of a conic, whose eccentricity is 1.

Definition 5

A plane curve formed by the intersection of a right circular cone and a plane parallel to an element of the cone is called parabola.



Definition 6

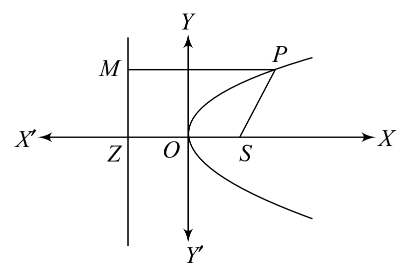
A conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a parabola if

(i) $\Delta \neq 0$

(ii) $h^2 - ab = 0$, where

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

8. Standard Equation of a Parabola



Let S be the focus and MN be the directrix of the parabola.

Draw SZ perpendicular to ZM and let O be the mid-point of FN.

Thus $OS = OZ$

So O lies on the parabola.

Consider O as the origin and OX and OY as x and y axes, respectively.

Let $OS = OZ = a$.

Then the co-ordinates of F is $(a, 0)$ and the equation of ZM is $x + a = 0$.

Now from the definition of the parabola, we get,

$$SP = PM$$

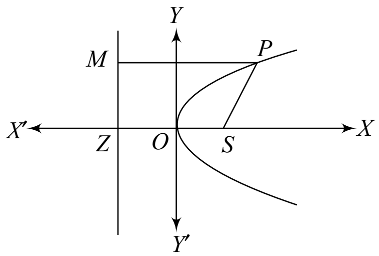
$$\Rightarrow \quad SP^2 = PM^2$$

$\Rightarrow \quad (x - a)^2 + (y - 0)^2 = (x + a)^2$

$\Rightarrow \quad y^2 = 4ax$

which is the required equation of a parabola. This is also known as horizontal parabola or right-ward parabola.

9. Important Terms Related to Parabola



Focus

It is the fixed point with reference to which the parabola is constructed. Here, S is the focus.

Directrix

It is a straight line outside the parabola. Here ZM is the directrix.

Axis of symmetry

It is the line which is perpendicular to the directrix and passes through the focus. It divides the parabola into two equal halves.

Vertex

It is the point on the axis of symmetry that intersects the parabola when the turn of the parabola is the sharpest.

The vertex is halfway between the directrix and the focus.

Focal chord

It is any chord that passes through the focus.

Latus rectum

It is that focal chord which is perpendicular to the axis of symmetry. The latus rectum is parallel to the directrix. Half of the latus rectum is called the semi-latus rectum.

Focal parameter

The distance from the focus to the directrix is called the focal parameter.

Focal distance

The distance between any point on the parabola to the focus is called the focal distance. Here, SP is the focal distance.

Parametric equation

From the equation of the parabola, we can write

$$\frac{y}{2a} = \frac{2x}{y} = t$$

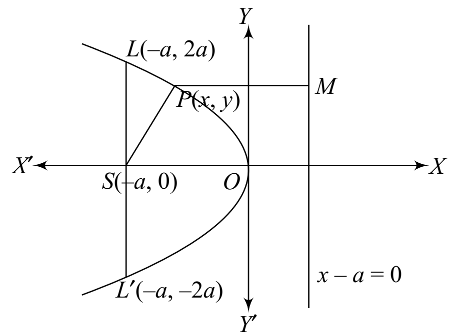
Then $x = at^2$, $y = 2at$, where t is a parameter.

The equations $x = at^2$ and $y = 2at$ are called the parametric equations and the point $(at^2, 2at)$ is also referred to as the point t.

(i) Vertex is: (0, 0)
(ii) Focus is: $(a, 0)$
(iii) Equation of the directrix is: $x + a = 0$.
(iv) Equation of the axis is: $y = 0$
(v) Equation of the tangent at the vertex is: $x = 0$
(vi) Length of the latus rectum: $4a$
(vii) Extremities of the latus rectum are: $L(a, 2a)$, $L'(a, -2a)$
(viii) Equation of the latus rectum is: $x = a$
(ix) Parametric equations of the parabola:
$y^2 = 4ax$ are $x = at^2$ and $y = 2at$
(x) Focal distance: $x + a$
(xi) Any point on the parabola can be considered as $(at^2, 2at)$.

Parabola openning leftwards

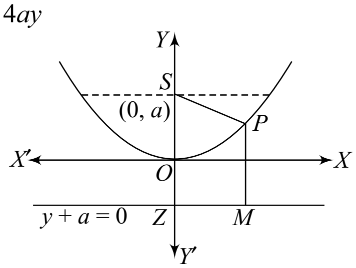
i.e $\quad y^2 = -4ax$



(i) Vertex is (0, 0)
(ii) Focus is $(-a, 0)$
(iii) Equation of the directrix is $x - a = 0$
(iv) Equation of the axis is $y = 0$
(v) Equation of the tangent at the vertex is $x = 0$
(vi) Length of the latus rectum is $4a$
(vii) Extremities of the latus rectum are:
$L(-a, 2a)$, $L'(a, -2a)$
(viii) Equation of the latus rectum is $x = -a$
(ix) Parametric equations of the parabola:
$y^2 = -4ax$ are $x = -at^2$ and $y = 2at$
(x) Focal distance is $x - a$
(xi) Any point on the parabola can be considered as $(-at^2, 2at)$.

Parabola opening upwards

i.e. $\quad x^2 = 4ay$

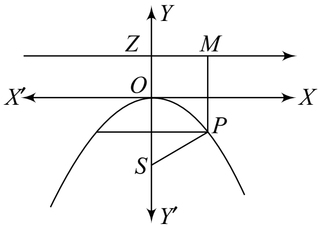


(i) Vertex is (0, 0)
(ii) Focus is $(0, a)$
(iii) Equation of the directrix is $y + a = 0$
(iv) Equation of the axis is $x = 0$
(v) Equation of the tangent at the vertex is $y = 0$
(vi) Length of the latus rectum $4a$
(vii) Extremities of the latus rectum are $L(2a, a)$, $L'(-2a, a)$
(viii) Equation of the latus rectum is $y = a$
(ix) Parametric equations of the parabola $x^2 = 4ay$ are $x = -2at$ and $y = at^2$

(x) Focal distance is $y + a$

(xi) Any point on the parabola can be considered as $(2at, at^2)$.

Parabola Opening Downwards
i.e. $x^2 = -4ay$



(i) Vertex is (0, 0)

(ii) Focus is $(0, -a)$

(iii) Equation of the directrix is $y - a = 0$

(iv) Equation of the axis is $x = 0$

(v) Equation of the tangent at the vertex is $y = 0$

(vi) Length of the latus rectum is $4a$

(vii) Extremities of the latus rectum are:

$$L(2a, -a), L'(-2a, -a)$$

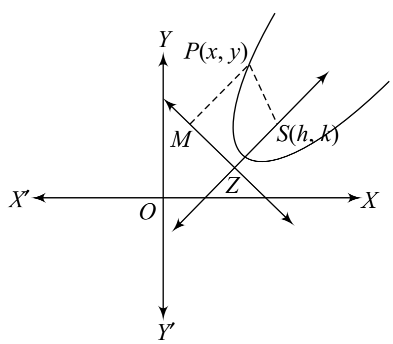
(viii) Equation of the latus rectum is $y = -a$

(ix) parametric equations of the parabola $x^2 = -4ay$ are $x = 2at$ and $y = -at^2$

(x) Any point on the parabola can be considered as $(2at, -at^2)$.

(xi) Focal distance is $y - a$

10. General Equation of a Parabola



Let $S(h, k)$ be the focus and $lx + my + n = 0$ is the equation of the directrix and $P(x, y)$ be any point on the parabola.

Then,

$$SP = PM$$

$$\Rightarrow \quad \sqrt{(x-h)^2 + (y-k)^2} = \left| \frac{lx + my + n}{\sqrt{(l^2 + m^2)}} \right|$$

$$\Rightarrow \quad (x-h)^2 + (y-k)^2 = \frac{(lx + my + n)^2}{(l^2 + m^2)}$$

$$\Rightarrow \quad m^2x^2 + l^2y^2 - 2lmxy + (\text{term})x + (\text{term})y + (\text{constant term}) = 0$$

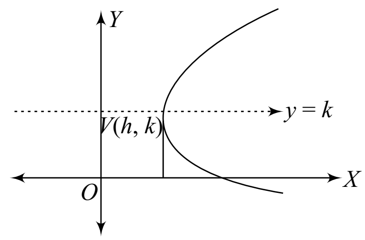
$$\Rightarrow \quad (mx - ly)^2 + 2gx + 2fy + c = 0$$

which is the general equation of a parabola.

11. Equation of a Parabola When the Vertex is (*h*, *k*) and Axis is Parallel to *x*-axis

The equation of the parabola $y^2 = 4ax$ can be written as $(y - 0)^2 = 4a(x - 0)$

The vertex of the parabola is $O(0, 0)$. Now the origin is shifted to $V(h, k)$ without changing the direction of axes, its equation becomes $(y - k)^2 = 4a(x - h)$

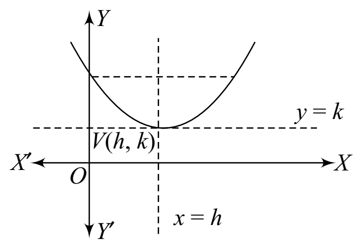


Thus its focus is $F(a + h, k)$, latus rectum $= 4a$ and the equation of the directrix is

$$x = h - a, \text{ i.e. } x + a - h = 0$$

The parametric equation of the curve $(y - k)^2 = 4a(x - h)$ are $x - h + at^2$ and $y = k + 2at$.

12. Equation of a Parabola when the Vertex is (*h*, *k*) and Axis is Parallel to *y*-axis



The equation of a parabola with the vertex $V(h, k)$ is

$$(x - h)^2 = 4a(y - k)$$

Thus, its focus is $F(h, a + k)$, latus rectum $= 4a$ and the equation of the directrix is

$$y = k - a, \text{ i.e. } y + a - k = 0$$

The parametric equation of the curve $(x - h)^2 = 4a(y - k)$ are $x = h + 2at$ and $y = k + at^2$.

***Note*:** The equation of a parabola, whose axis is parallel to y-axis can also be considered as $y = ax^2 + bx + c$.

Polar form of a Parabola

In polar coordinates, the equation of a parabola with parameters r and θ and the centre (0, 0) is given by

$$r = -\frac{2a}{1 + \cos\theta}$$

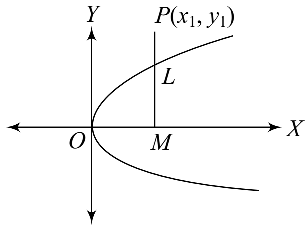
13. Focal Chord

Any line passing through the focus and intersects the parabola in two distinct points, it is known as focal chord of the parabola.

Any point on the parabola $y^2 = 4ax$ can be considered as $(at^2, 2at)$.

14. Position of a Point Relative to a Parabola

Consider the parabola $y^2 = 4ax$ and the point be (x_1, y_1).



The point (x_1, y_1) lies outside, on and inside of the parabola $y^2 = 4axy^2 = 4ax$ according as

$$y_1^2 - 4ax_1 > 0, = 0, < 0$$

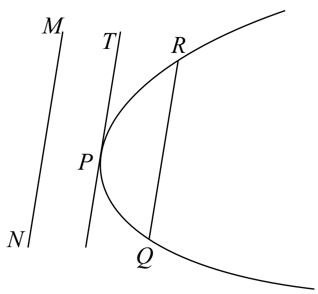
15. Intersection of a Line and a Parabola

Let the parabola be $y^2 = 4ax$ and the line be $y = mx + c$.

Eliminating x between these two equations, we get

$$y^2 = 4a\left(\frac{y-c}{m}\right)$$

$$\Rightarrow \quad my^2 - 4ay + 4ac = 0$$



The given line will cut the parabola in two distinct, coincident and imaginary points according as

$$D > 0, = 0, < 0$$

$$\Rightarrow \quad 16a^2 - 16amc > 0, = 0, < 0$$

$$\Rightarrow \quad a > cm, a = cm, a < cm$$

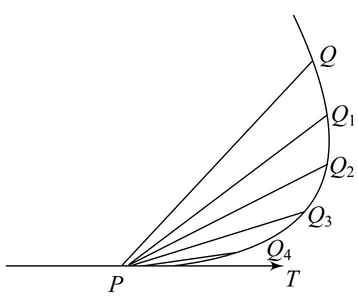
Condition of tangency: The line $y = mx + c$ will be a tangent to the parabola $y^2 = 4ax$, if $c = \dfrac{a}{m}$.

The equation of any tangent to the parabola can be considered as $y = mx + \dfrac{a}{m}$.

The co-ordinates of the point of contact in terms of m is $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$

16. Tangent

If a line intersects the parabola in two coincident points, it is known as the tangent to a parabola.



Equation of the Tangent to a Parabola in Different Forms

(i) Point form

The equation of the tangent to the parabola $y^2 = 4ax$ at (x_1, y_1) is

$$yy_1 = 2a(x + x_1)$$

Now the given equation is

$$y^2 = 4ax.$$

Differentiating with respect to x, we get

$$2y\frac{dy}{dx} = 4a$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2a}{y}$$

Now $m = \left(\dfrac{dy}{dx}\right)_{(x_1, y_1)} = \dfrac{2a}{y_1}$.

Thus the equation of tangent is

$$\Rightarrow \quad yy_1 - y_1^2 = 2ax - 2ax_1$$

$$\Rightarrow \quad yy_1 = 2ax - 2ax_1 + 4ax_1 = 2a(x + x_1)$$

which is the required equation of the tangent to the parabola $y^2 = 4ax$ at (x_1, y_1).

(ii) Parametric form

The equation of the tangent to the parabola $y^2 = 4ax$ at $(at^2, 2at)$ is

$$yt = x + at^2$$

The equation of the tangent to the parabola $y^2 = 4ax$ at (x_1, y_1) is

$$yy_1 = 2a(x + x_1)$$

$$\Rightarrow \quad y \cdot 2at = 2a(x + at^2)$$

$$\Rightarrow \quad yt = x + at^2$$

(iii) Slope form

The equation of the tangent to the parabola $y^2 = 4ax$ at $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$ is

$$y = mx + \frac{a}{m}, \text{ where } m = \text{slope}$$

The equation of the tangent to the parabola $y^2 = 4ax$ at (x_1, y_1) is

$$yy_1 = 2a(x + x_1) \qquad \ldots\text{(i)}$$

Here, $m = \dfrac{2a}{y_1} \Rightarrow y_1 = \dfrac{2a}{m}$

Since the point (x_1, y_1) lies on the parabola $y^2 = 4ax$, we have,

$$y_1^2 = 4ax_1.$$

$$\Rightarrow \quad 4ax_1 = \frac{4a^2}{m^2}$$

$$\Rightarrow \quad x_1 = \frac{a}{m^2}$$

Putting the values of x_1 and y_1 in Eq. (i), we get

$$y\cdot\left(\frac{2a}{m}\right)=2a\left(x+\frac{a}{m^2}\right)$$

$$\Rightarrow \quad y=mx+\frac{a}{m}$$

(iv) Condition of tangency

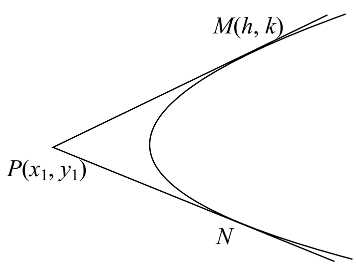
The line $y = mx + c$ will be a tangent to the parabola $y^2 = 4ax$ is $c=\frac{a}{m}$.

> ***Note:*** Any tangent to the parabola can be considered as $y=mx+\frac{a}{m}$.

(v) Director circle

The locus of the point of intersection of two perpendicular tangents to a parabola is known as the director circle.

(vi) The equation of the pair of tangents can be drawn to the parabola from the ***point*** (x_1, y_1).



Let (h, k) be any point on either of the tangents drawn from (x_1, y_1).

The equation of the line joining (x_1, y_1) and (h, k) is

$$y-y_1=\frac{k-y_1}{h-x_1}(x-x_1)$$

$$\Rightarrow \quad y=\frac{k-y_1}{h-x_1}x+\frac{hy_1-kx_1}{h-x_1}$$

If this be a tangent, it must be of the form $y=mx+\frac{a}{m}$.

Thus, $m=\frac{k-y_1}{h-x_1}$ and $\frac{a}{m}=\frac{hy_1-kx_1}{h-x_1}$

Therefore, by multiplication we get

$$a=\left(\frac{k-y_1}{h-x_1}\right)\left(\frac{hy_1-kx_1}{h-x_1}\right)$$

$\Rightarrow \quad a(h-x_1)^2=(k-y_1)(hy_1-kx_1)$

Hence, the locus of the point (h, k) is

$\Rightarrow \quad a(x-x_1)^2=(y-y_1)(xy_1-yx_1)$

$\Rightarrow \quad (y^2-4ax)(y_1^{\,2}-4ax_1)=\{yy_1-2a(x+x_1)\}^2.$

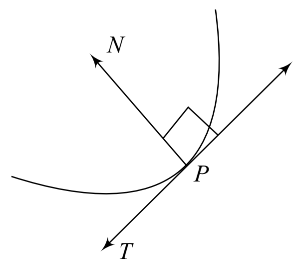
$\Rightarrow \quad SS_1=T^2$

where, $S: (y^2-4ax)$, $S_1: (y_1^{\,2}-4ax_1)$

and $\quad T: \{yy_1-2a(x+x_1)\}$.

17. Normal

It is a line which is perpendicular to the point of contact to the tangent.



Here PT is a tangent and PN is a normal.

Equation of Normals to the Parabola in Different Forms

(i) Point form

The equation of the normal to the parabola $y^2 = 4ax$ at (x_1, y_1) is

$$\frac{y-y_1}{y_1}=\frac{x-x_1}{2a}$$

The equation of the tangent to the parabola $y^2 = 4ax$ at (x_1, y_1) is

$$yy_1=2a(x+x_1) \qquad \ldots\text{(i)}$$

Slope of the tangent is $m(T)=\frac{2a}{y_1}$

Slope of the normal is $m(N)=-\frac{y_1}{2a}$

Thus the equation of the normal is

$$y-y_1=-\frac{y_1}{2a}(x-x_1)$$

(ii) Parametric form

The equation of the normal to the parabola

$y^2 = 4ax$ at $(at^2, 2at)$ is

$y = -tx + 2at + at^3$

As we know that the equation of the normal to the parabola $y^2 = 4ax$ at (x_1, y_1) is

$$\frac{y-y_1}{y_1}=-\frac{x-x_1}{2a}$$

Replacing x_1 by at^2 and y_1 by $2at$, we get

$$\frac{y-2at}{2at}=-\frac{x-at^2}{2a}$$

$\Rightarrow \quad y = -tx + 2at + at^3$

which is the required equation of the normal to the given parabola.

(iii) Slope form

The equation of the normal to the parabola

$y^2 = 4ax$ at $(ax^2, -2am)$ is

$y = mx - 2am - am^3$

As we know that the equation of the normal to the parabola

$y^2 = 4ax$ at $(at^2, 2at)$ is

$$y = -tx + 2at + at^3 \quad \ldots\text{(i)}$$

The slope of the normal is

$$m = -t$$

$$\Rightarrow \quad t = -m$$

Putting the values of m in Eq. (i), we get

$$y = mx - 2am - am^3$$

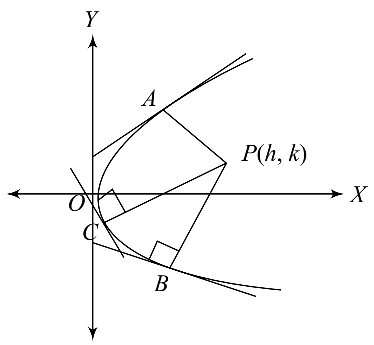
which is the required equation of the normal to the parabola $y^2 = 4ax$ at $(am^2, -2am)$.

(iv) Condition of normal

The line $y = mx + c$ will be a normal to the parabola $y^2 = 4ax$, if $c = -2am - am^3$ and the co-ordinates of the point of contact are $(am^2, -2am)$.

(v) Co-normal points

In general, three normals can be drawn from a point to a parabola and their feet (points) where they meet the parabola are called the co-normal points.



Here A, B and C are three co-normal points.

Let $P(h, k)$ be any given point $y^2 = 4ax$ be a parabola.

The equation of any normal to the parabola

$y^2 = 4ax$ is

$$y = mx - 2am - am^3$$

which passes through $P(h, k)$. Then

$$k = mh - 2am - am^3$$

$$\Rightarrow \quad am^3 + (2a - h)m + k = 0$$

which is a cubic equation in m. So it has three roots.

Thus, in total, three normals can be drawn from a point lies either outside or inside of a parabola.

Notes:

1. We can draw one and only one normal to a parabola, if a point lies on the parabola.
2. From an external point to a parabola, only one normal can be drawn.

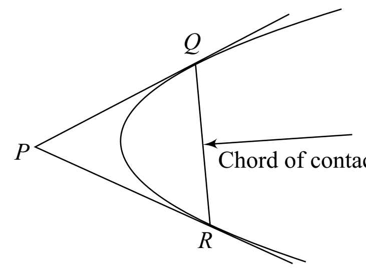
18. Chord of Contact

The chord joining the points of contact of two tangents drawn from an external point to a parabola is known as the chord of contact.

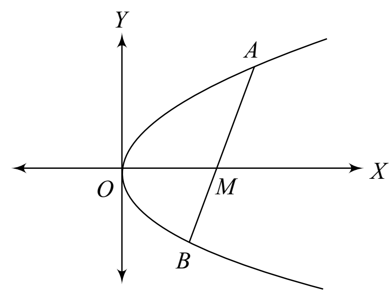
The equation of the chord of contact of tangents drawn from a point (x_1, y_1) to the parabola

$y^2 = 4ax$ is

$$yy_1 = 2a(x + x_1)$$



19. Chord Bisected at a Given Point



The equation of the chord of the parabola

$y^2 = 4ax$ is bisected at the point (x_1, y_1) is

$$T = S_1$$

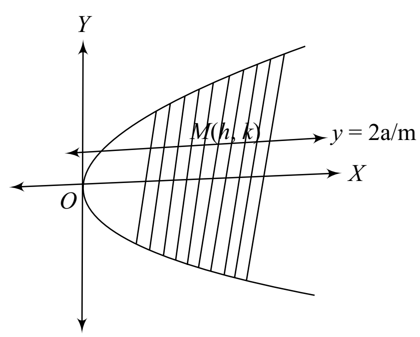
$$\Rightarrow \quad yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$$

where

$$T: y_1 - 2a(x + x_1),\ S: y_1^2 - 4ax_1$$

20. Diameter

The locus of the mid-points of a system of parallel chords to a parabola is known as the diameter of the parabola.



The equation of the diameter to the parabola $y^2 = 4ax$ bisecting a system of parallel chords with slope m is

$$y = \frac{2a}{m}$$

Let (h, k) be the mid-point of the chord $y = mx + c$ of the parabola $y^2 - 2ah$.

Then, $T = S_1$

$$\Rightarrow \quad ky - 2a(x + h) = k^2 - 2ah$$

Now slope = $\dfrac{2a}{k}$

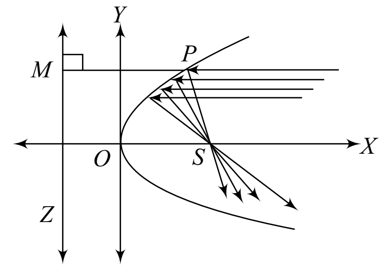
$$m = \frac{2a}{k} \Rightarrow k = \frac{2a}{m}$$

Hence the locus of the mid-point (h, k) is $y = \dfrac{2a}{m}$.

> ***Note:*** Any line which is parallel to the axis of the parabola drawn through any point on the parabola is called the diameter of the parabola and its equation is the y-coordinate of that point.

21. Reflection Property of a Parabola

All rays of light coming from the positive direction of x-axis and parallel to the axis of the parabola are reflected through the focus of the parabola



Exercises

Level I

(Problems based on Fundamentals)

1. What conic does $\sqrt{ax} + \sqrt{by} = 1$ represent?
2. If the conic $x^2 - 4xy + ly^2 + 2x + 4y + 10 = 0$ represents a parabola, find the value of λ.
3. If the conic $16(x^2 + (y-1)^2) = (x + \sqrt{3}y - 5)^2$ represents a non-degenerate conic, write its name and also find its eccentricity.
4. If the focus and the directrix of a conic be (1, 2) and $x + 3y + 10 = 0$ respectively and the eccentricity be $\dfrac{1}{\sqrt{2}}$, then find its equation.
5. Find the equation of a parabola, whose focus is (1, 1) and the directrix is $x - y + 3 = 0$.

ABC OF PARABOLA

6. Find the vertex, the focus, the latus rectum, the directrix and the axis of the parabolas
   (i) $y^2 = x + 2y + 2$
   (ii) $y^2 = 3x + 4y + 2$
   (iii) $x^2 = y + 4x + 2$
   (iv) $x^2 + x + y = 0$
7. If the focal distance on a point to a parabola $y^2 = 12x$ is 6, find the co-ordinates of that point.
8. Find the equation of a parabola, whose focus (–6, –6) and the vertex is (–2, –2).
9. The parametric equation of a parabola is $x = t^2 + 1$ and $y = 2t + 1$. Find its directrix.
10. If the vertex of a parabola be (–3, 0) and the directrix is $x + 5 = 0$, find its equation.
11 Find the equation of the parabola whose axis is parallel to y-axis and which passes through the points (0, 2), (–1, 0) and (1, 6).
12. Find the equation of a parabola whose vertex is (1, 2) and the axis is parallel to x-axis and also passes through the point (3, 4).
13. If the axis of a parabola is parallel to y-axis, the vertex and the length of the latus rectum are (3, 2) and 12 respectively, find its equation.

PROPERTIES OF THE FOCAL CHORD

14. If the chord joining $P(at_1^2, 2at_1)$ and is the focal chord, prove that $t_1 t_2 = -1$.
15. If the point $(at^2, 2at)$ be the extremity of a focal chord of the parabola $y^2 = 4ax$, prove that the length of the focal chord is $a\left(t + \dfrac{1}{t}\right)^2$.
16. If the length of the focal chord makes an angle θ with the positive direction of x-axis, prove that its length is $4a \operatorname{cosec}^2\theta$.
17. Prove that the semi-latus rectum of a parabola $y^2 = 4ax$ is the harmonic mean between the segments of any focal chord of the parabola.
18. Prove that the length of a focal chord of the parabola varies inversely as the square of its distance from the vertex.
19. Prove that the circle described on the focal chord as the diameter touches the tangent to the parabola.
20. Prove that the circle described on the focal chord as the diameter touches the directrix of the parabola.

POSITION OF A POINT RELATIVE TO A PARABOLA

21. If a point $(\lambda, -\lambda)$ lies in an interior point of the parabola $y^2 = 4x$, find the range of λ.
22. If a point $(\lambda, 2)$ is an exterior point of both the parabolas $y^2 = (x + 1)$ and $y^2 = -x + 1$, find the value of λ.

INTERSECTION OF A LINE AND A PARABOLA

23. If $2x + 3y + 5 = 0$ is a tangent to the parabola $y^2 = 8x$, find the co-ordinates of the point of contact.

24. If $3x + 4y + \lambda = 0$ is a tangent to the parabola $y^2 = 12x$, find the value of λ.
25. Find the length of the chord intercepted by the parabola $y^2 = 4ax$ and the line $y = mx + c$.

TANGENT AND TANGENCY

26. Find the point of intersection of tangents at $P(t_1)$ and $Q(t_2)$ on the parabola $y^2 = 4ax$.
27. Find the equation of tangent to the parabola $y^2 = 2x + 5y - 8$ at $x = 1$.
28. Find the equation of the tangent to the parabola $y^2 = 8x$ having slope 2 and also find its point of contact.
29. Two tangents are drawn from a point $(-1, 2)$ to a parabola $y^2 = 4x$. Find the angle between the tangents.
30. Find the equation of the tangents to the parabola $y = x^2 - 3x + 2$ from the point $(1, -1)$.
31. Find the equation of the common tangent to the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.
32. Find the equation of the common tangent to the parabola $y^2 = 4ax$ and $x^2 = 4by$.
33. Find the equation of the common tangent to the parabola $y^2 = 16x$ and the circle $x^2 + y^2 = 8$.
34. Find the equation of the common tangents to the parabolas $y = x^2$ and $y = -(x - 2)^2$.
35. Find the equation of the common tangents to the curves $y^2 = 8x$ and $xy = -1$.
36. Find the equation of the common tangent to the circle $x^2 + y^2 - 6y + 4 = 0$ and the parabola $y^2 = x$.
37. Find the equation of the common tangent touching the circle $x^2 + (y - 3)^2 = 9$ and the parabola $y^2 = 4x$ above the x-axis.
38. Find the points of intersection of the tangents at the ends of the latus rectum to the parabola $y^2 = 4x$.
39. Find the angle between the tangents drawn from a point $(1, 4)$ to the parabola $y^2 = 4x$.
40. Find the shortest distance between the line $y = x - 2$ and the parabola $y = x^2 + 3x + 2$.
41. Find the shortest distance from the line $x + y = 4$ and the parabola $y^2 + 4x + 4y = 0$.
42. If $y + b = m_1(x + a)$ and $y + b = m_2(x + a)$ are two tangents of the parabola $y^2 = 4ax$, find the value of m_1m_2.
43. The tangent to the curve $y = x^2 + 6$ at a point $(1, 7)$ touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at Q. Find the co-ordinates of Q.
44. Two straight lines are perpendicular to each other. One of them touches the parabola $y^2 = 4a(x + a)$ and the other touches $y^2 = 4b(x + b)$. Prove that the point of intersection of the lines lie on the line $x + a + b = 0$.
45. Prove that the area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.
46. Prove that the circle circumscribing the triangle formed by any three tangents to a parabola passes through the focus.
47. Prove that the orthocentre of any triangle formed by three tangents to a parabola lies on the directrix.
48. Prove that the equation of the director circle to the parabola $y^2 = 4ax$ is $x + a = 0$.
49. Find the equation of the director circle to the following parabolas:
    (i) $y^2 = x + 2$ (ii) $x^2 = 4x + 4y$
    (iii) $y^2 = 4x + 4y - 8$

NORMAL AND NORMALCY

50. Find the point of intersection of normals at $P(t_1)$ and $Q(t_2)$ on the parabola $y^2 = 4ax$.
51 Find the relation between t_1 and t_2, where the normal at t_1 to the parabola $y^2 = 4ax$ meets the parabola $y^2 = 4ax$ again at t_2.
52. If the normal at t_1 meets the parabola again at t_2, prove that the minimum value of t_2^2 is 8.
53. If two normals at t_1 and t_2 meet again the parabola $y^2 = 4ax$ at t_3, prove that $t_1t_2 = 2$.
54. Find the equation of the normal to the parabola $y^2 = 4x$ at the point $(1, 2)$.
55. Find the equation of the normal to the parabola $y^2 = 8x$ at $m = 2$.
56. If $x + y = k$ is a normal to the parabola $y^2 = 12x$, find the value of k.
57. If the normal at $P(18, 12)$ to the parabola $y^2 = 8x$ cuts it again at Q, prove that $9PQ = 80\sqrt{10}$.
58. Find the locus of the point of intersection of two normals to the parabola $y^2 = 4ax$, which are at right angles to one another.
59. If $lx + my + n = 0$ is a normal to the parabola $y^2 = 4ax$, prove that $al^3 + 2alm^2 + m^2n = 0$.
60. If a normal chord subtends a right angle at the vertex of the parabola $y^2 = 4ax$, prove that it is inclined at an angle of $\tan^{-1}(\sqrt{2})$ to the axis of the parabola.
61. At what point on the parabola $y^2 = 4x$, the normal makes equal angles with the axes?
62. Find the length of the normal chord which subtends an angle of 90° at the vertex of the parabola $y^2 = 4x$.
63. Prove that the normal chord of a parabola $y^2 = 4ax$ at the point (p, p) subtends a right angle at the focus.
64. Show that the locus of the mid-point of the portion of the normal to the parabola $y^2 = 4ax$ intercepted between the curve and the axis is another parabola.
65. Find the shortest distance between the curves $y^2 = 4x$ and $x^2 + y^2 - 12x + 31 = 0$.
66. Find the shortest distance between the curves $x^2 + y^2 + 12y + 35 = 0$ and $y^2 = 8x$.

CO-NORMAL POINT

67. Prove that the algebraic sum of the three concurrent normals to a parabola is zero.
68. Prove that the algebraic sum of the ordinates of the feet of three normals drawn to a parabola from a given point is also zero.

69. Prove that the centroid of the triangle formed by the feet of the three normals lies on the axis of the parabola. Also find the centroid of the triangle.
70. If three normals drawn from a given point (h, k) to any parabola be real, prove that $h > 2a$.
71. If three normals from a given point (h, k) to any parabola $y^2 = 4ax$ be real and distinct, prove that $27ak^2 < 4(h - 2a)^3$.
72. If a normal to a parabola $y^2 = 4ax$ makes an angle θ with the axis of the parabola, prove that it will cut the curve again at an angle of $\tan^{-1}\left(\frac{\tan\theta}{2}\right)$.
73. Prove that the normal chord to a parabola $y^2 = 4ax$ at the point whose ordinate is equal to its abscissa, which subtends a right angle at the focus of the parabola.
74. Prove that the normals at the end-points of the latus rectum of a parabola $y^2 = 4ax$ intersect at right angle on the axis of the parabola and their point of intersection is $(3a, 0)$.
75. If S be the focus of the parabola and the tangent and the normal at any point P meet the axes in T and G respectively, prove that $ST = SG = SP$.
76. From any point P on the parabola $y^2 = 4ax$, a perpendicular PN is drawn on the axis meeting at N, the normal at P meets the axis in G. Prove that the sub-normal NG is equal to its semi-latus rectum.
77. The normal to the parabola $y^2 = 4ax$ at a point P on it, meets the x-axis in G, prove that P and G are equidistant from the focus S of the parabola.
78. The normal at P to the parabola $y^2 = 4ax$ meets its axis at G. Q is another point on the parabola such that QG is perpendicular to the axis of the parabola. Prove that $QG^2 - PG^2 =$ constant.

CHORD OF CONTACT

79. Find the equation of the chord of contact to the tangents from the point (2, 3) to the parabola $y^2 = 4x$.
80. Find the chord of contact of the tangents to the parabola $y^2 = 12x$ drawn through the point $(-1, 2)$.
81. Prove that the locus of the point of intersection of two tangents to a parabola $y^2 = 4ax$ which make a given angle θ with one another is $y^2 - 4ax = (x + a)^2 \tan^2\theta$.
82. Prove that the length of the chord of contact of tangents drawn from (h, k) to the parabola $y^2 = 4ax$ is $\frac{1}{a}|(k^2 + 4a^2)(k^2 - 4ah)|^{1/2}$.
83. Prove that the area of the triangle formed by the tangents from the point (h, k) to the parabola $y^2 = 4ax$ and a chord of contact is $\frac{(k^2 - 4ah)^{3/2}}{2a}$.

CHORD BISECTED AT A POINT

84. Find the equation of the chord of the parabola $y^2 = 8x$ which is bisected at (2, 3).
85. Prove that the locus of the mid-points of the focal chord of the parabola is another parabola.
86. Prove that the locus of the mid-points of the chord of a parabola passes through the vertex is a parabola.
87. Prove that the locus of the mid-points of a normal chords of the parabola $y^2 = 4ax$ is $y^4 - 2a(x - 2a)y^2 + 8a^4 = 0$
88. Prove that the locus of the mid-point of a chord of a parabola $y^2 = 4ax$ which subtends a right angle at the vertex is $y^2 = 2a(x - 4a)$.
89. Prove that the locus of the mid-points of chords of the parabola $y^2 = 4ax$ which touches the parabola $y^2 = 4bx$ is $y^2(2a - b) = 4a^2x$.
90. Find the locus of the mid-point of the chord of the parabola $y^2 = 4ax$, which passes through the point $(3b, b)$.
91. Prove that the locus of the mid-points of all tangents drawn from points on the directrix to the parabola $y^2 = 4ax$ is $y^2(2x + a) = a(3x + a)^2$.

DIAMETER OF A PARABOLA

92. Prove that the tangent at the extremity of a diameter of a parabola is parallel to the system of chords it bisects.
93. Prove that tangents at the end of any chord meet on the diameter which bisects the chords.

REFLECTION PROPERTY OF A PARABOLA

94. A ray of light moving parallel to the x-axis gets reflected from a parabolic mirror whose equation is $(y - 4)^2 = 8(x + 1)$. After reflection, the ray passes through the point (α, β), find the value of $\alpha + \beta + 10$.
95. A ray of light is moving along the line $y = x + 2$, gets reflected from a parabolic mirror whose equation is $y^2 = 4(x + 2)$. After reflection, the ray does not pass through the focus of the parabola. Find the equation of the line which containing the reflected ray.

LEVEL II

(Mixed Problems)

1. Three normals to the parabola $y^2 = x$ are drawn through a point $(c, 0)$, then
(a) $c = 1/4$ (b) $c = 1/2$ (c) $c > 1/2$ (d) none
2. The line which is parallel to x-axis and crosses the curve $y = \sqrt{x}$ at an angle of 45° is
(a) $x = 1/4$ (b) $y = 1/4$ (c) $y = 1/2$ (d) $y = 1$
3. Consider a circle with its centre lying on the focus of the parabola $y^2 = 2px$ such that it touches the directrix of the parabola, the point of intersection of the circle and the parabola is
(a) $\left(\frac{p}{2}, p\right)$ (b) $\left(\frac{p}{2}, -p\right)$
(c) $\left(-\frac{p}{2}, p\right)$ (d) $\left(-\frac{p}{2}, -p\right)$

4. If the line $x - 1 = 0$ is the directrix of the parabola $y^2 - kx + 8 = 0$, one of the value of k is
(a) 1/8 (b) 8 (c) 4 (d) 1/4

5. If $x + y = k$ is a normal to the parabola $y^2 = 12x$, the value of k is
(a) 9 (b) 3 (c) –9 (d) –3

6. The equation of the directrix of the parabola $y^2 + 4y + 4x + 2 = 0$ is
(a) $x = -1$ (b) $x = 1$
(c) $x = -3/2$ (d) $x = 3/2$

7. The equation of the common tangent touching the circle $(x - 3)^2 + y^2 = 9$ and the parabola $y^2 = 4x$ above the x-axis is
(a) $\sqrt{3}y = 3x + 1$ (b) $\sqrt{3}y = -(x + 3)$
(c) $\sqrt{3}y = (x + 3)$ (d) $\sqrt{3}y = -(3x + 1)$

8. The locus of the mid-point of the line segment joining the focus to a moving point on the parabola $y^2 = 4ax$ is another parabola with directrix
(a) $x + a = 0$ (b) $x = -a/2$
(c) $x = 0$ (d) $x = a/2$

9. The focal chord of $y^2 = 16x$ is a tangent to $(x - 6)^2 + y^2 = 2$, the possible values of the slope of this chord are
(a) 1, –1 (b) –1/2, 2
(c) –2, 1/2 (d) 1/2, 2

10. The tangent to the parabola $y = x^2 + 6$ touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at the point
(a) (–6, –9) (b) (–13, –9)
(c) (–6, –7) (d) (13, 7)

11. The axis of a parabola is along the line $y = x$ and the distance of its vertex from the origin is $\sqrt{2}$ and that from its focus is $2\sqrt{2}$. If the vertex and the focus both lie in the first quadrant, then the equation of the parabola is
(a) $(x + y)^2 = (x + y - 2)$ (b) $(x - y)^2 = (x + y - 2)$
(c) $(x - y)^2 = 4(x + y - 2)$ (d) $(x - y)^2 = 8(x + y - 2)$

12. The equations of the common tangents to the parabola $y = x^2$ and $y = -(x - 2)^2$ is
(a) $y = 4(x - 1)$ (b) $y = 0$
(c) $y = -4(x - 1)$ (d) $y = -10(3x + 5)$

13. The tangent PT and the normal PN to the parabola $y^2 = 4ax$ at a point P on it meet its axis at points T and N, respectively. The locus of the centroid of the triangle PTN is a parabola whose
(a) vertex is $\left(\frac{2a}{3}, 0\right)$ (b) directrix is $x = 0$
(c) latus rectum is $2a$ (d) focus is $(a, 0)$

14. The normal at the point $(bt_1^2, 2bt_1)$ on a parabola meets the parabola again at $(bt_2^2, 2bt_2)$, then
(a) $t_2 = -t_1 + \frac{2}{t_1}$ (b) $t_2 = t_1 - \frac{2}{t_1}$
(c) $t_2 = t_1 + \frac{2}{t_1}$ (d) $t_2 = -t_1 - \frac{2}{t_1}$

15. If $a \neq 0$ and the line $2bx + 3cy + 4d = 0$ passes through the point of intersection of the parabolas $y^2 = 4ax$ and $x^2 = 4ay$, then
(a) $d^2 + (2b - 3c)^2 = 0$ (b) $d^2 + (2b + 3c)^2 = 0$
(c) $d^2 + (3b + 2c)^2 = 0$ (d) $d^2 + (3b - 2c)^2 = 0$

16. The locus of the vertices of the family of parabolas $y = \frac{a^2x^2}{2} + \frac{a^2x}{2} - 2a$ is
(a) $xy = \frac{105}{64}$ (b) $xy = \frac{3}{4}$
(c) $xy = \frac{35}{16}$ (d) $xy = \frac{64}{105}$

17. The angle between the tangents to the curve $y = x^2 - 5x + 6$ at the point (2, 0) and (3, 0) is
(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{4}$

18. The equation of a tangent to the parabola $y^2 = 8x$ is $y = x + 2$. The point on this from which the other tangent to the parabola is perpendicular to the given tangent is
(a) (–2, 0) (b) (–1, 1) (c) (0, 2) (d) (2, 4)

19. A parabola has the origin as its focus and the line $x = 2$ as its directrix. The vertex of the parabola is at
(a) (0, 1) (b) (2, 0)
(c) (0, 2) (d) (1, 0)

20. The length of the chord of the parabola $y^2 = x$ which is bisected at the point (2, 1) is
(a) $2\sqrt{3}$ (b) $4\sqrt{3}$ (c) $3\sqrt{2}$ (d) $2\sqrt{5}$

21. Two mutually perpendicular tangents to the parabola $y^2 = 4ax$ meet the axis in P_1 and P_2. If S be the focus of the parabola, $\frac{1}{l(SP_1)} + \frac{1}{l(SP_2)}$ is
(a) $\frac{4}{a}$ (b) $\frac{2}{a}$ (c) $\frac{1}{a}$ (d) $\frac{1}{4a}$

22. Which of the following equations represents a parabolic profile, represented parametrically, is
(a) $x = 3\cos t, y = 4\sin t$
(b) $x^2 - 2 = -2\cos t, y = 4\cos^2\left(\frac{t}{2}\right)$
(c) $\sqrt{x} = \tan t, \sqrt{y} = \sec t$
(d) $x = \sqrt{1 - \sin t}, y = \sin\left(\frac{t}{2}\right) + \cos\left(\frac{t}{2}\right)$

23. The points of contact Q and R of tangent from the point $P(2, 3)$ on the parabola $y^2 = 4x$ are
(a) (9, 6), (1, 2) (b) (1, 2), (4, 4)
(c) (4, 4), (9, 6) (d) $(9, 6), \left(\frac{1}{4}, 1\right)$

24. A tangent is drawn to the parabola $y^2 = 4x$ at the point P whose abscissa lies in [1, 4]. The maximum possible area of the triangle formed by the tangents at P ordinate of the point P and the x-axis is
(a) 8 (b) 16 (c) 24 (d) 32

25. The length of the normal chord $y^2 = 4x$ which makes an angle of $\frac{\pi}{4}$ with the axis of x is
(a) 8 (b) $8\sqrt{2}$ (c) 4 (d) $4\sqrt{2}$

26. The co-ordinates of the end points of a focal chord of a parabola $y^2 = 4ax$ are (x_1, y_1) and (x_2, y_2), then $(x_1x_2 + y_1y_2)$ is
(a) $2a^2$ (b) $-3a^2$ (c) $-a^2$ (d) $4a^2$

27. If the normal to a parabola $y^2 = 4ax$ at P meets the curve again at Q and if PQ and the normal at Q makes angles α and β, respectively with the x-axis, then $\tan\alpha + \tan\beta$ is
(a) 0 (b) –2 (c) –1/2 (d) –1

28. If the normal to the parabola $y^2 = 4ax$ at the point with parameter t_1 cuts the parabola again at the point with parameter t_2, then
(a) $2 \le t_2^2 \le 8$ (b) $2 \le t_2^2 \le 4$
(c) $t_2^2 \ge 4$ (d) $t_2^2 \ge 8$

29. A parabola $y = ax^2 + bx + c$ crosses the x-axis at $(\alpha, 0)$ and $(\beta, 0)$ both to the right of the origin.
A circle also passes through these two points. The length of a tangent from the origin to the circle is
(a) $\sqrt{\frac{bc}{a}}$ (b) ac^2 (c) $\frac{b}{a}$ (d) $\sqrt{\frac{c}{a}}$

30. Two parabolas have the same focus. If their directrices are the x-axis and the y-axis respectively, the slope of their common chord is
(a) 1, –1 (b) 4/3 (c) 3/4 (d) none

31. The straight line joining the point P on the parabola $y^2 = 4ax$ to the vertex and the perpendicular from the focus to the tangent at P intersect at R. Then the locus of R is
(a) $x^2 + 2y^2 - ax = 0$ (b) $x^2 + y^2 - 2ax = 0$
(c) $2x^2 + 2y^2 - ax = 0$ (d) $2x^2 + y^2 - 2ay = 0$

32. A normal chord of the parabola $y^2 = 4x$ subtending a right angle at the vertex makes an acute angle θ with the x-axis, then θ is
(a) $\tan^{-1}2$ (b) $\sec^{-1}(\sqrt{3})$
(c) $\cot^{-1}(\sqrt{3})$ (d) none

33. C is the centre of the circle with centre (0, 1) and the radius unity of the parabola $y = ax^2$. The set of values of a for which they meet at a point other than the origin is
(a) $a > 0$ (b) $0 < a < \frac{1}{2}$
(c) $\frac{1}{4} < a < \frac{1}{2}$ (d) $a > \frac{1}{2}$

34. TP and TQ are two tangents to the parabola $y^2 = 4ax$ at P and Q. If the chord PQ passes through the fixed point $(-a, b)$, the locus of T is
(a) $ay = 2b(x - b)$ (b) $by = 2a(x - a)$
(c) $by = 2a(x - a)$ (d) $ax = 2a(y - b)$

35. Through the vertex O of the parabola $y^2 = 4ax$ two chords OP and OQ are drawn and the circles on OP and OQ as diameters intersect in R. If θ_1, θ_2 and φ are the angles made with the axis by the tangents at P and Q on the parabola and by OR, the value of $\cot(\theta_1) + \cot(\theta_2)$ is
(a) $-2\tan(\varphi)$ (b) $-2\tan(\pi - \varphi)$
(c) 0 (d) $-2\cot(\varphi)$

36. The tangent at P to a parabola $y^2 = 4ax$ meets the directrix at U and the base of the latus rectum at V, then SUV (where S is the focus) must be a/an
(a) right Δ (b) equilateral Δ
(c) isosceles Δ (d) right isosceles Δ

37. Two parabolas $y^2 = 4a(x - m_1)$ and $x^2 = 4a(y - m_2)$ always touch one another, the quantities m_1 and m_2 are both variables. The locus of their points of contact has the equation
(a) $xy = a^2$ (b) $xy = 2a^2$
(c) $xy = 4a^2$ (d) none

38. If a normal to a parabola $y^2 = 4ax$ makes an angle φ with its axis, it will cut the curve again at an angle
(a) $\tan^{-1}(2\tan\varphi)$ (b) $\tan^{-1}\left(\frac{1}{2}\tan\varphi\right)$
(c) $\cot^{-1}\left(\frac{1}{2}\tan\varphi\right)$ (d) none

39. The vertex of a parabola is (2, 2) and the co-ordinates of its extremities of the latus rectum are (–2, 0) and (6, 0). The equation of the parabola is
(a) $y^2 - 4y + 8x - 12 = 0$ (b) $x^2 + 4x + 8y - 12 = 0$
(c) $x^2 - 4x + 8y - 12 = 0$ (d) $x^2 + 4x - 8y + 20 = 0$

40. The length of the chord of the parabola $y^2 = x$ which is bisected at the point (2, 1) is
(a) $2\sqrt{3}$ (b) $4\sqrt{3}$ (c) $3\sqrt{2}$ (d) $2\sqrt{5}$

41. If the tangent and the normal at the extremities of a focal chord of a parabola intersect at (x_1, y_1) and (x_2, y_2) respectively, then
(a) $x_1 = x_2$ (b) $x_1 = x_2$
(c) $y_1 = y_2$ (d) $x_2 = y_1$

42. If the chord of contact of tangents from a point P to the parabola $y^2 = 4ax$ touches the parabola $xy = 4by$, then the locus of P is a/an
(a) circle (b) parabola
(c) ellipse (d) hyperbola

43. The latus rectum of a parabola whose focal chord PSQ is such that $SP = 3$ and $SP = 2$ is given by
(a) 24/5 (b) 12/5 (c) 6/5 (d) none

44. If two normals to a parabola $y^2 = 4ax$ intersect at right angles, then the chord joining then feet passes through a fixed point whose co-ordinates are
(a) $(-2a, 0)$ (b) $(a, 0)$ (c) $(2a, 0)$ (d) None

45. The straight line passing through (3, 0) and cutting the curve $y = \sqrt{x}$ orthogonally is
(a) $4x + y = 18$ (b) $x + y = 9$
(c) $4x - y = 6$ (d) None

46. PQ is a normal chord of the parabola $y^2 = 4ax$ at P. A being the vertex of the parabola. Through P a line is drawn parallel to AQ meeting the x-axis in R. Then the length of AR is
(a) the length of latus rectum.
(b) the focal distance of the point P.
(c) $2 \times$ Focal distance of the point P.
(d) the distance of P from the directrix.

47. The locus of the point of intersection of the perpendicular tangents of the curve $y^2 + 4y - 6x - 2 = 0$ is
(a) $2x = 1$ (b) $2x + 3 = 0$
(c) $2y + 3 = 0$ (d) $2x + 5 = 0$

48. The length of the focal chord of the parabola $y^2 = 4ax$ at a distance p from the vertex is
(a) $\frac{2a^2}{p}$ (b) $\frac{a^3}{p^2}$ (c) $\frac{4a^3}{p^2}$ (d) $\frac{p^3}{a}$

49. The locus of a point such that two tangents drawn from it to the parabola $y^2 = 4ax$ are such that the slope of one is double the other is
(a) $y^2 = \frac{9}{2}ax$ (b) $y^2 = \frac{9}{4}ax$
(c) $y^2 = 9ax$ (d) $x^2 = 4ay$

50. The point on the parabola $y^2 = 4x$ which are closest to the curve $x^2 + y^2 - 24y + 128 = 0$ is
(a) $(0, 0)$ (b) $(2, \sqrt{2})$
(c) $(4, 4)$ (d) none

Level III

(Problems for JEE Advanced)

1. If two ends of a latus rectum of a parabola are the points $(3, 6)$ and $(-5, 6)$, find its focus.
2. Find the locus of the focus of the family of parabolas $y = \frac{a^2x^2}{3} + \frac{a^2x}{2} - 2a$.
3. Find the equation of the parabola whose vertex and the focus lie on the axis of x at distances a and a_1 from the origin, respectively.
4. A square has one vertex at the vertex of the parabola $y^2 = 4ax$ and the diagonal through the vertex lies along the axis of the parabola. If the ends of the other diagonal lie on the parabola, find the co-ordinates of the vertices of the square.
5. Find the equation of the parabola whose axis is $y = x$, the distance from origin to vertex is $\sqrt{2}$ and the distance from the origin to the focus is $2\sqrt{2}$.
6. If $a \neq 0$ and the line $2bx + 3cy + 4d = 0$ passes through the points of intersection of the parabola $y^2 = 4ax$ and $x^2 = 4ay$, prove that $d^2 + (2b + 3c)^2 = 0$.
7. Two straight lines are perpendicular to each other. One of them touches the parabola $y^2 = 4a(x + a)$ and the other $y^2 = 4b(x + b)$. Prove that their points of intersection lie on the line $x + a + b = 0$.
8. If the focal chord of $y^2 = 16x$ is a tangent to $(x - 6)^2 + y^2 = 2$, find the possible values of the slope of the chord.
9. Find the points of intersection of the tangents at the end of the latus rectum of the parabola $y^2 = 4x$.
10. If the tangent at the point $P(2, 4)$ to the parabola $y^2 = 8x$ meets the parabola $y^2 = 8x + 5$ at Q and R, find the mid-point of QR.
11. If $y + b = m_1(x + a)$ and $y + b = m_2(x + a)$ are two tangents to the parabola $y^2 = 4ax$, prove that $m_1m_2 = -1$.
12. Find the equation of the common tangent to the circle $(x - 3)^2 + y^2 = 9$ and the parabola $y^2 = 4x$ above the x-axis.
13. Find the equation of the common tangent to the curves $y^2 = 8x$ and $xy = -1$.
14. Find the common tangents of $y = x^2$ and $y = -x^2 + 4x - 4$.
15. Consider a circle with the centre lying on the focus of the parabola $y^2 = 2px$ such that it touches the directrix of that parabola. Find a point of intersection of the circle and the parabola.
16. If the normal drawn at a point $(at_1^2, 2at_1)$ of the parabola $y^2 = 4ax$ meets it again at $(at_2^2, 2at_2)$, prove that $t_1^2 + t_1t_2 + 2 = 0$.
17. Three normals to the parabola $y^2 = x$ are drawn through a point $(c, 0)$, find c.
18. A tangent to the parabola $y^2 = 8x$ makes an angle 45° with the straight line $y = 3x + 5$. Find the equation of the tangent and its point of contact.
19. Find the equations of the normal to the parabola $y^2 = 4ax$ at the ends of the latus rectum. If the normal again meets the parabola at Q and Q', prove that $QQ' = 12a$.
20. Prove that from any point $P(at^2, 2at)$ on the parabola $y^2 = 4ax$, two normals can be drawn and their feet Q and R have the parameters satisfying the equation $\lambda^2 + \lambda t + 2 = 0$.
21. Find the locus of the points of intersection of those normals to the parabola $x^2 = 8y$ which are at right angles to each other.
22. Two lines are drawn at right angles, one being a tangent to $y^2 = 4ax$ and the other to $x^2 = 4by$. Show that the locus of their points of intersection is the curve $(ax + by)(x^2 + y^2) + (bx - ay)^2 = 0$.
23. Prove that the locus of the centroid of an equilateral triangle inscribed in the parabola $y^2 = 4ax$ is $9y^2 = 4a(x - 8a)$.
24. Prove that the locus of the mid-points of chords of the parabola $y^2 = 4ax$ which subtends a right angle at the vertex is $y^2 = 2a^2(x - 4)$.
25. Prove that the locus of a point that divides a chord of slope 2 of the parabola $y^2 = 4x$ internally in the ratio 1 : 2 is a parabola. Find the vertex of the parabola.
26. From a point A, common tangents are drawn to the circle $x^2 + y^2 = \frac{a^2}{2}$ and the parabola $y^2 = 4ax$. Find the area of the quadrilateral formed by the common tangents, the chord of contact of the circle and the chord of contact of the parabola.

27. Normals are drawn from the point P with slopes m_1, m_2, m_3 to the parabola $y^2 = 4x$. If the locus of P with $m_1 m_2 = \alpha$ is a part of the parabola, find the value of α.
28. The tangent at a point P to the parabola $y^2 - 2y - 4x + 5 = 0$ intersects the directrix at Q. Find the locus of a point R such that it divides PQ externally in the ratio $\frac{1}{2}:1$.
29. Prove that the curve $y = -\frac{x^2}{2} + x + 1$ is symmetric with respect to the line $x = 1$. And also prove that it is symmetric about its axis.
30. Three normals are drawn from the point (14, 7) to the parabola $y^2 - 16x - 8y = 0$. Find the co-ordinates of the feet of the normals.
31. Find the locus of the foot of the perpendicular drawn from a fixed point to any tangent to a parabola.
32. Find the locus of the point of intersection of the normals to the parabola $y^2 = 4ax$ at the extremities of a focal chord.

Level IV

(Tougher Problems for JEE Advanced)

1. From the point (–1, 2), tangent lines are drawn to the parabola $y^2 = 4x$. Find the equation of the chord of contact. Also find the area of the triangle formed by the chord of contact and the tangents.
[Roorkee Main, 1994]
2. The equation $y^2 - 2x - 2y + 5 = 0$ represents
   (a) a circle with centre (1, 1)
   (b) a parabola with focus (1, 2)
   (c) a parabola with directrix $x = 3/2$
   (d) a parabola with directrix $x = -1/2$
[Roorkee, 1995]
3. A ray of light is coming along the line $y = b$ from the positive direction of x-axis and strikes a concave mirror whose intersection with the xy-plane is a parabola $y^2 = 4ax$. Find the equation of the reflected ray and show that it passes through the focus of the parabola. Both a and b are positive. **[Roorkee Main, 1995]**
4. If a tangent drawn at a point (t^2, t^2) on the parabola $y^2 = 4x$ is the same as the normal drawn at a point $(\sqrt{5}\cos\varphi, 2\sin\varphi)$ on the ellipse $4x^2 + 5y^2 = 20$, find the values of t and φ. **[Roorkee, 1996]**
5. The equations of normals to the parabola $y^2 = 4ax$ at the point $(5a, 2a)$ are
   (a) $y = x - 3a$ (b) $y = x + 3a$
   (c) $y + 2x = 12a$ (d) $3x + y = 33a$
[Roorkee, 1997]
6. Find the locus of the points of intersection of those normals to the parabola $x^2 = 8y$ which are at right angles to each other. **[Roorkee Main, 1997]**
7. The ordinates of points P and Q on the parabola $y^2 = 12x$ are in the ratio 1 : 2. Find the locus of the points of intersection of the normals to the parabola at P and Q.
[Roorkee Main, 1998]
8. Find the equations of the common tangents of the circle $x^2 + y^2 - 6y + 4 = 0$ and $y^2 = x$. **[Roorkee, 1999]**
9. Find the locus of points of intersection of tangents drawn at the ends of all normals chords to the parabola $y^2 = 4(x - 1)$. **[Roorkee Main, 2001]**
10. Find the locus of the trisection point of any double ordinate $y^2 = 4ax$.
11. Find the locus of the trisection point of any double ordinate $x^2 = 4by$.
12. Find the shortest distance between the curves $y^2 = 4x$ and $x^2 + y^2 - 12x + 31 = 0$.
13. Find the radius of the circle that passes through the origin and touches the parabola at $(a, 2a)$.
14. Find the condition if two different tangents of $y^2 = 4x$ are the normals to $x^2 = 4by$.
15. A circle is drawn to pass through the extremities of the latus rectum of the parabola $y^2 = 8x$. It is also given that the circle touches the directrix of the parabola. Find the radius of the circle.

Integer Type Questions

1. Find the maximum number of common chords of a parabola and a circle.
2. If the straight lines $y - b = m_1(x + a)$ and $y - b = m_2(x + a)$ are the tangents of $y^2 = 4ax$, find the value of $(m_1 m_2 + 4)$.
3. A normal chord of $y^2 = 4ax$ subtends an angle $\frac{\pi}{2}$ at the vertex of the parabola. If its slope is m, then find the value of $(m^2 + 3)$.
4. Find the slope of the normal chord of $y^2 = 8x$ that gets bisected at (8, 2).
5. Find the maximum number of common normals of $y^2 = 4ax$ and $x^2 = 4by$.
6. Find the length of the latus rectum of the parabola whose parametric equation are given by $x = t^2 + t + 1$ and $y = t^2 - t + 1$.
7. If the shortest distance between the curves $y^2 = x - 1$ and $x^2 = y - 1$ is d, find the value of $(8d^2 - 3)$.
8. If m_1 and m_2 be the slopes of the tangents that are drawn from (2, 3) to the parabola $y^2 = 4x$, find the value of $\left(\frac{1}{m_1} + \frac{1}{m_2} + 2\right)$.
9. Let $P(t_1)$ and $Q(t_2)$ are two points on the parabola $y^2 = 4ax$. If the normals at P and Q meet the parabola again at R, find the value of $(t_1 t_2 + 3)$.
10. Find the length of the chord intercepted between the parabola $y^2 = 4x$ and the straight line $x + y = 1$.
11. If $x + y = k$ is a normal to the parabola $y^2 = 12x$, find the value of k.
12. Find the number of distinct normals drawn from the point (–2, 1) to the parabola $y^2 - 4x - 2y - 3 = 0$.

Comprehensive Link Passages

Passage I

$y = x$ is a tangent to the parabola $y^2 = ax^2 + c$.

(i) If $a = 2$, the value of c is
(a) 1 (b) $-1/2$ (c) $1/2$ (d) $1/8$

(ii) If (1, 1) is a point of contact, the value of a is
(a) $1/4$ (b) $1/3$ (c) $1/2$ (d) $1/6$

(iii) If $c = 2$, the point of contact is
(a) (3, 3) (b) (2, 2) (c) (6, 6) (d) (4, 4).

Passage II

Consider, the parabola whose focus is at (0, 0) and the tangent at the vertex is $x - y + 1 = 0$.

(i) The length of the latus rectum is
(a) $4\sqrt{2}$ (b) $2\sqrt{2}$ (c) $8\sqrt{2}$ (d) $3\sqrt{2}$

(ii) The length of the chord of the parabola on the x-axis is
(a) $4\sqrt{2}$ (b) $2\sqrt{2}$ (c) $8\sqrt{2}$ (d) $3\sqrt{2}$

(iii) Tangents drawn to the parabola at the extremities of the chord $3x + 2y = 0$ intersect at an angle
(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) none

Passage III

Two tangents on a parabola are $x - y = 0$ and $x + y = 0$. If (2, 3) is the focus of the parabola.

(i) The equation of the tangent at the vertex is
(a) $4x - 6y + 5 = 0$ (b) $4x - 6y + 3 = 0$
(c) $4x - 6y + 1 = 0$ (d) $4x - 6y + 3/2 = 0$

(ii) The length of the latus rectum of the parabola is
(a) $\frac{6}{\sqrt{3}}$ (b) $\frac{10}{\sqrt{13}}$ (c) $\frac{2}{\sqrt{13}}$ (d) $\frac{9\sqrt{2}}{\sqrt{13}}$

(iii) If P and Q are ends of the focal chord of the parabola, then is
(a) $\frac{2\sqrt{13}}{3}$ (b) $2\sqrt{13}$ (c) $\frac{2\sqrt{13}}{3}$ (d) $\frac{2\sqrt{13}}{7}$

Passage IV

If l, m are variable real numbers such that $5l^2 + 6m^2 - 4lm + 3l = 0$, the variable line $lx + my = 1$ always touches a fixed parabola, whose axis is parallel to x-axis.

(i) The vertex of the parabola is
(a) $\left(-\frac{5}{3}, \frac{4}{3}\right)$ (b) $\left(-\frac{7}{4}, \frac{3}{4}\right)$
(c) $\left(\frac{5}{6}, -\frac{7}{6}\right)$ (d) $\left(\frac{1}{2}, -\frac{3}{4}\right)$

(ii) The focus of the parabola is
(a) $\left(\frac{1}{6}, -\frac{7}{6}\right)$ (b) $\left(\frac{1}{3}, \frac{4}{3}\right)$
(c) $\left(\frac{3}{2}, -\frac{3}{2}\right)$ (d) $\left(-\frac{3}{4}, \frac{3}{4}\right)$

(iii) The equation of the directrix of the parabola is
(a) $6x + 7 = 0$ (b) $4x + 11 = 0$
(c) $3x + 11 = 0$ (d) $2x + 13 = 0$

Passage V

The normals at three points P, Q, R on the parabola $y^2 = 4ax$ meet at (α, β).

(i) The centroid of ΔPQR is
(a) $\left(\frac{\alpha - 2a}{3}, 0\right)$ (b) $\left(\frac{2\alpha - 4a}{3}, 0\right)$
(c) $\left(\frac{\alpha}{3}, \frac{\beta}{3}\right)$ (d) $\left(\frac{\alpha + \beta}{3}, \frac{\beta - \alpha}{3}\right)$

(ii) The orthocentre of ΔPQR is
(a) $\left(\alpha + 6a, \frac{\beta}{2}\right)$ (b) $\left(\alpha + 3a, \frac{\beta}{2}\right)$
(c) $\left(\alpha - 6a, -\frac{\beta}{2}\right)$ (d) $\left(\alpha - 3a, \frac{\beta}{2}\right)$

(iii) The circumcentre of ΔPQR is
(a) $\left(\frac{\alpha + 2a}{2}, -\frac{\beta}{4}\right)$ (b) $\left(\frac{\alpha + 2a}{2}, \frac{\beta}{4}\right)$
(c) $\left(\frac{\alpha}{4}, \frac{\beta}{4}\right)$ (d) $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$

Passage VI

Consider the circle $x^2 + y^2 = 9$ and the parabola $y^2 = 8x$. They intersect at P and Q in the first and the fourth quadrants, respectively. The tangents to the circle at P and Q intersect the x-axis at R and tangents to the parabola at P and Q intersect the x-axis at S.

(i) The ratio of the areas of Δs PQS and PQR is
(a) $1 : \sqrt{2}$ (b) 1 : 2 (c) 1 : 4 (d) 1 : 8

(ii) The radius of the circumcircle of ΔPRS is
(a) 5 (b) $3\sqrt{3}$ (c) $3\sqrt{2}$ (d) $2\sqrt{3}$

(iii) The radius of the incircle of ΔPQR is
(a) 4 (b) 3 (c) 8/3 (d) 2

Passage VII

If P is a point moving on a parabola $y^2 = 4ax$ and Q is a moving point on the circle $x^2 + y^2 - 24ay + 128a^2 = 0$. The points P and Q will be closest when they lie along the normal to the parabola $y^2 = 4ax$ passing through the centre of the circle.

(i) If the normal at $(at^2, 2at)$ of the parabola passes through the centre of the circle, the value of t must be
(a) 1 (b) 2 (c) 3 (d) 4

(ii) The shortest distance between P and Q must be
(a) $a(\sqrt{2} - 1)$ (b) $2a(\sqrt{5} - 1)$
(c) $4a(\sqrt{5} - 1)$ (d) $4a(\sqrt{5} + 1)$

(iii) When P and Q are closest, the point P must be
(a) $(1, 2a)$ (b) $(2a, 2\sqrt{2}a)$
(c) $(4a, 4a)$ (d) $(5a, 4a)$

(iv) When P and Q are closest, the point Q must be

(a) $\left(\frac{4a}{\sqrt{5}}, 6a - \frac{8a}{\sqrt{5}}\right)$ (b) $\left(\frac{4a}{\sqrt{5}}, 12a - \frac{8a}{\sqrt{5}}\right)$

(c) $\left(\frac{4a}{\sqrt{5}}, -12a - \frac{8a}{\sqrt{5}}\right)$ (d) $\left(\frac{4a}{\sqrt{5}}, -8a - \frac{12a}{\sqrt{5}}\right)$

(v) When P and Q are closest and the diameter QR of the circle is drawn through Q, the x-cordinate of Q is

(a) $-\frac{2a}{\sqrt{5}}$ (b) $-\frac{8a}{\sqrt{5}}$ (c) $-\frac{4a}{\sqrt{5}}$ (d) $-\frac{6a}{\sqrt{5}}$

Passage VIII

If a source of light is placed at the fixed point of a parabola and if the parabola is reflecting surface, the ray will bounce back in a line parallel to the axis of the parabola.

(i) A ray of light is coming along the line $y = 2$ from the positive direction of x-axis and strikes a concave mirror whose intersection with the xy plane is a parabola $y^2 = 8x$, the equation of the reflected ray is

(a) $2x + 3y = 4$ (b) $3x + 2y = 6$
(c) $4x + 3y = 8$ (d) $5x + 4y = 10$

(ii) A ray of light moving parallel to the x-axis gets reflected from a parabolic mirror whose equation is $y^2 + 10y - 4x + 17 = 0$. After reflection, the ray must pass through the point

(a) $(-2, -5)$ (b) $(-1, -5)$
(c) $(-3, -5)$ (d) $(-4, -5)$

(iii) A ray of light is coming along the line $x = 2$ from the positive direction of y-axis and strikes a concave mirror whose intersection with the xy plane is a parabola $x^2 = 4y$, the equation of the reflected ray after second reflection is

(a) $2x + y = 1$ (b) $3x - 2y + 2 = 0$
(c) $y = 1$ (d) none

(iv) Two rays of light coming along the line $y = 1$ and $y = -2$ from the positive direction of x-axis and strikes a concave mirror whose intersection with the xy plane is a parabola $y^2 = x$ at A and B, respectively. The reflected rays pass through a fixed point C, the area of ΔABC is

(a) 21/ 8 s.u. (b) 19/2 s.u.
(c) 17/2 s.u. (d) 15/2 s.u.

Matrix Match
(For JEE-Advanced Examination Only)

1. Match the following columns: AB is a chord of the parabola $y^2 = 4ax$ joining $A(at_1^2, 2at_1)$ and $B(at_2^2, 2at_2)$.

Column I		Column II	
(A)	AB is a normal chord, if	(P)	$t_2 = 2 - t_1$
(B)	AB is a focal chord, if	(Q)	$t_1 t_2 = -4$
(C)	AB subtends 90° at (0, 0), if	(R)	$t_1 t_2 = -1$
(D)	AB is inclined at 45° to the axis of the parabola	(S)	$t_1^2 + t_1 t_2 + 2 = 0$

2. Match the following columns:

Column I		Column II	
(A)	The point, from which perpendicular tangents can be drawn to the parabola $y^2 = 4x$, is	(P)	$(-1, 2)$
(B)	The point, from which only one normal can be drawn to the parabola $y^2 = 4x$, is	(Q)	$(3, 2)$
(C)	The point, at which chord $x - y + 1 = 0$ of the parabola $y^2 = 4x$ is bisected, is	(R)	$(-1, -5)$
(D)	The point, from which tangents cannot be drawn to the parabola $y^2 = 4x$, is	(S)	$(5, -2)$

3. Match the following columns:

Column I		Column II	
(A)	The equation of the director circle to the parabola $y^2 = 12x$ is	(P)	$2y - 1 = 0$
(B)	The equation of the director circle to the parabola $x^2 = 16y$ is	(Q)	$x - 2 = 0$
(C)	The equation of the director circle to the parabola $y^2 + 4x + 4y = 0$ is	(R)	$y + 4 = 0$
(D)	The equation of the director circle to the parabola $y = x^2 + x + 1$ is	(S)	$x + 3 = 0$

4. Match the following columns:

Column I		Column II	
(A)	Number of distinct normals can be drawn from (–2, 1) to the parabola $y^2 - 4x - 2y - 3 = 0$ is	(P)	3
(B)	Number of distinct normals can be drawn from (2, 3) to the parabola $y = x^2 + x + 1$ is	(Q)	1
(C)	Number of distinct normals can be drawn from (–5, 3) to the parabola $y^2 - 4x - 6y - 1 = 0$ is	(R)	0
(D)	Number of tangents can be drawn from (1, 2) to the parabola $y^2 - 2x - 2y + 1 = 0$ is	(S)	2

5. Match the following columns:
Normals are drawn at points P, Q and R lying on the parabola $y^2 = 4x$ which intersect at (3, 0).

Column I		Column II	
(A)	Area of ΔPQR	(P)	2
(B)	Radius of the circumcircle ΔPQR	(Q)	5/2
(C)	Centroid of ΔPQR	(R)	(5/2, 0)
(D)	Circumcentre of ΔPQR	(S)	(2/3, 0)

6. Match the following columns:

Column I		Column II	
(A)	If $y + 3 = m_1(x + 2)$ and $y + 3 = m_2(x + 2)$ are two tangents to the parabola $y^2 = 8x$, then	(P)	$m_1 + m_2 = 0$
(B)	If $y = m_1x + c_1$ and $y = m_2x + c_2$ are two tangents to a parabola $y^2 = 4a(x + a)$, then	(Q)	$m_1 + m_2 = -1$
(C)	If $y = m_1(x + 4) + 2013$ and $y + 1 = m_2(x + 4) + 2014$ are two tangents to the parabola $y^2 = 16x$, then	(R)	$m_1, m_2 = -1$
(D)	If $y = m_1(x - 2) + 2010$ and $y = m_1(x - 2) + 2010$ are two tangents to the parabola $y^2 = -8x$, then	(S)	$m_1 - m_2 = 0$

7. Match the following columns:

Column I		Column II	
(A)	If $2x + y + \lambda = 0$ is a normal to the parabola $y^2 = -8x$, then λ is	(P)	9
(B)	If $x + y = k$ is a normal to the parabola $y^2 = 12x$, then k is	(Q)	24
(C)	If $2x - y - c = 0$ is a tangent to the parabola $y^2 = 16x$, then c is	(R)	17
(D)	If $y = 4x + d$ is a tangent to the parabola $y^2 = 4(x + 1)$, then $4d$ is	(S)	2

8. Match the following columns

Column I		Column II	
(A)	The equation of the directrix of the parabola $y^2 + 4x + 4y + 2 = 0$ is	(P)	$x - 2y + 4 = 0$
(B)	The equation of the axis of the parabola $x^2 + 4x + 4y + 2013 = 0$ is	(Q)	$2x - 3 = 0$
(C)	The equation of the tangent to the parabola $y^2 = 4x$ from the point (2, 3) is	(R)	$x + 3 = 0$
(D)	The equation of the directrix of the parabola $y^2 = 4x + 8$ is	(S)	$x + 2 = 0$

Questions asked in Previous Years' JEE-Advanced Examinations

1. Suppose that the normals drawn at three different points on the parabola $y^2 = 4x$ pass through the point (h, k). Show that $h > 2a$. **[IIT-JEE, 1981]**

2. A is a point on the parabola $y^2 = 4ax$. The normal at A cuts the parabola again at B. If AB subtends a right angle at the vertex of the parabola, find the slope of AB. **[IIT-JEE -1982]**

No questions asked in 1983.

3. Find the equation of the normal to the curve $x^2 = 4y$ which passes through the point (1, 2). **[IIT-JEE, 1984]**
4. Three normals are drawn from the point $(c, 0)$ to the curve $y^2 = x$. Show that c must be greater than 1/2. One normal is always the x-axis. Find c for which the other two normals are perpendicular to each other. **[IIT-JEE, 1991]**
5. Through the vertex O of the parabola $y^2 = 4x$, chords OP and OQ are drawn at right angles, show that for all positions of P, PQ cuts the axis at the parabola at a fixed point. Also find the locus of the mid-point of PQ. **[IIT-JEE, 1994]**
6. Consider a circle with its centre lying on the focus of the parabola $y^2 = 2px$ such that it touches the directrix of the parabola. Then a point of intersection of the circle and the parabola is

   (a) $\left(\frac{p}{2}, p\right)$ or $\left(-\frac{p}{2}, p\right)$ (b) $\left(\frac{p}{2}, \frac{p}{2}\right)$

   (c) $\left(-\frac{p}{2}, p\right)$ (d) $\left(-\frac{p}{2}, -\frac{p}{2}\right)$

   [IIT-JEE, 1995]
7. Show that the locus of a point that divides a chord of slope 2 of the parabola $y^2 = 4x$ internally in the ratio 1 : 2 is parabola. Also find its vertex. **[IIT-JEE, 1995]**
8. Points A, B, C lie on the parabola $y^2 = 4ax$. The tangent to the parabola at A, B and C taken in pair intersect at the points P, Q and R. Determine the ratio of the areas of Δs ABC and PQR. **[IIT-JEE, 1996]**
9. From a point A, common tangents are drawn to the circle $x^2 + y^2 = \frac{a^2}{2}$ and the parabola $y^2 = 4ax$. Find the area of the quadrilateral formed by the common tangents drawn from A and the chord of contact of the circle and the parabola. **[IIT-JEE, 1996]**

No questions asked in between 1997 to 1999.

10. If $x + y = k$ is normal to $y^2 = 12x$, then k is
    (a) 3 (b) 9 (c) –9 (d) –3
    [IIT-JEE, 2000]
11. If the line $x - 1 = 0$ is the directrix of the parabola $y^2 = kx - 8$, one of the value of the k is
    (a) 1/8 (b) 8 (c) 4 (d) 1/4
    [IIT-JEE, 2000]
12. The equation of the directrix of the parabola $y^2 + 4y + 4x + 2 = 0$ is
    (a) $x = -1$ (b) $x = 1$
    (c) $x = -3/2$ (d) $x = 3/2$
    [IIT-JEE, 2001]

13. The equation of the common tangent touching the circle $(x-3)^2+y^2=9$ and the parabola $y^2=4x$ above the x-axis is
(a) $y\sqrt{3}=(3x+1)$ (b) $y\sqrt{3}=-x-3$
(c) $y\sqrt{3}=x+3$ (d) $y\sqrt{3}=-(3x+1)$
[IIT-JEE, 2001]

14. The locus of the mid-point of the line segment joining the focus to a moving a point on the parabola $y^2=4x$ is another parabola with directrix
(a) $x=-a$ (b) $x=-a/2$
(c) $x=0$ (d) $x=a/2$
[IIT-JEE, 2002]

15. The equation of the common tangent to the curve $y^2=8x$ and $xy=-1$ is
(a) $3y=9x+2$ (b) $y=2x+1$
(c) $2y=x+8$ (d) $y=x+2$
[IIT-JEE, 2002]

16. The focal chord to $y^2=16x$ is tangent to $(x-6)^2+y^2=2$, the possible values of the slope of this chord, are
(a) $\{-1, 1\}$ (b) $\{-2, 2\}$
(c) $\{-2, 1/2\}$ (d) $\{2, -1/2\}$
[IIT-JEE, 2003]

17. Let C_1 and C_2 be, respectively, the parabolas $x^2=y-1$ and $y^2=x-1$, let P be any point on C_1 and Q be any point on C_2. Let P_1 and Q_1 be the reflections of P and Q respectively, with respect to the line $y=x$. Prove that P_1 lies on C_2, P_1 lies on C_1 and $PQ \geq \min[PP_1, QQ_1]$. Hence or otherwise determine points P_0 and Q_0 on the parabolas P_0 and Q_0, C_1 and C_2, respectively such that $P_0Q_0 \leq PQ$ for all pairs of points (P_1, Q) with P on C_1 and Q on C_2 **[IIT-JEE, 2003]**

18. Three normals with slopes m_1, m_2 and m_3 are drawn from a point P not on the axis of the parabola $y^2=4x$. If $m_1m_2=\alpha$, results in the locus of P being a part of the parabola, find the value of α. **[IIT-JEE, 2003]**

19. The angle between the tangents drawn from the point (1, 4) to the parabola $y^2=4x$ is
(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$
[IIT-JEE, 2004]

20. At any point P on the parabola $y^2-2y-4x+5=0$, a tangent is drawn which meets the directrix at Q. Find the locus of point R which divides QP externally in the ratio $\frac{1}{2}:1$. **[IIT-JEE, 2004]**

No questions asked in 2005.

21. The axis of a parabola is along the line $y=x$ and the distances of its vertex and focus from the origin are $\sqrt{2}$ and $2\sqrt{2}$, respectively. If the vertex and the focus both lie in the first quadrant, the equation of the parabola is
(a) $(x-y)^2=(x-y-2)$ (b) $(x-y)^2=(x+y-2)$
(c) $(x-y)^2=4(x+y-2)$ (d) $(x-y)^2=8(x+y-2)$
[IIT-JEE, 2006]

22. The equation of the common tangent to the parabola $y=x^2$ and $y=-(x-2)^2$ is/are
(a) $y=4(x-1)$ (b) $y=0$
(c) $y=-4(x-1)$ (d) $y=-30x-50$
[IIT-JEE, 2006]

26. **Match the following:**
Normals are drawn at points P, Q and R lying on the parabola $y^2=4x$ which intersect at (3, 0).

Column I		**Column II**	
(i)	Area of ΔPQR	(A)	2
(ii)	Radius of circumcircle of ΔPQR	(B)	5/2
(iii)	Centroid of ΔPQR	(C)	(1, 0)
(iv)	Circumcentre of ΔPQR	(D)	(2/3, 0)

[IIT-JEE, 2006]

24. Comprehension
Consider the circle $x^2+y^2=9$ and the parabola $y^2=8x$. They intersect at P and Q in the first and the fourth quadrants, respectively. The tangents to the circle at P and Q intersect the x-axis at R and tangents to the parabola at P and Q intersect the x-axis at S.
(i) The ratio of the areas of Δs PQS and PQR is
(a) $1:\sqrt{2}$ (b) 1 : 2 (c) 1 : 4 (d) 1 : 8
(ii) The radius of the circumcircle of ΔPRS is
(a) 5 (b) $3\sqrt{3}$ (c) $3\sqrt{2}$ (d) $2\sqrt{3}$
(iii) The radius of the incircle of ΔPQR is
(a) 4 (b) 3 (c) 8/3 (d) 2
[IIT-JEE, 2007]

25. **Statement 1:** The curve $y=-\frac{x^2}{2}+x+1$ is symmetric with respect to the line $x=1$.
Statement 2: A parabola is symmetric about its axis.
[IIT-JEE, 2007]

26. Let $P(x_1, y_1)$, $Q(x_2, y_2)$, $y_1<0$, $y_2<0$, be the end-points of the latus rectum of the ellipse $x^2+4y^2=4$. The equation of the parabola with the latus rectum PQ are
(a) $x^2+2\sqrt{3}y=3+\sqrt{3}$ (b) $x^2-2\sqrt{3}y=3+\sqrt{3}$
(c) $x^2+2\sqrt{3}y=3-\sqrt{3}$ (d) $x^2-2\sqrt{3}y=3-\sqrt{3}$
[IIT-JEE, 2008]

27. The tangent PT and the normal PN to the parabola $y^2=4ax$ at a point P on it meet axes at points T and N, respectively. The locus of the centroid of ΔPTN is a parabola whose
(a) vertex is $(2a/3, 0)$ (b) directrix is $x=0$
(c) latus rectum is $2a/3$ (d) focus is $(a, 0)$
[IIT-JEE, 2009]

28. Let A and B be two distinct points on the parabola $y^2=4x$. If the axis of the parabola touches a circle of radius r having AB as the diameter, the slope of the line joining A and B can be
(a) $-\frac{1}{r}$ (b) $\frac{1}{r}$ (c) $\frac{2}{r}$ (d) $-\frac{2}{r}$
[IIT-JEE, 2010]

29. Let L be a normal to the parabola $y^2 = 4x$. If L passes through the point (9, 6), then L is given by
(a) $y - x + 3 = 0$ (b) $y + 3x - 33 = 0$
(c) $y + x - 15 = 0$ (d) $y - 2x + 12 = 0$
[IIT-JEE, 2011]

30. Consider the parabola $y^2 = 8x$, Let Δ_1 be the area of the triangle formed by the end-points of its latus rectum and the point $P\left(\frac{1}{2}, 2\right)$ on the parabola, and Δ_2 be the area of the triangle formed by drawing tangents at P and at the end-points of the latus rectum. Then $\frac{\Delta_1}{\Delta_2}$ is...
[IIT-JEE, 2011]

31. Let S be the focus of the parabola $y^2 = 8x$ and let PQ be the common chord of the circle $x^2 + y^2 - 2x - 4y = 0$ and the given parabola. The area of ΔPQS is ...
[IIT-JEE, 2012]

Comprehension

Let PQ be a focal chord of a parabola $y^2 = 4ax$. The tangents to the parabola at P and Q meet at a point lying on the line $y = 2x + a$, $a > 0$.

32. The length of the chord PQ is
(a) $7a$ (b) $5a$ (c) $2a$ (d) $3a$

33. If the chord PQ subtends an angle θ at the vertex of $y^2 = 4ax$, then $\tan\theta$ is
(a) $\frac{2\sqrt{7}}{3}$ (b) $-\frac{2\sqrt{7}}{3}$ (c) $\frac{2\sqrt{5}}{3}$ (d) $-\frac{2\sqrt{5}}{3}$
[IIT-JEE, 2013]

37. **Match Matrix**
A line L: $y = mx + 3$ meets y-axis at $E(0, 3)$ and the arc of the parabola $y^2 = 16x$, $0 \le y \le 6$ at the point $F(x_0, y_0)$. The tangent to the parabola at $F(x_0, y_0)$ intersects the y-axis at $G(0, y_1)$. The slope m of the line L is chosen such that the area of ΔEFG has a local maximum.
Match List I and List II and select the correct answers using the code given below the lists.

List I		List II
P	$m =$	1. 1/2
Q	Maximum area of ΔEFG is	2. 4
R	$y_0 =$	3. 2
S	$y_1 =$	4. 1

[IIT-JEE, 2013]

35. The common tangents to the circle $x^2 + y^2 = 2$ and the parabola $y^2 = 8x$ touch the circle at the points P, Q and the parabola at the points R, S. Then the area of the quadrilateral $PQRS$ is
(a) 3 (b) 6 (c) 9 (d) 15
[IIT-JEE, 2014]

Comprehension

Let a, r, s, t be non-zero real numbers.
Let $P(at^2, 2at)$, Q, $R(ar^2, 2ar)$ and $S(as^2, 2as)$ be distinct points on the parabola $y^2 = 4ax$. Suppose that PQ is the focal chord and lines QR and PK are parallel, where K is the point $(2a, 0)$.

36. The value of r is
(a) $-\frac{1}{t}$ (b) $\frac{t^2+1}{t}$ (c) $\frac{1}{t}$ (d) $\frac{t^2-1}{t}$
[IIT-JEE, 2104]

37. If $st = 1$, the tangent at P and the normal at S to the parabola meet at a point whose ordinate is
(a) $\frac{(t^1+1)^2}{2t^3}$ (b) $\frac{a(t^2+1)^2}{2t^3}$
(c) $\frac{a(t^2+1)^2}{t^3}$ (d) $\frac{a(t^2+2)^2}{t^3}$
[IIT-JEE, 2014]

38. Let the curve C be the mirror image of the parabola $y^2 = 4x$ with respect to the line $x + y + 4 = 0$. If A and B are the points of intersection of C with the line $y = -5$, then the distance between A and B is... **[IIT-JEE-2015]**

39. If the normals of the parabola $y^2 = 4x$ drawn at the end points of its latus rectum are tangents to the circle $(x - 3)^2 + (y + 2)^2 = r^2$, then the value of r^2 is...
[IIT-JEE-2015]

40. Let P and Q be distinct points on the parabola $y^2 = 2x$ such that a circle with PQ as diameter passes through the vertex O of the parabola. If P lies in the first quadrant and the area of the triangle ΔOPQ is 3: 2, then which of the following is (are) the coordinates of P?
(a) $(4, 2\sqrt{2})$ (b) $(9, 3\sqrt{2})$
(c) $\left(\frac{1}{4}, \frac{1}{\sqrt{2}}\right)$ (d) $(1, \sqrt{2})$
[IIT-JEE-2015]

41. The circle $C_1 : x^2 + y^2 = 3$, with centre at O, intersects the parabola $x^2 = 2y$ at the point P in the first quadrant. Let the tangent to the circle C_1 at P touches other two circles C_2 and C_3 at R_2 and R_3, respectively.
Suppose C_2 and C_3 have equal radii $2\sqrt{3}$ and centres Q_2 and Q_3, respectively.
If Q_2 and Q_3 lie on the y-axis, then
(a) $Q_2Q_3 = 12$ (b) $R_2R_3 = 4\sqrt{6}$
(c) $ar(\Delta OR_2R_3) = 6\sqrt{2}$ (d) $ar(\Delta PQ_2Q_3) = 4\sqrt{2}$
[IIT-JEE-2016]

42. Let P be the point on the parabola $y^2 = 4x$ which is at the shortest distance from the center S of the circle $x^2 + y^2 - 4x - 16y + 64 = 0$. Let Q be the point on the circle dividing the line segment SP internally. Then
(a) $SP = 2\sqrt{5}$
(b) $SQ:SP = (\sqrt{5}+1):2$
(c) the x-intercept of the normal to the parabola at P is 6.
(d) the slope of the tangent to the circle at Q is $\frac{1}{2}$
[IIT-JEE-2016]

ANSWERS

LEVEL I

1. Parabola
2. $\lambda = 4$
3. $\frac{1}{2}$
4. $20\{x^2 + y^2 - 2x - 4y + 5\}$
   $= (x^2 + 9y^2 + 100 + 6xy + 20x + 6y)$
5. $x^2 + 2xy + y^2 + 2x + 2y + 4 = 0$
6. (i) V: (–3, 1), S: $\left(-\frac{11}{4}, 1\right)$, $L.R. = 1$, A: $y = 1$
   (ii) V: (–2, 2).; S: $\left(-\frac{5}{4}, 2\right)$; $L.R. = 3$, A: $y = 2$
   (iii) V: (2, -6), S: $\left(2, -\frac{23}{4}\right)$; $L.R. = 1$, A: $x = 2$
   (iv) V: $\left(-\frac{1}{2}, \frac{1}{2}\right)$; S: (–1/2, 3/4) ; $L.R.$ =1,
   A: $2y - 1 = 0$.
7. (3, 6) and (3, –6)
8. $x^2 - 2xy + y^2 + 32x + 32y + 76 = 0$
9. $x = 0$
10. $y^2 = 8x + 24 = 8(x + 3)$
11. $y = x^2 + 3x + 2$
12. $(y - 2)^2 = 2(x - 1)$
13. $(x - 3)^2 = 12(y - 1)$
14. $t_1 t_2 = -1$
15. $a\left(t + \frac{1}{2}\right)^2$
16. $4a \operatorname{cosec}^2 \theta$
18. $(0, at)$
21. (0, 4)
22. $-3 < x < 3$
23. (1/2, –2) or (25/2, –10)
24. 16
25. $\frac{4}{m^2}\sqrt{1 + m^2}\sqrt{a(a - mc)}$
26. $(at_1t_2, a(t_1 + t_2))$
27. $2x + y - 4 = 0$ and $2x - y + 1 = 0$
28. $\left(\frac{1}{2}, 2\right)$ or $\left(\frac{1}{2}, -2\right)$
29. $\theta = \frac{\pi}{2}$
30. $y = x - 2$ and $y = -3x + 2$
31. $x + y + a = 0$
32. $y = \left(-\frac{a^{1/3}}{b^{1/3}}\right)x + a\left(-\frac{b^{1/3}}{a^{1/3}}\right)$
33. $y = \pm x \pm 4$
34. $y = 4x - 4$
35. $y = x + 2$
36. $x + 2y = 1$
37. $x - \sqrt{3}y + 3 = 0$
38. $L(1, 2)$ and $L'(1, -2)$ is (1, 0)
39. $\frac{\pi}{3}$
40. $\frac{3}{\sqrt{2}}$
41. $2\sqrt{2}$
42. –1
43. (–2, –9)
44. $x + a + b = 0$
45. $\frac{1}{2}a^2(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)$
46. $x^2 + y^2 - a(1 + t_2t_3 + t_3t_1 + t_1t_2)x$
    $- a(t_1 + t_2 + t_3 - t_1t_2t_3)y$
    $+ a^2(t_2t_3 + t_3t_1 + t_1t_2) = 0$
47. $\left(-a, a\left(\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} + \frac{1}{m_1m_2m_3}\right)\right)$
48. $x + a = 0$
49. (i) $4x + 9 = 0$
    (ii) $y + 2 = 0$
    (iii) $x = 0$
50. $x = 2a + a(t_1^2 + t_2^2 + t_1t_2)$
    and $y = -at_1t_2(t_1 + t_2)$
51. $t_2 = -t_1 - \frac{2}{t_1}$
52. ≥ 8
53. 2
54. $x + y = 3$
55. $y = 2x - 24$
56. $k = 9$
57. $9PQ = 80\sqrt{10}$
58. $y^2 = a(x - 3a)$
59. $al^3 + 2alm^2 + nm^2 = 0$
60. $\theta = \tan^{-1}(\sqrt{2})$
61. $(m^2, -2m) = (1, \mp 2)$
62. $PQ = 6\sqrt{3}$
64. $y^2 = a(x - a)$
65. $\sqrt{5}$
66. $2\sqrt{5} - 1$
69. $\left(\frac{2h - 4a}{3}, 0\right)$

70. $h > 2a$
71. $27ak^2 < 4(h - 2a)^3$
72. $\varphi = \tan^{-1}\left(\frac{\tan\theta}{2}\right)$
74. $(3a, 0)$
79. $2x - 3y + 4 = 0$
80. $y = 3x - 3$
81. $(y^2 - 4ax) = (x + a)^2 \tan^2\theta$
82. $\frac{1}{|a|} \times \sqrt{(k^2 - 4ah)(k^2 + 4a^2)}$
83. $\frac{(k^2 - 4ah)^{3/2}}{2a}$
84. $4x - 3y + 1 = 0$
85. $y^2 = 2a(x - a)$
86. $y^2 = 2ax$
87. $y^4 - 2a(x - 2a)y^2 + 8a^4 = 0$
88. $y^4 = 2a(x - 4a)$
89. $(2a - b)y^2 = 4a^2x$
90. $y^2 - 2ax - by + 6ab = 0$
91. $y^2(2x + a) = a(3x + a)^2$
93. $y = a(t_1 + t_2)$
94. 15
95. $x - 7y + 10 = 0$

Level II

1. (c)	2. (c)	3. (b)	4. (c)	5. (a)
6. (d)	7. (c)	8. (c)	9. (a)	10 ()
11. (d)	12. (b)	13. (a)	14. (d)	15. (b)
16. (a)	17. (a)	18. (a)	19. ()	20. (d)
21. (c)	22. (b)	23. (b)	24. (b)	25. (b)
26. (b)	27. (b)	28. (d)	29. (d)	30. (a)
31. (b)	32. (b)	33. (d)	34. (c)	35. (a)
36. (c)	37. (c)	38. (b)	39. (c)	40. (d)
41. (c)	42. (d)	43. (a)	44. (b)	45. (a)
46. (c)	47. (d)	48. (c)	49. (a)	50. (c)

Level III

1. $(4, 0)$
2. $4ya^2 + 8a^3 - xa^4 + 4x = 0$
3. $y^2 = 4(a_1 - a)(x - a)$
4. $(0, 0), (4a, 4a), (4a, -4a), (8a, 0)$
5. $(x - y)^2 = 8(x + y - 2)$
8. $m = 1, -1$
9. $(-1, 0)$
10. $(2, 4)$
12. $y\sqrt{3} = x + 3$
13. $y = x + 2$
14. $y = 0, y = 4(x - 1)$
15. $\left(\frac{p}{2}, p\right), \left(\frac{p}{2}, -p\right)$
17. $c > 1/2$
18. $x - 2y + 8 = 0, (8, 8)$
19. $12a$
21. $x^2 = 2(y - 6)$
22. $(ax + by)(x^2 + y^2) + (bx - ay)^2 = 0$
23. $9y^2 = 4a(x - 8a)$
24. $y^2 = 2a^2(x - 4)$
25. $\left(\frac{2}{9}, \frac{8}{9}\right)$
26. $\frac{15a^2}{4}$
27. $\alpha = 2$
28. $(y - 1)^2(x + 1) + 4 = 0$
30. $(3, -4), (0, 0), (8, 16)$
31. $y = -\frac{(x - h)}{(y - k)} - \frac{a(y - k)}{(x - h)}$
32. $y^2 = a(x - 3a)$

Level IV

1. $8\sqrt{2}$ s.u.
2. (c)
3. $(b^2 - 4a^2)y - 4abx + 4a^2b = 0$
4. $t = \pm\frac{1}{\sqrt{5}}, \varphi = \cos^{-1}\left(-\frac{1}{\sqrt{5}}\right)$
5. (a, c)
6. $x^2 = 2(y - 6)$
7. $y + 18\left(\frac{x - 6}{21}\right)^{3/2} = 0$
8. $x - 2y + 1 = 0$
9. $(2 - x)y^2 = (5y^2 + 32)$
10. $9y^2 = 4ax$
11. $9x^2 = 4by$
12. $(\sqrt{21} - \sqrt{5})$
13. $\frac{5a}{\sqrt{2}}$
14. $|b| < \frac{1}{2\sqrt{2}}$
15. 4

INTEGER TYPE QUESTIONS

1. 6
2. 3
3. 5

4. 2
5. 6
6. 2
7. 6
8. 5
9. 5
10. 8
11. 9
12. 3

COMPREHENSIVE LINK PASSAGES

Passage I: (i) (d) (ii) (c) (iii) (d)
Passage II: (i) (b) (ii) (a) (iii) (c)
Passage III: (i) (d) (ii) (b) (iii) (c)
Passage IV: (i) (a) (ii) (b) (iii) (c)
Passage V: (i) (b) (ii) (c) (iii) (a)
Passage VI: (i) (c) (ii) (b) (iii) (d)
Passage VII: (i) (a) (ii) (c) (iii) (c) (iv) (b) (v) (c)
Passage VII: (i) (c) (ii) (b) (iii) (d)

MATRIX MATCH

1. (A) → (S); (B) → (R); (C) → (Q); (D) → (P)
2. (A) → (P, R); (B) → (P,R); (C) → (Q); (D) → (Q, S)
3. (A) → (S); (B) → (R); (C) → (Q); (D) → (P)
4. (A) → (Q); (B) → (P); (C) → (Q); (D) → (R)
5. (A) → (P); (B) → (Q); (C) → (S); (D) → (R)
6. (A) → (R); (B) → (R); (C) → (R); (D) → (R)
7. (A) → (Q); (B) → (P); (C) → (S); (D) → (R)
8. (A) → (Q); (B) → (S);(C) → (P); (D) → (R)

QUESTIONS ASKED IN PREVIOUS YEARS' JEE-ADVANCED EXAMINATIONS

2. $\pm\sqrt{2}$
3. $x + y = 3$
4. 3/4
5. $y^2 = 2x - 8$
6. (a)
7. $\left(\frac{2}{9}, \frac{8}{9}\right)$
8. 2 : 1
9. $\frac{15a^2}{4}$
10. (b)
11. (c)
12. (d)
13. (c)
14. (c)
15. (d)
16. (a)
17.
18. $\alpha = 2$
19. (c)
20. $(y - 1)^2(x + 1) + 4 = 0$
21. (d)
22. (a, b)
23. (i) → A; (ii) → B; (iii) → D; (iv) → C
24. (i) - (c) (ii)-(b) (iii)-(d)
25. (a)
26. (b, c)
27. (a, d)
28. (c, d)
29. (a, b, d)
30. 2
31. 4
32. (b)
33. (d)
34. $m = 1$
35. (d)
36. (d)
37. (b)
38. 4
39. 2
40. (a, c)
41. (a, b, c)
42. (a, c, d)

HINTS AND SOLUTIONS

LEVEL I

1. The given equation is

$$\sqrt{ax} + \sqrt{by} = 1$$

$$\Rightarrow (\sqrt{ax} + \sqrt{by})^2 = 1$$

$$\Rightarrow ax + by + 2\sqrt{abxy} = 1$$

$$\Rightarrow (ax + by - 1)^2 = (2\sqrt{abxy})^2$$

$$\Rightarrow a^2x^2 + b^2y^2 + 1 + 2abxy - 2ax - 2by = 4abxy$$

$$a^2x^2 - 2abxy + b^2y^2 - 2ax - 2by + 1 = 0$$

Here $A = a^2$, $B = b^2$ and $H = -ab$

Now, $H^2 - AB = a^2b^2 - a^2b^2 = 0$

Hence, it represents a parabola.

2. Here, $a = 1$, $h = -2$ and $b = \lambda$.

Since, it represents a parabola, so

$$h^2 - ab = 0$$

$$\Rightarrow 4 - \lambda = 0$$

$$\Rightarrow \lambda = 4$$

3. The given conic is

$$16(x^2+(y-1)^2)=(x+\sqrt{3}y-5)^2$$

$$\Rightarrow \quad (x^2+(y-1)^2)=\frac{1}{4}\left(\frac{x+\sqrt{3}y-5}{\sqrt{1+3}}\right)^2 \qquad \ldots\text{(i)}$$

which represents an ellipse.
Now, Eq. (i) can also be written as

$$SP^2 = e^2 \times PM^2$$

Thus, the eccentricity is $\frac{1}{2}$.

4. Let S be the focus, PM be the directrix and the eccentricity = e
From the definition of conic section, we get

$$\frac{SP}{PM}=e$$

$$\Rightarrow \quad SP = e \times PM$$

$$\Rightarrow \quad SP^2 = e^2 \times PM^2$$

$$\Rightarrow \quad (x-1)^2+(y-2)^2=\frac{1}{2}\times\left(\frac{x+3y+10}{\sqrt{1+9}}\right)^2$$

$$\Rightarrow \quad 20\{(x-1)^2+(y-2)^2\}=(x+3y+10)^2$$

$$\Rightarrow \quad 20\{x^2+y^2-2x-4y+5\}$$
$$= (x^2+9y^2+100+6xy+20x+6y)$$

5. Let S be the focus and PM be the directrix.
From the definition of conic section, it is clear that,

$$SP = PM$$

$$\Rightarrow \quad SP^2 = PM^2$$

$$\Rightarrow \quad (x-1)^2+(y-1)^2=\left(\frac{x-y+3}{\sqrt{1+1}}\right)^2$$

$$\Rightarrow \quad 2\{(x-1)^2+(y-1)^2\}=(x-y+3)^2$$

$$\Rightarrow \quad 2(x^2+y^2-2x-2y+2)$$
$$= (x^2+y^2+9-2xy-6x-6y)$$

$$\Rightarrow \quad x^2+2xy+y^2+2x+2y+4=0$$

6. (i) The given equation is

$$y^2 = x+2y+2$$
$$\Rightarrow \quad y^2-2y = x+2$$
$$\Rightarrow \quad (y-1)^2 = x+3$$
$$\Rightarrow \quad Y^2 = X, \text{ where } X = x+3,\ Y = y-1$$

Vertex: $V(0, 0)$

$$\Rightarrow \quad X=0,\ Y=0$$
$$\Rightarrow \quad x+3=0,\ y-1=0$$
$$\Rightarrow \quad x=-3,\ y=1$$

Hence, the vertex is $(-3, 1)$
Focus: $(a, 0)$

$$\therefore \quad X=a,\ Y=0$$
$$\Rightarrow \quad x+3=\frac{1}{4},\ y-1=0$$
$$\Rightarrow \quad x=\frac{1}{4}-3,\ y=1$$
$$\Rightarrow \quad x=-\frac{11}{4},\ y=1$$

Hence, the focus is $\left(-\frac{11}{4}, 1\right)$

Latus rectum: $4a = 1$
Directrix: $X + a = 0$

$$\Rightarrow \quad x+3=\frac{1}{4}$$
$$\Rightarrow \quad x=-\frac{11}{4}$$
$$\Rightarrow \quad 4x+11=0$$

Axis: $Y = 0$

$$\Rightarrow \quad y-1=0$$
$$\Rightarrow \quad y=1$$

(ii) The given equation is

$$y^2 = 3x+4y+2$$
$$\Rightarrow \quad y^2-4y = x^2+2$$
$$\Rightarrow \quad y^2-4y+4 = 2x+6$$
$$\Rightarrow \quad (y-2)^2 = 3(x+2)$$
$$\Rightarrow \quad Y^2 = 3X,$$

where $X = x + 2$ and $Y = y - 2$
Vertex: $(0, 0)$

$$\Rightarrow \quad X=0,\ Y=0$$
$$\Rightarrow \quad x+2=0 \text{ and } y-2=0$$
$$\Rightarrow \quad x=-2 \text{ and } y=2$$

Hence, the vertex is $(-2, 2)$
Focus: $(a, 0)$

$$\Rightarrow \quad X=a,\ Y=0$$
$$\Rightarrow \quad x+2=\frac{3}{4},\ y-2=0$$
$$\Rightarrow \quad x=-\frac{5}{4} \text{ and } y=2$$

Hence, the focus is $\left(-\frac{5}{4}, 2\right)$

Latus rectum: $4a = 3$
Directrix: $X + a = 0$

$$\Rightarrow \quad x+2=3/4$$
$$\Rightarrow \quad x=-5/3$$
$$\Rightarrow \quad 3x+5=0$$

Axis: $Y = 0$

$$\Rightarrow \quad y-2=0$$
$$\Rightarrow \quad y=2$$

(iii) The given equation is

$$x^2 = y+4x+2$$
$$\Rightarrow \quad x^2-4x = y+2$$
$$\Rightarrow \quad x^2-4x+4 = y+6$$
$$\Rightarrow \quad (x-2)^2 = y+6$$
$$\Rightarrow \quad X^2 = Y,$$

where $X = x - 2$ and $Y = y + 6$
Vertex: $(0, 0)$

$$\Rightarrow \quad X=0,\ Y=0$$
$$\Rightarrow \quad x-2=0 \text{ and } y+6=0$$
$$\Rightarrow \quad x=2 \text{ and } y=-6$$

Hence, the vertex is $(2, -6)$
Focus: $(0, a)$

$$\Rightarrow \quad X=0,\ Y=a$$
$$\Rightarrow \quad x-2=0 \text{ and } y+6=1/4$$
$$\Rightarrow \quad x=2 \text{ and } y=-23/4$$

Hence, the focus is $\left(2, -\frac{23}{4}\right)$

Latus rectum: $4a = 1$

Directrix: $Y + a = 0$

$\Rightarrow \quad y + 6 = 1/4$

$\Rightarrow \quad 4y + 23 = 0$

Axis: $X = 0$

$\Rightarrow \quad x - 2 = 0$

$\Rightarrow \quad x = 2$

(iv) The given equation is

$$x^2 + x + y = 0$$

$\Rightarrow \quad x^2 + x = -y$

$\Rightarrow \quad \left(x + \frac{1}{2}\right)^2 = -y + \frac{1}{4} = -\left(y - \frac{1}{4}\right)$

$\Rightarrow \quad X^2 = -Y$, where

$$X = x + \frac{1}{2}, Y = y - \frac{1}{2}$$

Vertex: $(0, 0) \Rightarrow X = 0, Y = 0$

$\Rightarrow \quad x + \frac{1}{2} = 0, y - \frac{1}{2} = 0$

$\Rightarrow \quad x = -\frac{1}{2}, y = \frac{1}{2}$

Hence, the vertex is $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Focus : $(0, -a)$

$\Rightarrow \quad X = 0, Y = -a$

$\Rightarrow \quad x + 1/2 = 0, y - 1/2 = 1/4$

$\Rightarrow \quad x = -1/2, y = 3/4$

Hence, the focus is $(-1/2, 3/4)$

Latus rectum: $4a = 1$

Directrix: $Y - a = 0$

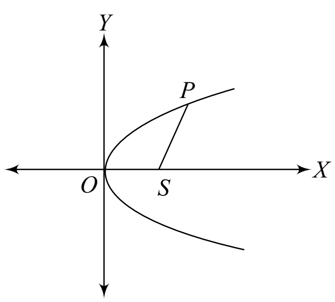
$\Rightarrow \quad y - 1/2 - 1/4 = 0$

$\Rightarrow \quad y - 3/4 = 0$

$\Rightarrow \quad 4y - 3 = 0$

Axis: $Y = 0 \Rightarrow y - 1/2 = 0 \Rightarrow 2y - 1 = 0$

7. Let the point be (x, y)



The given equation is

$$y^2 = 12x$$

$\Rightarrow \quad 4a = 12$

$\Rightarrow \quad a = 12/4 = 3$

Given focal distance = 6

$\therefore \quad x + a = 6$

$\Rightarrow \quad x + 3 = 6$

$\Rightarrow \quad x = 6 - 3 = 3$

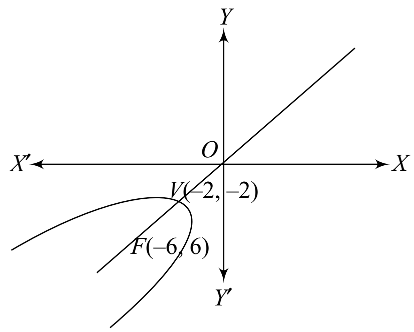
When $x = 3$, $y^2 = 12 \times 3 = 36$

$\Rightarrow \quad y = \pm 6$

Hence, the co-ordinates of the points are (3, 6) and (3, –6).

8. Let the vertex be V and the focus be S.

The equation of axis is $x - y = 0$.



Let the point Q is the point of intersection of the axis and the directrix.

Clearly, V is the mid-point of Q and S.

Then Q is (2. 2).

As we know that the directrix is perpendicular to the axis of the parabola. So, the equation of the directrix is

$$x + y - k = 0$$

which is passing through (2, 2).

Therefore, $k = 4$.

Hence, the equation of the directrix is

$$x + y - 4 = 0$$

Thus the equation of the parabola is

$$\sqrt{(x+6)^2 + (y+6)^2} = \left(\frac{x + y - 4}{\sqrt{1+1}}\right)$$

$\Rightarrow \quad 2((x + 6)^2 + (y + 6)^2) = (x + y - 4)^2$

$\Rightarrow \quad 2(x^2 + y^2 + 12x + 12y + 36)$

$$= (x^2 + y^2 + 16 + 2xy + 8x - 8y)$$

$\Rightarrow \quad x^2 - 2xy + y^2 + 32x + 32y + 76 = 0$

9. The given equations are $x = t^2 + 1$ and $y = 2t + 1$

Eliminating t, we get

$$(y - 1)^2 = 4(x - 1)$$

$\Rightarrow \quad Y^2 = 4X$, where $X = (x - 1)$

and $\quad Y = (y - 1)$

Hence, the equation of the directrix is

$$X + a = 0$$

$\Rightarrow \quad x - 1 + 1 = 0$

$\Rightarrow \quad x = 0.$

10. Let the vertex be V and the focus be S.

Let Q be the point of intersection of the axis and the directrix.

Clearly, Q be $(-5, 0)$ and V be the mid-point of S and Q.

Then focus S is $(-1, 0)$.

Hence, the equation of the parabola is

$$\sqrt{(x+1)^2 + y^2} = \left(\frac{x+5}{\sqrt{1^2}}\right)$$

$\Rightarrow \quad (x + 1)^2 + y^2 = (x + 5)^2$

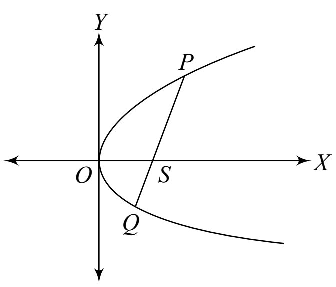
$\Rightarrow \quad y^2 = 8x + 24 = 8(x + 3)$

11. Let the equation of the parabola be
$$y = ax^2 + bx + c \quad \ldots\text{(i)}$$
which is passing through (0, 2), (–1. 0) and (1, 6). So
$$c = 2,\ a + c = b,\ a + b + c = 6$$
Solving, we get
$$a = 1,\ b = 3 \text{ and } c = 2$$
Hence, the equation of the parabola is $y = x^2 + 3x + 2$.

12. Let the equation of the parabola be $(y - k)^2 = 4a(x - h)$, where vertex is (h, k).
Then the equation becomes
$$(y - 2)^2 = 4a(x - 1)$$
which is passing through (3, 4).
Therefore, $8a = 4 \Rightarrow a = \frac{1}{2}$
Hence, the equation of the parabola is
$$(y - 2)^2 = 2(x - 1)$$

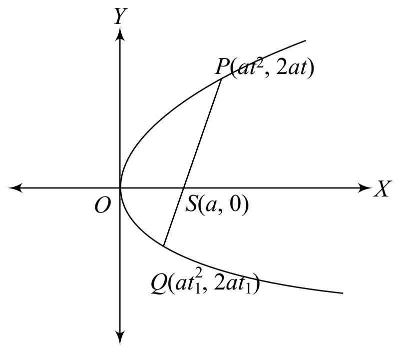
13. Let the equation of the parabola be $(x - H)^2 = 4a(y - k)$, where vertex is (h, k).
Thus the equation becomes
$$(x - 3)^2 = 4a(y - 2)$$
Also it given that, the length of the latus rectum = 12
$\Rightarrow \quad 4a = 12 \qquad \Rightarrow a = 3.$
Hence, the equation of the parabola is
$$(x - 3)^2 = 12(y - 1).$$

14. Let $y^2 = 4ax$ be a parabola, if PQ be a focal chord.



Consider any two points on the parabola $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$.
Since PQ passes through the focus $S(a, 0)$, so P, S, Q are collinear.
Thus, $m(PS) = m(QS)$
$$\Rightarrow \quad \frac{2at_1 - 0}{at_1^2 - a} = \frac{0 - 2at_2}{a - at_2^2}$$
$$\Rightarrow \quad \frac{2at_1}{at_1^2 - a} = \frac{2at_2}{at_2^2 - a}$$
$$\Rightarrow \quad \frac{2t_1}{t_1^2 - 1} = \frac{2t_2}{t_2^2 - 1}$$
$$\Rightarrow \quad \frac{t_1}{t_1^2 - 1} = \frac{t_2}{t_2^2 - 1}$$
$$\Rightarrow \quad t_1(t_2^2 - 1) = t_2(t_1^2 - 1)$$
$$\Rightarrow \quad t_1t_2(t_2 - t_1) + (t_2 - t_1) = 0$$
$$\Rightarrow \quad t_1t_2 + 1 = 0$$
$$\Rightarrow \quad t_1t_2 = -1$$
which is the required relation.

15. Since one extremity of the focal chord is $P(at^2, 2at)$, then the other extremity will be $Q\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$.

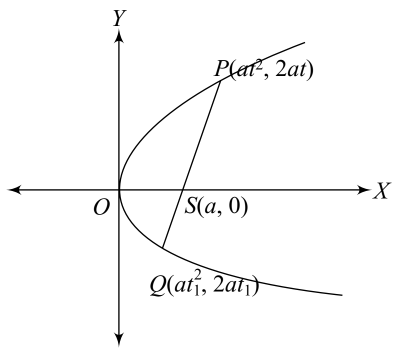


Thus, $PQ = SP + SQ$
$$= (at^2 + a) + \left(\frac{a}{t^2} + a\right)$$
$$= a\left(t^2 + \frac{1}{t^2} + 2\right) = a\left(t + \frac{1}{t}\right)^2$$

16. Now, slope of $PQ = \dfrac{2}{t - \frac{1}{t}} = \tan\theta$
$$\Rightarrow \quad 2\cot\theta = t - \frac{1}{t}$$
Thus, $PQ = a\left(t + \frac{1}{t}\right)^2$
$$= a\left[\left(t - \frac{1}{t}\right)^2 + 4\right]$$
$$= a(4\cot^2\theta + 4)$$
$$= 4a\,\text{cosec}^2\,\theta$$

17. $S = (a, 0),\ P = (at^2, 2at)$ and $Q = \left(\frac{a}{t^2}, -\frac{2a}{t}\right)$
Thus, $SP = a + at^2,\ SQ = a + \frac{a}{t^2}$
Now, Harmonic mean of SP and SQ
$$= \frac{2SP \cdot SQ}{SP + SQ} = \frac{2}{\frac{1}{SP} + \frac{1}{SQ}}$$
$$= \frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a + at^2} + \frac{1}{a + \frac{a}{t^2}} = \frac{1 + t^2}{a(1 + t^2)} = \frac{1}{a}$$
Thus, $\dfrac{2}{\frac{1}{SP} + \frac{1}{SQ}} = \dfrac{2}{\frac{1}{a}} = 2a$
= semi-latus rectum.

18.



The equation of the focal chord SP:

$$y-0=\frac{2at-0}{at^2-a}(x-a)$$

$\Rightarrow$ $y(t^2-1)=2tx-2at$

$\Rightarrow$ $2tx-(t^2-1)y-2at=0$

Let d be the distance of the focal chord SP from the vertex (0, 0) to the parabola $y^2=4ax$.

Then $d=\left|\dfrac{(0-0-2at)}{\sqrt{4t^2+(t^2-1)^2}}\right|$

$$=\frac{2at}{(t^2+1)}=\frac{2a}{\left(t+\frac{1}{t}\right)}$$

Also, $PQ=a\left(t+\dfrac{1}{t}\right)^2=a\times\dfrac{4a^2}{d^2}=\dfrac{4a^3}{d^2}$

Thus, $PQ\ \alpha\ \dfrac{1}{d^2}$

Hence the length of the focal chord varies inversely as the square of its distance from the vertex of the given parabola.

19. Let the circle described on the focal chord SP, where $S=(a,0)$ and $P=(at^2,2at)$.

The equation of the circle is

$$(x-at^2)(x-a)+(y-2at)(y-0)-0$$

Solving it with y-axis, $x=0$, we have

$$y^2-2aty+a^2t^2=0$$

Clearly, it has equal roots.

So the circle touches the y-axis.

Also, the point of contact is $(0, at)$.

20. The equation of the circle described on AB as diameter is

$$(x-at^2)\left(x-\frac{a}{t^2}\right)+(y-2at)\left(y+\frac{2a}{t}\right)=0$$

Put $x=a$, we have

$$y^2-2a\left(t-\frac{1}{t}\right)y+a^2\left(t-\frac{1}{t}\right)^2=0$$

Clearly, it has equal roots.

Hence the circle touches the directrix at $x=-a$.

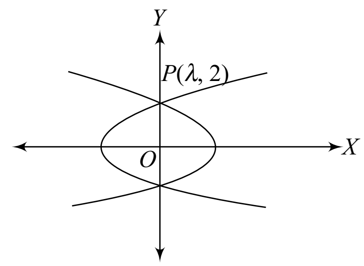
21. Since, the point $(\lambda,-\lambda)$ lies inside of the parabola $y^2=4x$, then $\lambda^2-4\lambda<0$

$\Rightarrow$ $\lambda(\lambda-4)<0$

$\Rightarrow$ $0<\lambda<4$

Hence, the range of λ is (0, 4).

22. Since the point $(\lambda, 2)$ is an exterior point of both the parabolas



$y^2=(x+1)$ and $y^2=-(x-1)$,

So we have

$4-x-1>0$ and $4+x-1>0$

$\Rightarrow$ $3-x>0$ and $3+x>0$

$\Rightarrow$ $x-3<0$ and $x+3>0$

$\Rightarrow$ $x<3$ and $x>-3$

$\Rightarrow$ $-3<x<3$

23. The given line is

$2x+3y+5=0$ …(i)

and the parabola is $y^2=8x$ …(ii)

Since (i) is a tangent to the parabola $y^2=8x$, so

$$\left(\frac{-2x-5}{3}\right)^2=8x$$

$\Rightarrow$ $(2x+5)^2=72x$

$\Rightarrow$ $4x^2+20x+25=72x$

$\Rightarrow$ $4x^2-52x+25=0$

$\Rightarrow$ $4x^2-2x-50x+25=0$

$\Rightarrow$ $2x(2x-1)-25(2x-1)=0$

$\Rightarrow$ $(2x-1)(2x-25)=0$

$\Rightarrow$ $x=1/2, x=25/2$

When $x=1/2$, then $y=\left(\dfrac{-1-5}{3}\right)=-2$

Also, when $x=25/2$, then $y=-10$

Hence, the points of contact are

(1/2, –2) or (25/2, –10)

24. The given parabola is

$y^2=12x$ …(i)

$\Rightarrow$ $4a=12$

$\Rightarrow$ $a=3$

The given line is $3x+4y+\lambda=0$

$\Rightarrow$ $y=-\dfrac{3}{4}x-\dfrac{\lambda}{4}$ …(ii)

Since, the line (ii) is a tangent to the parabola (i), so

$$c=\frac{a}{m}$$

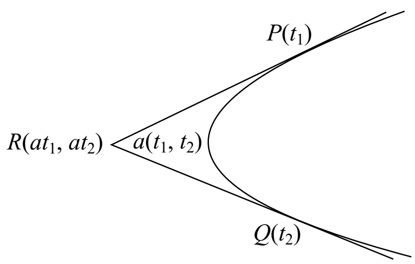
$\Rightarrow$ $-\dfrac{\lambda}{4}=\dfrac{3}{\left(-\frac{3}{4}\right)}=-4$

$\Rightarrow$ $\lambda=16$

Hence, the value of λ is 16.

25. Let the equation of the parabola be
$y^2 = 4ax$
and the line be $y = mx + c$.
Solving the above equations, we get
$(mx + c)^2 = 4ax$
$\Rightarrow \quad m^2x^2 + (2mc + 4a)x + c^2 = 0$
Let the line $y = mx + c$ intersects the parabola in two real and distinct points, say (x_1, y_1) and (x_2, y_2).
Thus $(x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2$
$$= \frac{4(mc - 2a)^2}{m^4} - \frac{4c^2}{m^2} = \frac{16a(a - mc)}{m^4},$$
and $y_1 - y_2 = m(x_1 - x_2)$
Thus, the required length
$$= \sqrt{(y_1 - y_2)^2 + (x_1 - x_2)^2}$$
$$= \sqrt{(x_1 - x_2)^2 m^2 + (x_1 - x_2)^2}$$
$$= \sqrt{1 + m^2}\,(x_1 - x_2)$$
$$= \frac{4}{m^2}\sqrt{1 + m^2}\,\sqrt{a(a - mc)}$$

26. Let the parabola be $y^2 = 4ax$ and the two points on the parabola are $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$, respectively.



The equation of the tangent at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ are
$$t_1 y = x + at_1^2 \quad \ldots\text{(i)}$$
and $\quad t_2 y = x + at_2 \quad \ldots$(ii)
Solving these equations, we get
$x = at_1t_2,\ y = a(t_1 + t_2)$
Hence the co-ordinates of the point of intersection of the tangents are $(at_1t_2, a(t_1 + t_2))$.

Notes:
1. x-co-ordinate is the geometric mean of the x-co-ordinates of P and Q.
2. y-co-ordinate is the arithmetic mean of the y-co-ordinates of P and Q.

27. The given parabola is $y^2 = 2x + 5y - 8$.
when $x = 1$, $y^2 = 5y - 6$
$\Rightarrow \quad y^2 - 5y + 6 = 0$
$\Rightarrow \quad (y - 2)(y - 3)$
$\Rightarrow \quad y = 2, 3$
Thus, the points are (1, 2) and (1, 3).
Hence, the equations of tangents can be at (1, 2) and (1, 3) be
$$2y = (x + 1) + \frac{5}{2}(y + 2) - 8$$
and $\quad 3y = (x + 1) + \frac{5}{2}(y + 3) - 8$
$\Rightarrow \quad 2x + y - 4 = 0$ and $2x - y + 1 = 0$

28. Let the equation of the tangent be
$y = 2x + c \quad \ldots$(i)
If the equation (i) be a tangent to the parabola, then
$$c = \frac{a}{m} = \frac{2}{2} = 1.$$
Thus, the equation of the tangent is
$y = 2x + 1 \quad \ldots$(ii)
The given parabola is $y^2 = 8x \quad \ldots$(iii)
Solving (ii) and (iii), we get
$(2x + 1)^2 = 8x$
$\Rightarrow \quad 4x^2 + 4x + 1 = 8x$
$\Rightarrow \quad (2x - 1)^2 = 0$
$\Rightarrow \quad x = \frac{1}{2}$
When $x = 1/2$, then $y = \pm 2$
Hence, the point of contacts are
$$\left(\frac{1}{2}, 2\right) \text{ or } \left(\frac{1}{2}, -2\right)$$

29. The equation of line from (–1, 2) is
$(y - 2) = m(x + 1)$
$\Rightarrow \quad mx - y + (m + 2) = 0$
$\Rightarrow \quad y = mx + (m + 2) \quad \ldots$(i)
The line (i) will be a tangent to the parabola $y^2 = 4x$, if
$$(m + 2) = \frac{1}{m}$$
$\Rightarrow \quad m^2 + 2m - 1 = 0$
which is a quadratic in m.
Let its roots are m_1, m_2.
Thus, $m_1 + m_2 = -2$ and $m_1m_2 = -1$
Let θ be the angle between them.
Then, $\tan(\theta) = \left|\frac{m_2 - m_1}{1 + m_1m_2}\right|$
$$= \left|\frac{\sqrt{(m_1 + m_2)^2 - 4m_1m_2}}{1 + m_1m_2}\right|$$
$\Rightarrow \quad \tan(\theta) = \infty = \tan\frac{\pi}{2}$
$\Rightarrow \quad \theta = \frac{\pi}{2}$
Hence, the angle between the tangents is $\theta = \frac{\pi}{2}$.

30. Let the equation of the tangent be

$$(y + 1) = m(x - 1)$$

$$\Rightarrow \quad y = mx - (m + 1) \quad \text{...(i)}$$

Equation (i) be a tangent to the parabola

$$y = x^2 - 3x + 2,$$

then $mx - m - 1 = x^2 - 3x + 2$

$$\Rightarrow \quad x^2 - (m + 3)x + (m + 3) = 0$$

Since it has equal roots, so

$$D = 0$$

$$\Rightarrow \quad (m + 3)^2 - 4(m + 3) = 0$$

$$\Rightarrow \quad (m + 3)(m + 3 - 4) = 0$$

$$\Rightarrow \quad (m + 3)(m - 1) = 0$$

$$\Rightarrow \quad m = 1, -3$$

Hence, the equation of the tangents are

$y = x - 2$ and $y = -3x + 2$

31. The given parabolas are

$y^2 = 4ax$ and $x^2 = 4ay$

Let the equation of the tangent be $y = mx + \dfrac{a}{m}$.

If it is a tangent to the parabola $x^2 = 4ay$, then

$$x^2 = 4a\left(mx + \frac{a}{m}\right)$$

$$\Rightarrow \quad mx^2 + 4am^2x + 4a^2$$

$$\Rightarrow \quad mx^2 - 4am^2x - 4a^2 = 0$$

Now $D = 0$ gives,

$$16a^2m^2 + 16a^2m = 0$$

$$\Rightarrow \quad 16a^2m\,(m^3 + 1) = 0$$

$$\Rightarrow \quad m(m^3 + 1) = 0$$

$\Rightarrow \quad m = 0$ and -1

Since $m = 0$ will not satisfy the given tangent, so

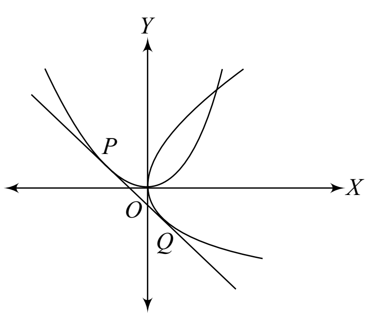
$$m = -1$$

Hence, the equation of the common tangent be

$$y = -x - a$$

$$\Rightarrow \quad x + y + a = 0$$

32. The given parabolas are $y^2 = 4ax$ and $x^2 = 4by$.



Let the equation of the tangent be

$$y = mx + \frac{a}{m} \quad \text{...(i)}$$

Since the equation (i) is also a tangent to the parabola $x^2 = 4by$, so

$$x^2 = 4b\left(mx + \frac{a}{m}\right)$$

$$\Rightarrow \quad mx^2 = 5m^2x + 4ab$$

$$\Rightarrow \quad mx^2 - 4bm^2x - 4ab = 0$$

Since it has equal roots, so

$$D = 0$$

$$16b^2m^4 + 16abm = 0$$

$$\Rightarrow \quad 16bm(bm^3 + a) = 0$$

$$\Rightarrow \quad m^3 = -\frac{a}{b}$$

$$\Rightarrow \quad m = -\frac{a^{1/3}}{b^{1/3}}$$

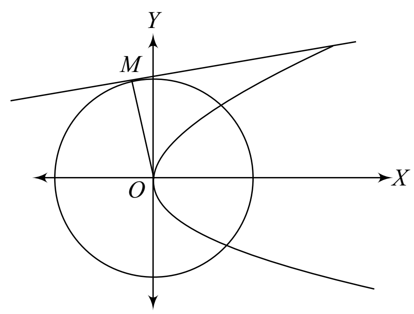
Hence, the equation of the common tangent be

$$y = \left(-\frac{a^{1/3}}{b^{1/3}}\right)x + a\left(-\frac{b^{1/3}}{a^{1/3}}\right)$$

33. Let the equation of the tangent to the parabola $y^2 = 16x$ is

$$y = mx + \frac{4}{m}$$

$$\Rightarrow \quad m^2x - my + 4 = 0 \quad \text{...(i)}$$



If the Eq. (i) be a tangent to the circle $x^2 + y^2 = 8$, the length of the perpendicular from the centre to the tangent is equal to the radius of the circle.

Therefore,

$$\left|\frac{0 - 0 + 4}{\sqrt{m^4 + m^2}}\right| = 2\sqrt{2}$$

$$\Rightarrow \quad 8(m^4 + m^2) = 16$$

$$\Rightarrow \quad (m^4 + m^2 - 2) = 0$$

$$\Rightarrow \quad (m^4 + 2m^2 - m^2 - 2) = 0$$

$$\Rightarrow \quad (m^2 + 2)(m^2 - 1) = 0$$

$$\Rightarrow \quad (m^2 - 1) = 0$$

$$\Rightarrow \quad m = \pm 1$$

Hence, the common tangents are $y = \pm x \pm 4$.

34. Any point on the parabola $y = x^2$ is (t, t^2).

Now the tangent at (t, t^2) is

$$xx_1 = \frac{1}{2}(y + y_1)$$

$$\Rightarrow \quad tx = \frac{1}{2}(y + t^2)$$

$$\Rightarrow \quad 2tx - y - t^2 = 0$$

If it is a tangent to the parabola,

$y = -(x - 2)^2$, then

$2tx - t^2 = -(x - 2)^2$

$\Rightarrow \quad 2tx - t^2 = -x^2 + 4x - 4$

$\Rightarrow \quad x^2 + 2(2 - t)x + (t^2 - 4) = 0$

Since it has equal roots,

$D = 0$

$4(2 - t)^2 - 4(t^2 - 4) = 0$

$\Rightarrow \quad (2 - t)^2 - (t^2 - 4) = 0$

$\Rightarrow \quad t = 2$

Hence, the equation of the common tangent is

$y = 4x - 4$

35. Let the equation of the tangent to the parabola $y^2 = 8x$ is

$$y = mx + \frac{2}{m} \quad \ldots\text{(i)}$$

If it is a tangent to the curve $xy = -1$, then

$$x\left(mx + \frac{2}{m}\right) = -1$$

$\Rightarrow \quad m^2x^2 + 2x + m = 0$

Since it has equal roots, so,

$D = 0$

$\Rightarrow \quad 4 - 4m^3 = 0$

$\Rightarrow \quad m^3 = 1$

$\Rightarrow \quad m = 1$

Hence, the equation of the common tangent is $y = x + 2$.

36. Any point on the parabola $y^2 = x$ can be considered as (t^2, t).

The equation of the tangent to the parabola $y^2 = x$ at (t^2, t) is

$$yy_1 = \frac{1}{2}(x + x_1).$$

$$\Rightarrow \quad yt = \frac{1}{2}(x + t^2)$$

$\Rightarrow \quad x + 2yt - t^2 = 0 \quad \ldots\text{(i)}$

If it is a tangent to the circle $x^2 + y^2 - 6y + 4 = 0$, then

$(2yt - t^2)^2 + y^2 - 6y + 4 = 0$

$\Rightarrow \quad 4y^2t^2 + t^4 - 4yt^3 + y^2 - 6y + 4 = 0$

$\Rightarrow \quad (4t^2 + 1)y^2 - 2(2t^3 + 3)y + (t^4 + 4) = 0$

Since it has equal roots, so

$D = 0$

$\Rightarrow \quad 4(2t^3 + 1)^2 - 4(4t^2 + 1)(t^4 + 4) = 0$

$\Rightarrow \quad (2t^3 + 1)^2 - (4t^2 + 1)(t^4 + 4) = 0$

$\Rightarrow \quad t^4 - 12t^3 + 16t^2 - 5 = 0$

$\Rightarrow \quad t = 1$

Hence, the equation of the common tangent is $x + 2y = 1$.

37. Any tangent to the parabola $y^2 = 4x$ is

$$y = mx + \frac{a}{m}$$

$$\Rightarrow \quad y = mx + \frac{1}{m}$$

$\Rightarrow \quad m^2x - my + 1 = 0 \quad \ldots\text{(i)}$

If it is a tangent to the circle $x^2 + (y - 3)^2 = 9$ the length of the perpendicular from the centre to the tangent is equal to the radius of the circle.

Therefore,

$$\left|\frac{3m^2 + 1}{\sqrt{m^4 + m^2}}\right| = 3$$

$\Rightarrow \quad (3m^2 + 1)^2 = 9(m^4 + m^2)$

$\Rightarrow \quad (9m^4 + 6m^2 + 1) = 9(m^4 + m^2)$

$\Rightarrow \quad 3m^2 = 1$

$$\Rightarrow \quad m = \pm\left(\frac{1}{\sqrt{3}}\right)$$

Since, the tangent touches the parabola above x-axis, so it will make an acute angle with x-axis, so that m is positive.

Thus $m = \dfrac{1}{\sqrt{3}}$.

Hence, the common tangent is $x - \sqrt{3}y + 3 = 0$.

38. The equation of the given parabola is $y^2 = 4x$.

We have, $4a = 4 \Rightarrow a = 1$

Let the end-points of the latus rectum are $L(a, 2a)$ and $L'(a, -2a)$.

Therefore $L = (1, 2)$ and $L' = (1, -2)$.

As we know that the point of intersection to the tangents at $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ to the parabola $y^2 = 4ax$ is

$$\left(at_1t_2, a\left(\frac{t_1 + t_2}{2}\right)\right)$$

Thus, the point of intersection of the tangents at $L(1, 2)$ and $L'(1, -2)$ is $(1, 0)$.

39. The equation of the tangent to the parabola

$y^2 = 4x \quad \ldots\text{(i)}$

from $(1, 4)$ is

$y - 4 = m(x - 1)$

$\Rightarrow \quad y = mx + (4 - m) \quad \ldots\text{(ii)}$

Since (ii) is a tangent to the parabola $y^2 = 4x$, so

$$c = \frac{a}{m}$$

$$\Rightarrow \quad (4 - m) = \frac{1}{m}$$

$\Rightarrow \quad 4m - m^2 - 1 = 0$

$\Rightarrow \quad m^2 - 4m + 1 = 0$

It has two roots, say m_1 and m_2.

Therefore, $m_1 + m_2 = 4$ and $m_1m_2 = 1$

Let θ be the angle between the tangents

Then $\tan(\theta) = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$

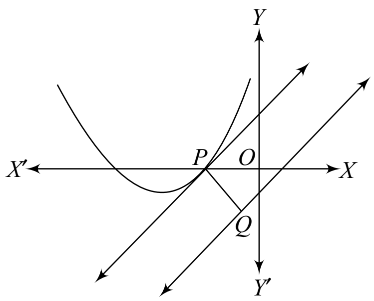
$$= \left|\frac{\sqrt{(m_1 + m_2)^2 - 4m_1 m_2}}{1 + m_1 m_2}\right|$$

$$= \left|\frac{\sqrt{12}}{2}\right| = \sqrt{3} = \tan\frac{\pi}{3}$$

$\Rightarrow \quad \theta = \frac{\pi}{3}$

Hence, the angle between the tangents is $\frac{\pi}{3}$.

40. The shortest distance between a line and the parabola means the shortest distance between a line and a tangent to the parabola parallel to the given line.



Thus, the slopes of the tangent and the line will be the same.

Therefore,

$2x + 3 = 1$

$\Rightarrow \quad x = -1$

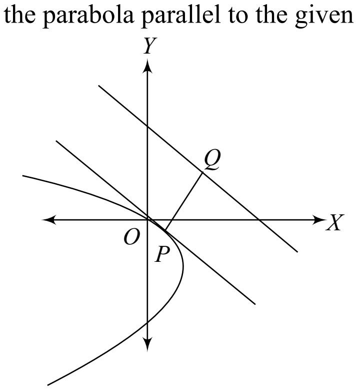
When $x = -1$, then $y = 0$.

Hence, the point on the parabola is $(-1, 0)$.

Thus, the required shortest distance

$$= \left|\frac{-1 - 0 - 2}{\sqrt{1+1}}\right| = \frac{3}{\sqrt{2}}$$

41. The shortest distance between a line and the parabola means the shortest distance between a line and a tangent to the parabola parallel to the given line.



Thus, the slopes of the tangent and the line will be the same.

Therefore, $-\frac{4}{2y+4} = -1 \Rightarrow y = 0$

When $y = 0$, then $x = 0$.

Thus, the point on the parabola is $(0, 0)$.

Hence, the required shortest distance

$$= \left|\frac{0 + 0 - 4}{\sqrt{1+1}}\right| = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

42. The given tangents are $y + b = m_1(x + a)$

and $\quad y + b = m_2(x + a)$

Therefore, both the tangents pass through $(-a, -b)$ which is a point lying on the directrix of the parabola.

Thus, the angle between them is 90°.

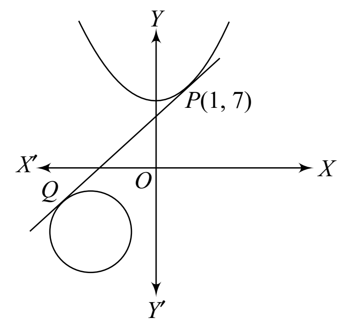
Hence, the value of $m_1 m_2$ is -1.

43. The equation of the tangent to the curve $y = x^2 + 6$ at $(1, 7)$ is

$$\frac{1}{2}(y + y_1) = xx_1 + 6$$

$\Rightarrow \quad \frac{1}{2}(y + 7) = x + 6$

$\Rightarrow \quad 2x - y - 5 = 0 \qquad \ldots\text{(i)}$



The given circle is

$$(x + 8)^2 + (y + 6)^2 = (\sqrt{100 - c})^2 \qquad \ldots\text{(ii)}$$

If the line (i) be a tangent to the circle (ii), the length of the perpendicular from the centre of the circle is equal to the radius of the circle.

Therefore, $\left|\frac{-16 + 6 - 5}{\sqrt{2^2 + 1}}\right| = \sqrt{100 - c}$

$\Rightarrow \quad 45 = 100 - c$

$\Rightarrow \quad c = 100 - 45 = 5$

Thus, the equation of the circle is

$(x + 8)^2 + (y + 6)^2 = 45$

$\Rightarrow \quad x^2 + y^2 + 16x + 12y + 55 = 0 \qquad \ldots\text{(iii)}$

Solving Eqs (i) and (iii), we get

$x^2 + (2x - 5)^2 + 16x + 12(2x - 5) + 55 = 0$

$\Rightarrow \quad x^2 + 4x^2 + 25 - 20x + 16x + 24x - 60 + 55 = 0$

$\Rightarrow \quad 5x^2 + 20x + 20 = 0$

$\Rightarrow \quad x^2 + 4x + 4 = 0$

$\Rightarrow \quad (x + 2)^2 = 0$

$\Rightarrow \quad x = -2$

When $x = -2$, then $y = 2x - 5 = -4 - 5 = -9$.

Hence, the point Q is $(-2, -9)$.

44. Any tangent to $y^2 = 4(x + a)$ is

$$y = m_1(x + a) + \frac{a}{m_1} \quad \ldots\text{(i)}$$

Also, any tangent to $y^2 = 4b(x + b)$ is

$$y = m_2(x + b) + \frac{b}{m_2} \quad \ldots\text{(ii)}$$

Since, two tangents are perpendicular, so

$m_1 m_2 = -1$

$\Rightarrow \quad m_2 = -\frac{1}{m_1}$

From Eq. (ii), we get

$$y = -\frac{1}{m_1}(x + b) - bm_1 \quad \ldots\text{(iii)}$$

Now subtracting Eq. (i) and Eq. (iii), we get

$$m_1(x + a) + \frac{1}{m_1}(x + b) + \frac{a}{m_1} + bm_1 = 0$$

$$\Rightarrow \quad \left(m_1 + \frac{1}{m_1}\right)x + \left(m_1 + \frac{1}{m_1}\right)a + \left(m_1 + \frac{1}{m_1}\right)b = 0$$

$\Rightarrow \quad x + a + b = 0$

Hence, the result.

45. Let the three points of the parabola be

$$P(at_1^2, 2at_1), Q(at_2^2, 2at_2) \text{ and } R(at_3^2, 2at_3)$$

and the points of intersection of the tangents at these points are $A(t_2\, t_3,\ a(t_2 + t_3))$, $B(t_1 t_3,\ a(t_1 + t_3))$ and $A(t_1 t_2, a(t_1 + t_2))$

Now,

$$\text{ar}(\Delta PQR) = \frac{1}{2}\begin{vmatrix} at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ at_3^2 & 2at_3 & 1 \end{vmatrix}$$

$$= a^2(t_1 - t_2)(t_2 - t_3)(t_3 - 1)$$

Also,

$$\text{ar}(\Delta ABC) = \frac{1}{2}\begin{vmatrix} at_2t_3 & a(t_2 + t_3) & 1 \\ at_3t_1 & a(t_3 + t_1) & 1 \\ at_1t_2 & a(t_1 + t_2) & 1 \end{vmatrix}$$

$$= \frac{1}{2}a^2(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)$$

Hence, the result.

46. Let P, Q and R be the points on the parabola $y^2 = 4ax$ at which tangents are drawn and let their co-ordinates be

$$P(at_1^2, 2at_1), Q(at_2^2, 2at_2) \text{ and } R(at_3^2, 2at_3).$$

The tangents at Q and R intersect at the point

$A[at_2t_3, a(t_2 + t_3)]$

Similarly, the other pairs of tangents at the points $B[at_1t_3, a(t_1 + t_3)]$ and $C[at_1t_2, a(t_1 + t_2)]$

Let the equation to the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \ldots\text{(i)}$$

Since it passes through the above three points, we have

$$a^2t_2^2t_3^2 + a^2(t_2 + t_3)^2 + 2gat_2t_3 + 2fa(t_2 + t_3) + c = 0 \quad \ldots\text{(ii)}$$

$$a^2t_1^2t_3^2 + a^2(t_1 + t_3)^2 + 2gat_1t_3 + 2fa(t_1 + t_3) + c = 0 \quad \ldots\text{(iii)}$$

and

$$a^2t_1^2t_2^2 + a^2(t_1 + t_2)^2 + 2gat_1t_2 + 2fa(t_1 + t_2) + c = 0 \quad \ldots\text{(iv)}$$

Subtracting Eq. (iii) from Eq. (ii) and dividing by $a(t_2 - t_1)$, we get

$$a(t_3^2(t_1 + t_2) + t_1 + t_2 + 2t_3) + 2gt_3 + 2f = 0$$

similarly from Eqs (iii) and (iv), we get

$$a(t_1^2(t_3 + t_2) + t_3 + t_2 + 2t_1) + 2gt_1 + 2f = 0$$

From these two equations, we get

$2g = -a(1 + t_2t_3 + t_3t_1 + t_1t_2)$,

$2f = -a(t_1 + t_2 + t_3 - t_1t_2t_3)$

Substituting these values of g and f in Eq. (ii), we get

$c = a^2(t_2t_3 + t_3t_1 + t_1t_2)$.

Thus, the equation of the circle is

$$x^2 + y^2 - a(1 + t_2t_3 + t_3t_1 + t_1t_2)x$$
$$-a(t_1 + t_2 + t_3 - t_1t_2t_3)y$$
$$+ a^2(t_2t_3 + t_3t_1 + t_1t_2) = 0$$

47. Let the equations of the three tangents be

$$y = m_1x + \frac{a}{m_1} \quad \ldots\text{(i)}$$

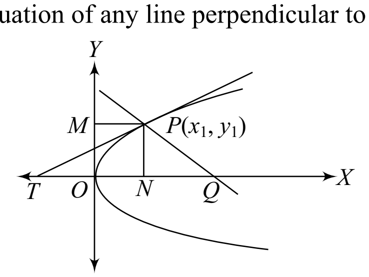
$$y = m_2x + \frac{a}{m_2} \quad \ldots\text{(ii)}$$

and $\quad y = m_3x + \frac{a}{m_3} \quad \ldots\text{(iii)}$

The point of intersection of (ii) and (iii) is

$$\left(\frac{a}{m_2m_3}, a\left(\frac{1}{m_2} + \frac{1}{m_3}\right)\right)$$

The equation of any line perpendicular to (i) and



The equation of any tangent to the parabola at (x_1, y_1) is

$(y - y_1) = m(x - x_1)$,

where $m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$

passes through the point of intersection of tangents (ii) and (iii) is

$$y - a\left(\frac{1}{m_2} + \frac{1}{m_3}\right) = -\frac{1}{m_1}\left(x - \frac{a}{m_2 m_3}\right)$$

i.e. $$y + \frac{x}{m_1} = a\left[\frac{1}{m_2} + \frac{1}{m_3} + \frac{1}{m_1 m_2 m_3}\right] \quad \ldots\text{(iv)}$$

Similarly the equation to the straight line through the point of intersection of (iii) and (i) and perpendicular to (ii) is

$$y + \frac{x}{m_2} = a\left[\frac{1}{m_3} + \frac{1}{m_1} + \frac{1}{m_1 m_2 m_3}\right] \quad \ldots\text{(v)}$$

and the equation of the straight line through the point of intersection of (i) and (ii) and perpendicular to (iii) is

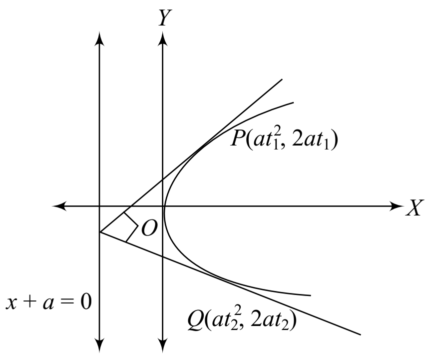
$$y + \frac{x}{m_3} = a\left[\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_1 m_2 m_3}\right] \quad \ldots\text{(vi)}$$

The point which is common to the straight lines (iv), (v) and (vi), i.e. the orthocentre of the triangle is

$$\left(-a, a\left(\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} + \frac{1}{m_1 m_2 m_3}\right)\right)$$

Hence, the point lies on the directrix.

48.



The equations of tangents at P and Q are $yt_1 = x + at_1^2$ and $yt_2 = x + at_2$.

The point of intersection of these tangents is

$[at_1t_2, a(t_1 + t_2)]$

Let this point be (h, k).

The slope of the tangents are $m_1 = \frac{1}{t_1}$ and $m_2 = \frac{1}{t_2}$.

Since these two tangents are perpendicular, so

$m_1 m_2 = -a$

$\Rightarrow \quad \frac{1}{t_1} \cdot \frac{1}{t_2} = -1$

$\Rightarrow \quad t_1 \cdot t_2 = -1$

$\Rightarrow \quad h = -a$

Thus the locus of the points of intersection is $x + a = 0$ which is the directrix of the parabola $y^2 = 4ax$.

49. (i) The given parabola is

$y^2 = x + 2$

$\Rightarrow \quad Y^2 = X,$

where $X = x + 2$ and $Y = y$

We have,

$4a = 1$

$\Rightarrow \quad a = \frac{1}{4}$

Hence, the equation of the director circle is

$X + a = 0$

$\Rightarrow \quad x + 2 + \frac{1}{4} = 0$

$\Rightarrow \quad 4x + 9 = 0$

(ii) The given parabola is

$x^2 = 4x + 4y$

$\Rightarrow \quad (x^2 - 4x + 4) = 4y + 4 = 4(y + 1)$

$\Rightarrow \quad (x - 2)^2 = 4(y + 1)$

$\Rightarrow \quad X^2 = 4Y,$

where $X = x - 2$

and $\quad Y = y + 1$

We have, $4a = 4$

$\Rightarrow \quad a = 1$

Hence, the equation of the director circle is

$Y + a = 0$

$\Rightarrow \quad y + 1 + 1 = 0$

$\Rightarrow \quad y + 2 = 0$

(iii) The given parabola is

$y^2 = 4x + 4y - 8$

$\Rightarrow \quad y^2 - 4y + 4 = 4x - 8 + 4$

$\Rightarrow \quad (y - 2)^2 = 4x - 4 = 4(x - 1)$

$\Rightarrow \quad Y^2 = 4X,$

where $X = (x - 1)$

and $\quad Y = (y - 2)$

We have $4a = 4 \Rightarrow a = 1$

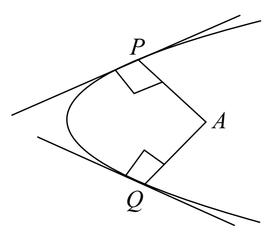
Hence, the equation of the director circle is

$X + a = 0$

$\Rightarrow \quad x - 1 + 1 = 0$

$\Rightarrow \quad x = 0$

50. Let the parabola be $y^2 = 4ax$ and two points on the parabola are $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$.



The equation of the normals at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ are

$$y = -t_1 x + 2at_1 + at_1^3 \quad \ldots\text{(i)}$$

and

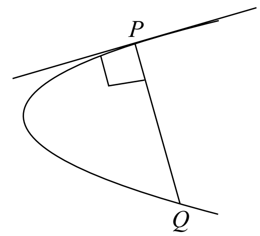
$$y = -t_2 x + 2at_2 + at_2^3 \quad \ldots\text{(ii)}$$

Solving Eqs (i) and (ii), we get

$x = 2a + a(t_1^2 + t_2^2 + t_1 t_2)$

and $\quad y = -at_2 t_2(t_1 + t_2)$.

51. Let the parabola be $y^2 = 4ax$ and the two points on the parabola are $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$.



The equation of the normal to the parabola at $P(at_1^2, 2at_1)$ is $y = -t_1x + 2at_1 + at_1^3$ which meets the parabola again at $Q(at_2^2, 2at_2)$.

Thus, $2at_2 = -at_1t_2^2 + 2at_1 + at_1^3$

$\Rightarrow \quad 2a(t_2 - t_1) + at_1(t_2^2 - t_1^2) = 0$

$\Rightarrow \quad (t_2 - t_1)[2a + at_1(t_2 + t_1)] = 0$

$\Rightarrow \quad [2 + t_1(t_2 + t_1)] = 0$

$\Rightarrow \quad t_1^2 + t_1t_2 + 2 = 0$

$\Rightarrow \quad t_2 = -t_1 - \dfrac{2}{t_1}$

which is the required condition.

52. As we know that if the normal at t_1 meets the parabola again at t_2, then

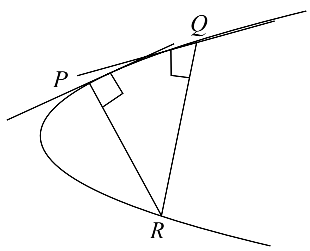
$$t_2 = -t_1 - \frac{2}{t_1}$$

$$\Rightarrow \quad t_2^2 = \left(-t_1 - \frac{2}{t_1}\right)^2$$

$$= t_1^2 + \frac{4}{t_1^2} + 4 \geq 4 + 4 = 8$$

Hence, the result.

53. Since the normal at t_1 meets the parabola at t_3, so $t_3 = -t_1 - \dfrac{2}{t_1}$.



Similarly, $t_3 = -t_2 - \dfrac{2}{t_2}$

Thus, $-t_1 - \dfrac{2}{t_1} = -t_2 - \dfrac{2}{t_2}$

$$\Rightarrow \quad (t_1 - t_2) = \left(\frac{2}{t_1} - \frac{2}{t_2}\right) = \frac{2(t_1 - t_2)}{t_1t_2}$$

$\Rightarrow \quad t_1t_2 = 2$

54. The equation of the normal is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1), \text{ where } a = 1$$

$$\Rightarrow \quad y - 2 = -\frac{2}{2}(x - 1) = -x + 1$$

$\Rightarrow \quad x + y = 3$

55. The equation of the normal to the parabola $y^2 = 4ax$ is

$y = mx - 2am - am^3$

Here, $a = 2$ and $m = 2$.

Therefore, $y = 2x - 8 - 16 = 2x - 24$

Hence, the equation of the normal is $y = 2x - 24$.

56. The given parabola is $y^2 = 12x$.

We have, $4a = 12 \Rightarrow a = 3$.

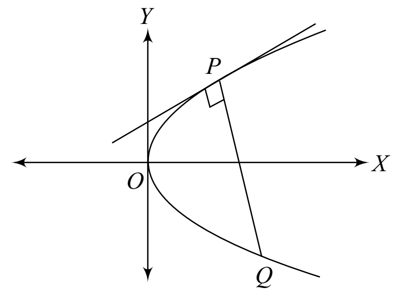
The given line $x + y = k \Rightarrow y = -x + k$ …(i)

The line (i) will be a normal to the given parabola, if $k = -2am - am^3 = 6 + 3 = 9$.

Hence, the value of k is 9.

57. The equation of the parabola is $y^2 = 8x$.

We have, $4a = 8 \Rightarrow a = 2$



The equation of the normal to the given parabola at $P(8, 12)$ is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

$$\Rightarrow \quad y - 12 = -\frac{12}{4}(x - 18)$$

$\Rightarrow \quad y - 12 = -3x + 54$

$\Rightarrow \quad 3x + y = 66$

Solving $y = 66 - 3x$ and the parabola $y^2 = 8x$, we get $x = -\dfrac{44}{3}$ and $y = \dfrac{242}{9}$

Hence, the co-ordinates of Q is $\left(-\dfrac{44}{3}, \dfrac{242}{9}\right)$.

$$\text{Thus, } PQ = \sqrt{\left(\frac{242}{9} - 18\right)^2 + \left(-\frac{44}{3} - 12\right)^2}$$

$$= \sqrt{\frac{6400}{81} + \frac{6400}{9}}$$

$$= 80\sqrt{\frac{1}{81} + \frac{1}{9}}$$

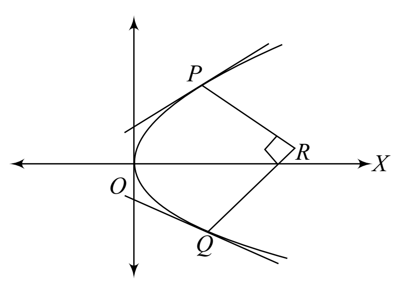
$$= \frac{80}{9} \times \sqrt{10}$$

$$\Rightarrow \quad 9PQ = 80\sqrt{10}$$

58. The equations of normals at $P(t_1)$ and $Q(t_2)$ are

$y = -t_1x + 2at_1 + at_1^3$

and $y = -t_2x + 2at_2 + at_2^3$



Since these two normals are at right angles, so $t_1t_2 = -1$.

Let $M(h, k)$ be the point of intersection of two normals.

Then, $h = 2a + a(t_1^2 + t_1t_2 + t_2^2)$

and $k = -at_1t_2(t_1 + t_2)$

$\Rightarrow h = 2a + a\{(t_1 + t_2)^2 - 2t_1t_2\}$

and $k = -at_1t_2(t_1 + t_2)$

$\Rightarrow h = 2a + a\{(t_1 + t_2)^2 + 2\}$

and $k = a(t_1 + t_2)$

Eliminating t_1 and t_2, we get,

$k^2 = a(h - 3a)$

Hence, the locus of $M(h, k)$ is $y^2 = a(x - 3a)$.

59. The given line is

$lx + my = 0$

$\Rightarrow my = -lx - n$

$\Rightarrow y = \left(-\frac{l}{m}\right)x + \left(-\frac{n}{m}\right)$

As we know that the line $y = mx + c$ will be a normal to the parabola $y^2 = 4ax$ if

$c = -2am - am^3$

$\Rightarrow \left(-\frac{n}{m}\right) = -2a\left(-\frac{l}{m}\right) - a\left(-\frac{l}{m}\right)^3$

$\Rightarrow al^3 + 2alm^2 + nm^2 = 0$

Hence, the result.

60. The equation of the parabola is $y^2 = 4ax$.

If the normal at $P(t_1)$ meets the parabola again at $Q(t_2)$, then

$t_2 = -t_1 - \frac{2}{t_1}$

$\Rightarrow t_1t_2 = -t_1^2 - 2$

$\Rightarrow t_1^2 + t_1t_2 + 2 = 0$ …(i)

The chord joining t_1, t_2 subtends a right angle at the vertex, so the product of their slopes = –1

$\Rightarrow \frac{2}{t_1} \cdot \frac{2}{t_2} = -1$

$\Rightarrow t_1t_2 = -4$ …(ii)

From Eqs (i) and (ii), we get

$t_1^2 - 4 + 2 = 0$

$\Rightarrow t_1^2 = 2$

$\Rightarrow t_1 = \sqrt{2}$

$\Rightarrow \tan\theta = \sqrt{2}$

$\Rightarrow \theta = \tan^{-1}(\sqrt{2})$

61. The given equation of the parabola is $y^2 = 4x$.

We have $4a = 4 \Rightarrow a = 1$.

The equation of the normal to the parabola

$y^2 = 4x$ at $(am^2, -2am)$ is

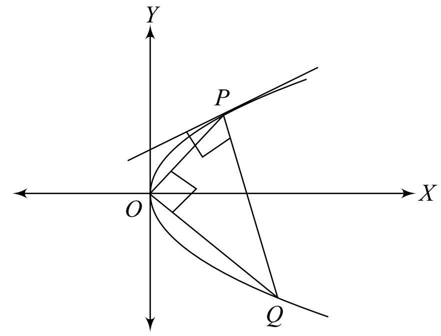
$y = mx - 2am - am^3 = mx - 2m - m^3$

Since, the normal makes equal angles with the axes, so $m = \pm 1$

Thus, the points are $(m^2, -2m) = (1, \mp 2)$

62. If the normal at $P(t_1)$ meets the parabola at $Q(t_2)$, then

$t_2 = -t_1 - \frac{2}{t_1}$ …(i)



Since the normal chord subtends an angle of 90° at the vertex, then

$t_1t_2 = -4$

From Eq. (i), we get

$t_1^2 + t_1t_2 + 2 = 0$

$\Rightarrow t_1^2 - 4 + 2 = 0$

$\Rightarrow t_1^2 - 2 = 0$

Also, $t_2^2 = \left(-t_1 - \frac{2}{t_1}\right)^2$

$= t_1^2 + \frac{4}{t_1^2} + 4$

$= 2 + 2 + 4 = 8$

Therefore,

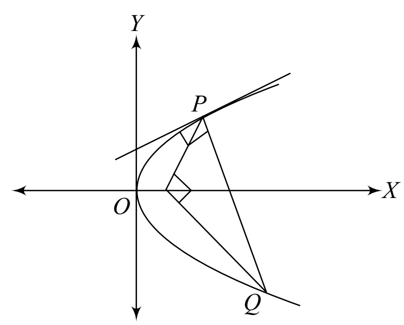
$PQ^2 = a^2(t_1^2 - t_2^2)^2 + 4a^2(t_1 - t_2)^2$

$= 1 \cdot (2 - 8)^2 + 4(\sqrt{2} + 2\sqrt{2})^2$

$= 36 + 72 = 108$

$\Rightarrow PQ = 6\sqrt{3}$

63. The equation of the parabola is $y^2 = 4ax$.



Let the normal chord be PQ, where $P(t_1)$ and $Q(t_2)$.

Since the abscissa and ordinate of the point (p, p) are same, then

$2at_1 = at_1^2$

$\Rightarrow t_1 = 2$

If the normal at $P(t_1)$ meets the parabola $Q(t_2)$, then

$t_2 = -t_1 - \frac{2}{t_1}$

$\Rightarrow t_2 = -2 - 1 = -3$

Let $S(a, 0)$ be the focus of the parabola $y^2 = 4ax$.

Then the slope of $SP = \dfrac{2at_1}{at_1^2 - a} = \dfrac{4}{4-1} = \dfrac{4}{3}$

and the slope of $SQ = \dfrac{2at_2}{at_2^2 - a}$

$$= \frac{-6}{9-1} = \frac{-6}{8} = -\frac{3}{4}$$

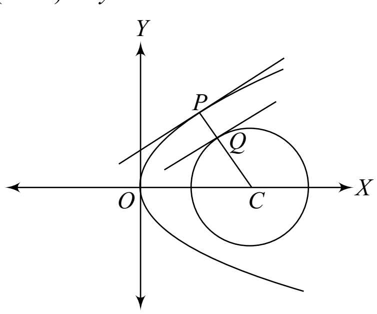
It is clear that

$$m(SP) \times m(SQ) = \frac{4}{3} \times -\frac{3}{4} = -1$$

Hence, the result.

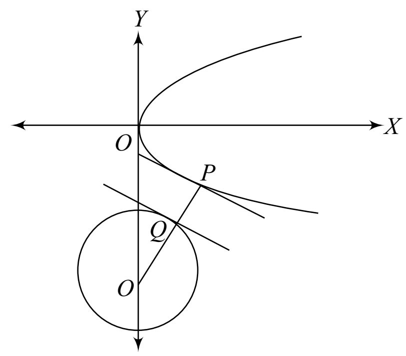
64. The given equation of the parabola is $y^2 = 4ax$.
The equation of the normal at $P(am^2, -2am)$ is
$$y = mx - 2am - am^3 \quad \ldots\text{(i)}$$
Let Q be a point on the axis of the parabola.
Put $y = 0$ in Eq. (i), we get
$$x = 2a + am^2$$
Hence, the co-ordinates of the point Q is $(2a + am^2, 0)$.
Let $M(h, k)$ be the mid-point of the normal PQ.
Then, $h = \dfrac{am^2 + 2a + am^2}{2}$
and $k = -\dfrac{2am}{2} = -am$
Eliminating m, we get
$$h = a + \frac{k^2}{a}$$
$\Rightarrow \quad a^2 + k^2 = ah$
Hence, the locus of $M(h, k)$ is
$$a^2 + y^2 = ax$$
$\Rightarrow \quad y^2 = a(x - a)$

65. The given equation of the parabola is $y^2 = 4x$.
The equation of the normal at $P(m^2, -2m)$ to the parabola $y^2 = 4x$ is
$$y = mx - 2m - m^3 \quad \ldots\text{(i)}$$
The given equation of the circle is
$$x^2 + y^2 - 12x + 31 = 0$$
$\Rightarrow \quad (x - 6)^2 + y^2 = 5 \quad \ldots\text{(ii)}$



The shortest distance between the parabola and the circle lies along the common normal.
Therefore, the centre of a circle passes through the normal, so we have
$$0 = 6m - 2m - m^3$$
$\Rightarrow \quad m^3 - 4m = 0$
$\Rightarrow \quad m = 0, -2, 2$
Therefore, P is $(4, -4)$ or $(4, 4)$ and let $C(6, 0)$ be the centre of the circle and Q be a point on the circle.
Therefore,
$$CP = \sqrt{(6-4)^2 + (4-0)^2} = \sqrt{20} = 2\sqrt{5}$$
and $CQ = \sqrt{5}$
Thus, the shortest distance $= CP - CQ$
$$= 2\sqrt{5} - \sqrt{5} = \sqrt{5}$$

66. The given equation of the parabola is $y^2 = 8x$.
We have, $4a = 8 \Rightarrow a = 2$.



The equation of the normal to the parabola
$y^2 = 8x$ at $P(4m^2, -4m)$ is
$$y = mx - 4m - 2m^3 \quad \ldots\text{(i)}$$
The given equation of the circle is
$$x^2 + y^2 + 12y + 35 = 0$$
$\Rightarrow \quad x^2 + (y + 6)^2 = 1$
Thus, the centre of the circle is $C(0, -6)$.
As we know that the shortest distance between a circle and the parabola lies along the common normal.
Therefore, the normal always passes through the centre of the circle. So
$$-6 = -4m - 2m^3$$
$\Rightarrow \quad m^3 + 2m - 3 = 0$
$\Rightarrow \quad m = 1$
Thus, the point P is $(4, -4)$.
Let Q be any point on the circle.
Then $CQ = 1$ and
$$CP = \sqrt{(4-0)^2 + (-4+6)^2} = \sqrt{20} = 2\sqrt{5}$$
Hence, the shortest distance $= PQ$
$$= CP - CQ = 2\sqrt{5} - 1.$$

67. The equation of any normal to a parabola $y^2 = 4ax$ is $y = mx - 2am - am^3$
which meets at a point, say (h, k).
Thus, $am^3 + (2a - h)m + k = 0$.
which is a cubic equation in m.
So it has three roots, say m_1, m_2 and m_3.
Therefore, $m_1 + m_2 + m_3 = 0$
Hence the result.

68. Let the ordinates of A, B and C be y_1, y_2 and y_3 respectively.

Then, $y_1 = -2am_1, y_2 = -2am_2, y_3 = -2am_3$.

Thus,

$$y_1 + y_2 + y_3 = -2am_3 - 2am_2 - 2am_3$$
$$= -2a(m_1 + m_2 + m_3) = -2a.0 = 0$$

69. If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a ΔABC, then its centroid is

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$
$$= \left(\frac{x_1 + x_2 + x_3}{3}, 0\right)$$

Hence the centroid lies on the axis of the parabola.
Also,

$$\left(\frac{x_1 + x_2 + x_3}{3}\right) = \frac{1}{3}(am_1^2 + am_2^2 + am_3^2)$$
$$= \frac{a}{3}\left\{0 - 2\left(\frac{2a - h}{a}\right)\right\} = \frac{2h - 4a}{3}$$

Thus, the centroid of ΔABC is $\left(\frac{2h - 4a}{3}, 0\right)$.

70. When three normals are real, then all the three roots of $am^3 + (2a - h)m + k = 0$ are real.

Let its three roots are m_1, m_2, m_3.

For any real values of m_1, m_2, m_3,

$$m_1^2 + m_2^2 + m_3^2 > 0$$
$$\Rightarrow \quad (m_1 + m_2 + m_3)^2 - 2(m_1m_2 + m_2m_3 + m_3m_1) > 0$$
$$\Rightarrow \quad 0 - 2\left(\frac{2a - h}{a}\right) > 0$$
$$\Rightarrow \quad h - 2a > 0$$
$$\Rightarrow \quad h > 2a$$

71. Let $f(m) = am^3 + (2a - h)m + k$

$\Rightarrow \quad f'(m) = 3am^2 + (2a - h)\ 2a$

If $f(m)$ has three distinct roots, so $f'(m)$ has two distinct roots.

Let two distinct roots of $f'(m) = 0$ are α and β.

Thus, $\alpha = \sqrt{\left(\frac{h - 2a}{3}\right)}$ and $\beta = -\sqrt{\left(\frac{h - 2a}{3}\right)}$.

Now, $f(\alpha)f(\beta) = 0$

$$f(\alpha)\, f(-\alpha) = 0$$
$$\Rightarrow \quad (a\alpha^3 + (2a - h)\alpha + k)(-a\alpha^3 - (2a - h)\alpha + k) < 0$$
$$\Rightarrow \quad k^2 - ((a\alpha^2 + (2a - h))^2\, \alpha^2) < 0$$
$$\Rightarrow \quad k^2 - \left(\frac{h - 2a}{3} + (2a - h)\right)^2\left(\frac{h - 2a}{3}\right) < 0$$
$$\Rightarrow \quad k^2 - \left(\frac{4a - 2h}{3}\right)^2\left(\frac{h - 2a}{3}\right) < 0$$
$$\Rightarrow \quad k^2 - \frac{4(h - 2a)^3}{27a} < 0$$
$$\Rightarrow \quad 27ak^2 < 4(h - 2a)^3$$

Hence, the result.

72. Let the normal at $P(at_1^2, 2at_1)$ be $y = -t_1x + 2at_1 + at_1^3$

Thus slope of the normal = $\tan\theta = -t_1$

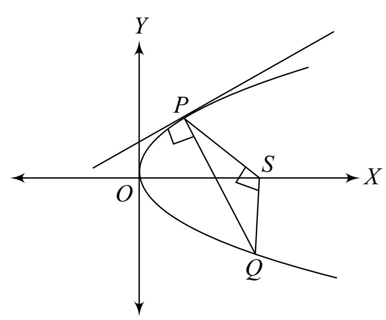
It meets the parabola again at $Q(at_2^2, 2at_2)$

Then $t_2 = -t_1 - \frac{2}{t_1}$.

Now the angle between the normal and the parabola = angle between the normal and the tangent at Q. If φ be the angle between them, then

$$\tan\varphi = \frac{m_1 - m_2}{1 + m_1m_2}$$
$$= \frac{-t_1 - \frac{1}{t_2}}{1 + (-t_1)\left(\frac{1}{t_2}\right)}$$
$$= -\frac{t_1t_2 + 1}{t_2 - t_1}$$
$$= -\frac{t_1\left(-t_1 - \frac{2}{t_1}\right) + 1}{-t_1 - \frac{2}{t_1} - t_1}$$
$$= -\frac{-t_1^2 - 1}{-2\left(\frac{1 + t_1^2}{t_1}\right)}$$
$$= -\frac{t_1}{2}$$
$$= \frac{\tan\theta}{2}$$
$$\Rightarrow \quad \varphi = \tan^{-1}\left(\frac{\tan\theta}{2}\right)$$

73. The equation of the normal to the parabola $y^2 = 4ax$ at $P(at^2, 2at)$ is
$y = -tx + 2at + at^3$



It is given that $at^2 = 2at \Rightarrow t = 2$.
Thus, the co-ordinates of P is $(4a, 4a)$ and focus is $S(a, 0)$.
Also, the normal chord meets the parabola at some point, say Q. Then the co-ordinates of Q is $(9a, -6a)$.
Now,

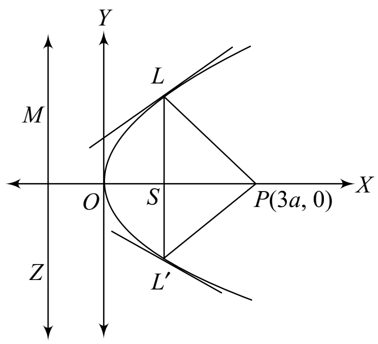
Slope of $SP = m_1 = \frac{4a}{3a} = \frac{4}{3}$

and the slope of $SQ = m_2 = \frac{-6a}{8a} = -\frac{3}{4}$.

Thus, $m_1 \times m_2 = \frac{4}{3} \times -\frac{4}{3} = -1$

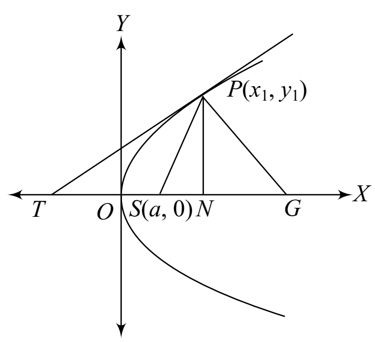
Hence, the result.

74. Let the latus rectum be LSL', where $L = (a, 2a)$ and $L' = (a, -2a)$.



Normal at $L(a, 2a)$ is $x + y = 3a$ …(i)
Normal at $L'(a, -2a)$ is $x - y = 3a$ …(ii)
Clearly, (ii) is perpendicular to (i).
Solving, we get
$x = 3a$ and $y = 0$
Hence, the point of intersection is $(3a, 0)$.

75.



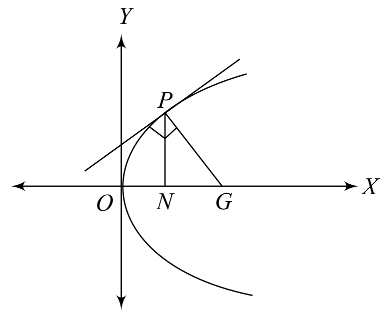
Let $P(x_1, y_1)$ be any point on the parabola $y^2 = 4ax$.
The equation of any tangent and any normal at $P(x_1, y_1)$ are

$$yy_1 = 2a(x + x_1) \text{ and } \frac{y - y_1}{y_1} = -\frac{x - x_1}{2a}$$

Since the tangent and the normal meet its axis at T and G, respectively, so the co-ordinates of T and G are $(-x_1, 0)$ and $(x_1 + 2a, 0)$, respectively.

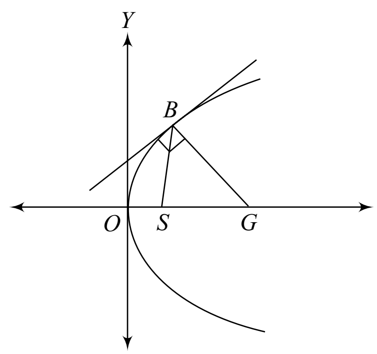
Thus, $SP = PM = x_1 + a$,
$SG = AG - AS = x_1 + 2x - a = x_1 + a$.
and $ST = AS + AT = a + x_1$.
Hence, $SP = SG = ST$.

76. The normal at $P(t)$ is $y = -tx + 2at + at^3$.



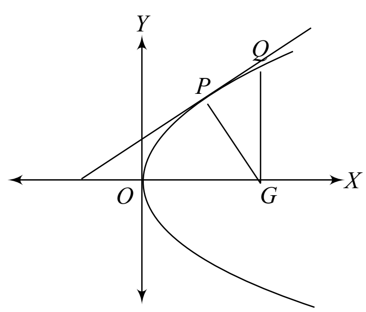
It meets the x-axis at G.
Thus the co-ordinates of G be $(2a + at^2, 0)$.
Also N is $(at^2, 0)$.
Thus $NG = 2a + at^2 - at^2 = 2a$ = semi-latus rectum.

77. The normal at $P(t)$ is $y = -tx + 2at + at^3$



Thus, S is $(a, 0)$, G is $(2a + at^2, 0)$ and P is $(at^2, 2at)$.
Now, $SP = a + x = a + at^2 = a(1^2 + t^2)$
$SG = 2a + at^2 - a = a + at^2 = a(1 + t^2) = SP$
Thus, P and G are equidistant from the focus.

78.



The normal at $P(t)$ is $y = -tx + 2at + at^3$
Thus G is $(2a + at^2, 0)$ and P is $(at^2, 2at)$.
Now, $PG^2 = 4a^2 + 4a^2t^2$ …(i)
Q is a point on the parabola such that QG is perpendicular to axis so that its ordinate is QG and abscissa is the same as of G.
Hence, the point Q is $(2a + at^2, QG)$.

But Q lies on the parabola $y^2 = 4ax$.

Now,

$$QG^2 = 4a(2a + at^2)$$
$$= 8a^2 + 4a^2t^2$$
$$= (4a^2 + 4a^2t^2) + 4a^2$$
$$= PG^2 + 4a^2$$

$\Rightarrow$ $QG^2 - PG^2 = 4a^2 =$ constant.

Hence, the result.

79. The equation of the chord of contact of the tangents from the point (2, 3) to the parabola $y^2 = 4x$ is

$$yy_1 = 2(x + x_1)$$
$\Rightarrow$ $3y = 2(x + 2)$
$\Rightarrow$ $2x - 3y + 4 = 0$

80. The equation of the chord of contact of the tangents to the parabola $y^2 = 12x$ drawn through the point (–1, 2) is

$$2y = 6(x - 1)$$
$\Rightarrow$ $y = 3x - 3$

81. Let the point of intersection of the tangents be $R(\alpha, \beta)$. The equation of the tangent to the parabola at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ are

Thus, the slopes of the tangents are

$$m_1 = \frac{1}{t_1} \text{ and } m_2 = \frac{1}{t_2}$$

Then the point of intersection of the two tangents be $[at_1t_2, a(t_1 + t_2)]$.

Therefore $\alpha = at_1t_2$ and $\beta = a(t_1 + t_2)$.

Let θ be the angle between the two tangents. Then

$$\tan(\theta) = \left|\frac{\frac{1}{t_1} - \frac{1}{t_2}}{1 + \frac{1}{t_1t_2}}\right| = \left|\frac{t_2 - t_1}{t_1t_2 + 1}\right|$$

$\Rightarrow$ $(1 + t_1t_2)\tan(\theta) = \left(\sqrt{(t_1 + t_2)^2 - 4t_1t_2}\right)$

$\Rightarrow$ $(1 + t_1t_2)^2 \tan^2\theta = (t_1 + t_2)^2 - 4t_1t_2$

$\Rightarrow$ $\left(1 + \frac{\alpha}{a}\right)^2 \tan^2\theta = \frac{\beta^2}{a^2} - \frac{4\alpha}{a} = \frac{\beta^2 - 4a\alpha}{a^2}$

$\Rightarrow$ $(\alpha + a)^2 \tan^2\theta = (\beta^2 - 4a\alpha)$

Hence, the locus of (α, β) is

$$(y^2 - 4ax) = (x + a)^2 \tan^2\theta$$

82. The equation of the parabola is $y^2 = 4ax$ and the point (h, k) be P.

Let the tangents from P touch the parabola at $Q(at_1^2, 2at_1)$ and $R(at_2^2, 2at_2)$, then P is the point of intersection of the tangents.

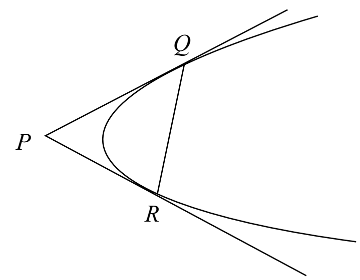
Therefore, $h = at_1t_2$ and $k = a(t_1 + t_2)$

$\Rightarrow$ $t_1t_2 = \frac{h}{a}$ and $(t_1 + t_2) = \frac{k}{a}$

Now,

$$QR = \sqrt{(at_1^2 - at_2^2)^2 + (2at_1 - 2at_2)^2}$$
$$= |a(t_1 - t_2)|\sqrt{(t_1 + t_2)^2 - 4}$$
$$= |a|\sqrt{\left(\frac{k^2}{a^2} - \frac{4h}{a}\right)}\sqrt{\left(\frac{k^2}{a^2} + 4\right)}$$
$$= \frac{1}{|a|} \times \sqrt{(k^2 - 4ah)(k^2 + 4a^2)}$$

83. Let tangents are drawn from $P(h, k)$ to the parabola $y^2 = 4ax$, intersects the parabola at Q and R.



Then the chord of contact of the tangents to the given parabola is QR.

Then QR is

$$yk = 2a(x + h)$$
$\Rightarrow$ $2ax - yk + 2ah = 0$

Therefore $PM =$ the length of perpendicular from $P(h, k)$ to QR

$$= \left|\frac{2ah - k^2 + 2ah}{\sqrt{k^2 + 4a^2}}\right|$$
$$= \left|-\frac{k^2 - 4ah}{\sqrt{k^2 + 4a^2}}\right|$$
$$= \left|\frac{k^2 - 4ah}{\sqrt{k^2 + 4a^2}}\right|$$

Thus, the area (ΔPQR)

$$= \frac{1}{2} \cdot QR \cdot PM$$
$$= \frac{1}{2} \times \frac{1}{|a|}\sqrt{(k^2 - 4ah)(k^2 + 4a^2)} \times \frac{(k^2 - 4ah)}{\sqrt{k^2 + 4a^2}}$$
$$= \frac{(k^2 - 4ah)^{3/2}}{2a} \text{ if } a > 0$$

84. The equation of the chord of the parabola $y^2 = 4x$, which is bisected at (2, 3) is

$$T = S_1$$
$\Rightarrow$ $yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$
$\Rightarrow$ $3y - 4(x + 2) = 9 - 8.2 = 7.$
$\Rightarrow$ $3y - 4x - 1 = 0$
$\Rightarrow$ $4x - 3y + 1 = 0$

85. Let the equation of the parabola be $y^2 = 4ax$.

The equation of the chord of the parabola, whose mid-point (x_1, y_1) is

$T = S_{1.}$

$\Rightarrow \quad yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$

If it is a focal chord, then it will pass through the focus $(a, 0)$ of the parabola.

Therefore,

$0. y_1 - 2a(x + x_1) = y_1^2 - 4ax_1$

$\Rightarrow \quad y_1^2 = -2a^2 - 2ax_1 + 4ax_1 = 2a(x_1 - a)$

Hence, the locus of (x_1, y_1) is $y^2 = 2a(x - a)$.

86. Let the equation of the parabola be $y^2 = 4ax$.

Let VP be any chord of the parabola through the vertex and $M(h, k)$ be the mid-pont of it.

Then, the co-ordinates of P becomes $(2h, 2k)$.

Since P lies on the parabola, so

$(2k)^2 = 4a \cdot (2h)$

$\Rightarrow \quad 4k^2 = 8ah$

$\Rightarrow \quad k^2 = 2ah$

Hence, the locus of (h, k) is $y^2 = 2ax$.

87. Equation of the normal at any point $(at^2, 2at)$ of the parabola $y^2 = 4ax$ is

$y = -tx + 2at + at^3$ …(i)

Lep PQ be the normal, whose mid-point is $M(\alpha, \beta)$.

Therefore,

$T = S_1$

$\Rightarrow \quad y\beta - 2a(x + \alpha) = \beta^2 - 3a\alpha$

$\Rightarrow \quad y\beta = 2a(x + \alpha) = (\beta^2 - 4a\alpha)$ …(ii)

Equations (i) and (ii) are identical.

Therefore, $\dfrac{1}{\beta} = \dfrac{t}{-2a} = \dfrac{2at + at^3}{\beta^2 - 2a\alpha}$

$\Rightarrow \quad t = -\dfrac{2a}{\beta}$ and $\dfrac{t}{-2a} = \dfrac{2at + at^3}{\beta^2 - 2a\alpha}$

From the above two relations, eliminating t, we get

$$\frac{\left(-\dfrac{2a}{\beta}\right)}{-2a} = \frac{2a\left(\dfrac{-2a}{\beta}\right) + a\left(\dfrac{-2a}{\beta}\right)^3}{\beta^2 - 2a\alpha}$$

$$\Rightarrow \quad (\beta^2 - 2a\alpha) = -2a\left(2a + a\left(\frac{-2a}{\beta}\right)^2\right)$$

$\Rightarrow \quad \beta^2(\beta^2 - 2a\alpha) = -2a(2a\beta^2 + 4a^3)$

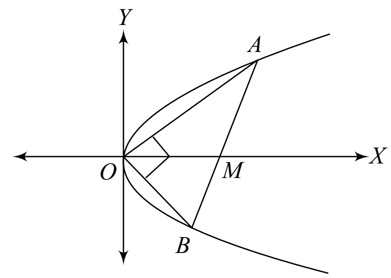
$\Rightarrow \quad \beta^2(\beta^2 - 2a\alpha) = -4a^2\beta^2 - 8a^4)$

$\Rightarrow \quad \beta^4 - 2a(\alpha - 2a)\beta^2 + 8a^4 = 0$

Hence the locus of $M(\alpha, \beta)$ is

$y^4 - 2a(x - 2a)y^2 + 8a^4 = 0$

88. Let QR be the chord and $M(\alpha, \beta)$ be the mid-point of it.



Then the equation of the chord of a parabola

$y^2 = 4ax$ at $M(\alpha, \beta)$ is

$T = S_1$

$\Rightarrow \quad y\beta - 2a\ (x + \alpha) = \beta^2 - 2a\alpha$

$\Rightarrow \quad y\beta - 2ax = \beta^2 - 2a\alpha$ …(i)

Let $V(0, 0)$ be the vertex of the parabola.

The combined equation of VQ and VR, making homogeneous by means of (i), we have

$$y^2 = 4ax \times \left(\frac{y\beta - 2ax}{\beta^2 - 2a\alpha}\right)$$

$\Rightarrow \quad y^2(\beta^2 - 2a\alpha) - 4a\beta xy + 8a^2x^2 = 0$

Since, the chord QR subtends a right angle at the vertex, so we have

co-oefficient of x^2 +co-efficient of $y^2 = 0$

$\Rightarrow \quad (\beta^2 - 2a\alpha) + 8a^2 = 0$

$\Rightarrow \quad \beta^2 = 2a(a - 4a)$

Hence, the locus of $M(\alpha, \beta)$ is

$y^2 = 2a(x - 4a)$

89. Let QR be the chord of the parabola and $M(\alpha, \beta)$ be its mid-point. Then the equation of the chord QR bisected at $M(\beta, \beta)$ is

$T = S_1$

$\Rightarrow \quad y\beta - 2a(x + a) = \beta^2 - 4a\alpha$

$\Rightarrow \quad y\beta = 2a(x + a) + (\beta^2 - 4a\alpha)$

$\Rightarrow \quad y\beta = 2ax + (\beta^2 - 2a\alpha)$ …(i)

If the Eq. (i) be a tangent to the parabola $y^2 = 4bx$, then

$c = \dfrac{b}{m}$

$$\Rightarrow \quad \left(\frac{\beta^2 - 2a\alpha}{\beta}\right) = \frac{b}{(2a/\beta)} = \frac{b\beta}{2a}$$

$\Rightarrow \quad (b - 2a)\beta^2 + 4a^2\alpha = 0$

Hence, the locus of $M(\alpha, \beta)$ is

$(2a - b)y^2 = 4a^2x$

90. Let the mid-point of the chord be $M(h, k)$.

Then the equation of the chord at $M(h, k)$ is

$T = S_1$

$\Rightarrow \quad yk - 2a(x + h) = k^2 - 4ah$

which passes through the point $(3b, b)$.

Then, $bk - 2a(3b + h) = k^2 - 4ah$.

$\Rightarrow \quad k^2 - 2ah - bk + 6ab = 0$

Hence, the locus of $M(h, k)$ is

$$y^2 - 2ax - by + 6ab = 0$$

91. The equation of the given parabola is

$$y^2 = 4ax \quad \ldots\text{(i)}$$

The equation of the tangent at $P\,(at^2, 2at)$ is

$$yt = x + at^2 \quad \ldots\text{(ii)}$$

The equation of the directrix of the parabola

$$y^2 = 4ax \text{ is } x + a = 0 \quad \ldots\text{(iii)}$$

Solving Eqs (ii) and (iii), we get

$$x = -a \text{ and } y = \frac{a(t^2-1)}{t}$$

Thus, the point on the directrix, say Q, whose co-ordinates are $\left(-a, \frac{a(t^2-1)}{t}\right)$.

Let $M(h, k)$ be the mid-point of P and Q. Then

$$h = \frac{at^2 - a}{2} \text{ and } k = \frac{a(t^2-1)}{2t} + \frac{2at}{2}$$

$$\Rightarrow \quad t^2 = \frac{2h+a}{a} \text{ and } 4k^2t^2 = a^2(3t^2-1)^2$$

Eliminating t, we get

$$4k^2\left(\frac{2h+a}{a}\right) = a^2\left(3\left(\frac{2h+a}{a}\right) - 1\right)^2$$

$$\Rightarrow \quad 4k^2(2h+a) = a(6h+3a-a)^2$$

$$\Rightarrow \quad k^2(2h+a) = a(3h+a)^2$$

Hence, the locus of $M(h, k)$ is

$$y^2(2x+a) = a(3x+a)^2$$

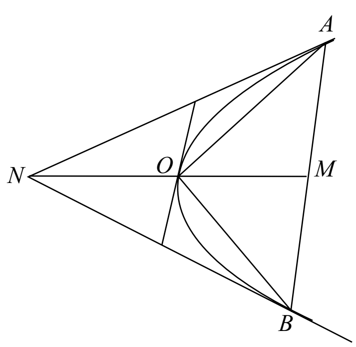
92. Let $y = mx + c$ represents the system of parallel chords. The equation of the diameter to the parabola $y^2 = 4ax$ is $y = \frac{2a}{m}$.

The diameter meets the parabola $y^2 = 4ax$ at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

The equation of the tangent to the parabola $y^2 = 4ax$ at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ is $y = mx + \frac{a}{m}$.

which is parallel to $y = mx + c$.

93.



Let AB be the chord, where $A = (at_1^2, 2at_1)$ and $B = (at_2^2, 2at_2)$.

Now,

Slope of AB

$$= m(AB) = \frac{2a(t_2 - t_1)}{a(t_2^2 - t_1^2)} = \frac{2}{t_1 + t_2}$$

The equation of the diameter is

$$y = \frac{2a}{m} \quad \Rightarrow \quad y = a(t_1 + t_2) \quad \ldots\text{(i)}$$

Now the tangents at $A = (at_1^2, 2at_1)$ and $B = (at_2^2, 2at_2)$ meet at $N[at_1t_2, a(t_1 + t_2)]$.

Thus, N lies on $y = a(t_1 + t_2)$.

94. As we know that all rays of light parallel to x-axis of the parabola are reflected through the focus of the parabola.

The equation of the given parabola is

$$(y-4)^2 = 8(x+1)$$

$$\Rightarrow \quad Y^2 = 8X,$$

where $Y = y - 4$ and $X = x + 1$

Now the focus of the parabola is $(a, 0)$.

Therefore,

$$X = a \text{ and } Y = 0$$

$$\Rightarrow \quad x + 1 = 2 \text{ and } y - 4 = 0$$

$$\Rightarrow \quad x = 1 \text{ and } y = 4$$

Hence, the focus is $(1, 4)$.

Thus $\alpha = 1$ and $\beta = 4$

Now, $\alpha + \beta + 10 = 1 + 4 + 10 = 15$.

95. Let the line $y = x + 2$ intersects the parabola at P.

Solving the line $y = x + 2$ and the parabola $y^2 = 4(x + 2)$, we get

the point P is $(2, 4)$.

Now the equation of the tangent to the parabola

$$y^2 = 4(x+2) \text{ at } P(2, 4) \text{ is}$$

$$yy_1 = 2(x + x_1) + 8$$

$$\Rightarrow \quad 4y = 2(x+2) + 8$$

$$\Rightarrow \quad x - 2y + 6 = 0$$

Let IP be the incident ray, PM be the reflected ray and PN be the normal

As we know that the normal is equally inclined with the incident ray as well as the reflected ray.

Now the slope of $IP = 1$, slope of normal $PN = -2$ and let the slope of the reflected ray $= m$. Then

$$\frac{1+2}{1-2} = \frac{-2-m}{1-2m}$$

$$\Rightarrow \quad m = \frac{1}{7}$$

Hence, the equation of the reflected ray is

$$y - 2 = \frac{1}{7}(x - 4)$$

$$\Rightarrow \quad 7y - 14 = x - 4$$

$$\Rightarrow \quad x - 7y + 10 = 0$$

Level III

1. Clearly, focus $= \left(\dfrac{3-5}{2}, \dfrac{6+6}{2}\right) = (-1, 6)$

2. Given $y = \dfrac{a^2x^2}{3} + \dfrac{a^2x}{2} - 2a$

$\Rightarrow \quad 6y = 2a^2x^2 + 3a^2x - 12a$

$\Rightarrow \quad 2a^2\left(x^2 + \dfrac{3}{2}x\right) = 6y + 12a = 6(y + 2a)$

$\Rightarrow \quad \left(x^2 + \dfrac{3}{2}x\right) = \dfrac{3}{a^2}(y + 2a)$

$\Rightarrow \quad \left(x + \dfrac{3}{4}\right)^2 = \dfrac{3}{a^2}(y + 2a) + \dfrac{9}{16}$

$\Rightarrow \quad \left(x + \dfrac{3}{4}\right)^2 = \dfrac{3}{a^2}\left(y + 2a + \dfrac{3a^2}{16}\right)$

$\Rightarrow \quad X^2 = \dfrac{3}{a^2}Y,$

where $X = x + \dfrac{3}{4}, Y = y + 2a + \dfrac{3a^2}{16}$

Let focus $= (h, k)$

$X = 0$ and $Y = \dfrac{3}{4a^2}$

$\Rightarrow \quad x + \dfrac{3}{4} = 0$ and $y + 2a + \dfrac{3a^2}{16} = \dfrac{3}{4a^2}$

$\Rightarrow \quad h = -\dfrac{3}{4}, k + 2a + \dfrac{3a^2}{16} = \dfrac{3}{4a^2}$

$\Rightarrow \quad k + 2a + \dfrac{-4ha^2}{16} = \dfrac{-4h}{4a^2}$

$\Rightarrow \quad 16ka^2 + 32a^3 - 4ha^4 + 16h = 0$

$\Rightarrow \quad 4ka^2 + 8a^3 - ha^4 + 4h = 0$

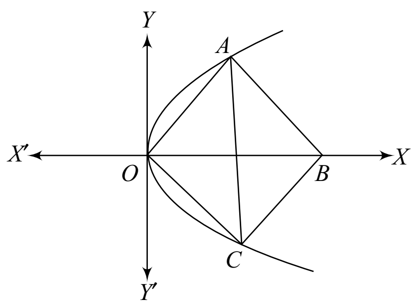
Hence, the locus of (h, k) is

$4ya^2 + 8a^3 - xa^4 + 4x = 0$

3. Clearly, the equation of the parabola is

$y^2 = 4(a_1 - a)(x - a)$

4. Let $A = (at^2, 2at)$, $C = (at^2, -2at)$



and $B = (2at^2, 0)$

Now, $m(OA) \times m(OC) = -1$

$\Rightarrow \quad \dfrac{2at}{at^2} \times \dfrac{-2at}{at^2} = -1$

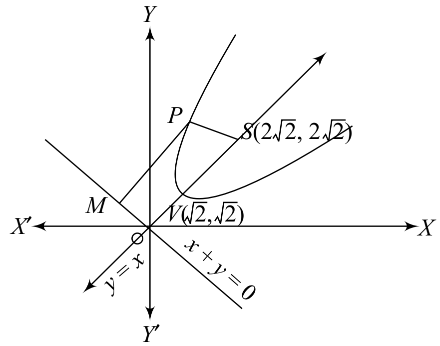
$\Rightarrow \quad \dfrac{2t}{t^2} \times \dfrac{-2t}{t^2} = -1$

$\Rightarrow \quad t^2 = 4$

$\Rightarrow \quad t = 2$

Hence, the co-ordinates of the vertices are $O = (0, 0)$; $A = (4a, 4a)$; $B = (8a, 0)$, and $C = (4a, -4a)$.

5. Let $P = (x, y)$.



We have,

$SP = PM$

$\Rightarrow \quad SP^2 = PM^2$

$\Rightarrow \quad (x - 2\sqrt{2})^2 + (y - 2\sqrt{2})^2 = \left(\dfrac{x + y}{\sqrt{2}}\right)^2$

$\Rightarrow \quad 2[(x - 2\sqrt{2})^2 + (y - 2\sqrt{2})^2] = (x + y)^2$

$\Rightarrow \quad (x - y)^2 = 8\sqrt{2}(x + y - 2\sqrt{2})$

6. Clearly, the point of intersection is $(4a, 4a)$ which satisfies the straight line

$2bx + 3cy + 4d = 0$

$\Rightarrow \quad 2b(4a) + 3c(4a) + 4d = 0$

$\Rightarrow \quad 2ab + 3ac + d = 0$

$\Rightarrow \quad a(2b + 3c) + d = 0$

$\Rightarrow \quad a(2b + 3c) = -d$

$\Rightarrow \quad a^2(2b + 3c)^2 = d^2$

$\Rightarrow \quad (2b + 3c)^2 = \left(\dfrac{d}{a}\right)^2$

$\Rightarrow \quad d^2 = a^2(2b + 3c)^2$

7. Any tangent to the parabola $y^2 = 4a(x + a)$ is

$$y = m_1(x + a) + \frac{a}{m_1} \quad \ldots\text{(i)}$$

and to the parabola $y^2 = 4b(x + b)$ is

$$y = m_2(x + b) + \frac{b}{m_2} \quad \ldots\text{(ii)}$$

Since two tangents are perpendicular, so

$$m_1 m_2 = -1 \quad \ldots\text{(iii)}$$

Thus, $y = m_2(x+b) + \frac{b}{m_2}$

$$y = -\frac{1}{m_1}(x+b) - bm_1 \quad \ldots(iv)$$

Eliminating m_1 from Eqs (i) and (iv), we get

$$m_1(x+a) + \frac{a}{m_1} = -\frac{1}{m_1}(x+b) - bm_1$$

$\Rightarrow \quad m_1(x+a+b) = -\frac{1}{m_1}(x+a+b)$

$\Rightarrow \quad (x+a+b)\left(m_1 + \frac{1}{m_1}\right) = 0$

$\Rightarrow \quad (x+a+b) = 0 \; (\because m_1 \neq 0)$

$\Rightarrow \quad (x+a+b) = 0$

Hence, the result.

8. Let the focal chord be $y = mx + c$ which is passing through the focus. So

$0 = 4m + c$

$\Rightarrow \quad c = -4m$

Thus, the focal chord is $y = mx - 4m$

$mx - y - 4m = 0 \quad \ldots(i)$

(i) is tangent to the circle $(x-6)^2 + y^2 = 2$, so

$$\left|\frac{6m - 4m}{\sqrt{m^2+1}}\right| = \sqrt{2}$$

$\Rightarrow \quad \left|\frac{2m}{\sqrt{m^2+1}}\right| = \sqrt{2}$

$\Rightarrow \quad 4m^2 = 2(m^2+1)$

$\Rightarrow \quad 2m^2 = 2$

$\Rightarrow \quad m^2 = 1$

$\Rightarrow \quad m = \pm 1$

9. The co-ordinates of the latus rectum are

$L = (a, 2a) = (1, 2)$ and $L' = (a, -2a) = (1, -2)$

The equation of the tangent at L is

$yy_1 = 2(x + x_1)$

$\Rightarrow \quad 2y = 2(x+1)$

$\Rightarrow \quad y = (x+1) \quad \ldots(i)$

The equation of the tangent at L is

$-2y = 2(x+1)$

$\Rightarrow \quad y = -(x+1) \quad \ldots(ii)$

Solving Eqs (i) and (ii), we get

$x = -1, y = 0$

Hence, the point of intersection is $(-1, 0)$.

10. The equation of the tangent at P to the parabola

$y^2 = 8x$ is

$4y = 4(x+2)$

$\Rightarrow \quad y = (x+2) \quad \ldots(i)$

Given parabola is $y^2 = 8x + 5 \quad \ldots(ii)$

Solving Eqs (i) and (ii), we get

$(x+2)^2 = 8x + 5$

$\Rightarrow \quad x^2 + 4x + 4 = 8x + 5$

$\Rightarrow \quad x^2 - 4x - 1 = 0$

$\Rightarrow \quad (x-2)^2 = (\sqrt{5})^2$

$\Rightarrow \quad x = 2 \pm \sqrt{5}$

Thus, $y = 4 \pm \sqrt{5}$

Therefore, $Q = (2+\sqrt{5}, 4+\sqrt{5})$

and $\quad R = (2-\sqrt{5}, 4-\sqrt{5})$

Thus, the mid-point of Q and R is $(2, 4)$.

11. Clearly, both the lines pass through $(-a, -b)$ which a point lying on the directrix of the parabola.

Thus, $m_1 m_2 = -1$, since tangents drawn from any point on the directrix are always mutually perpendicular.

12. The equation of any tangent to the parabola $y^2 = 4x$ is

$$y = mx + \frac{1}{m}$$

$\Rightarrow \quad my = m^2x + 1$

$\Rightarrow \quad m^2x - my + 1 = 0 \quad \ldots(i)$

Also (i) is a tangent to the circle $(x-3)^2 + y^2 = 9$

Thus, the length of the perpendicular from the centre to the tangent is equal to the radius of the circle.

So, $\left|\frac{3m^2+1}{\sqrt{m^4+m^2}}\right| = 3$

$\Rightarrow \quad (3m^2+1)^2 = 9(m^4+m^2)$

$\Rightarrow \quad 9m^4 + 6m^2 + 1 = 9(m^4+m^2)$

$\Rightarrow \quad 3m^2 = 1$

$\Rightarrow \quad m = \pm\frac{1}{\sqrt{3}}$

Hence, the equation of the common tangent which lies above x-axis is $y = \frac{x}{\sqrt{3}} + \sqrt{3}$

$\Rightarrow \quad x - \sqrt{3}y + 3 = 0$

13. The equation of any tangent to the parabola $y^2 = 8x$ is

$$y = mx + \frac{2}{m} \quad \ldots(i)$$

Since (i) is also a tangent of $xy = -1$, so

$$x\left(mx + \frac{2}{m}\right) = -1$$

$\Rightarrow \quad mx^2 + \frac{2x}{m} + 1 = 0$

$\Rightarrow \quad m^2x^2 + 2x + m = 0$

Since it will provide us equal roots, so

$D = 0$

$\Rightarrow \quad 4 - 4m^3 = 0$

$\Rightarrow \quad m^3 = 1$

$\Rightarrow \quad m = 1$

Hence, the equation of the common tangent is $y = x + 2$.

14. Given parabola is $y = x^2$.

So, $4a = 1$

$a = 1/4$

The equation of any tangent is

$$x = my + \frac{a}{m} = my + \frac{1}{4m}$$

which is also a tangent of

$y = -x^2 + 4x - 4$

$\Rightarrow \quad x = m(-x^2 + 4x - 4) + \frac{1}{4m}$

$\Rightarrow \quad 4mx = 4m^2(-x^2 + 4x - 4) + 1$

$\Rightarrow \quad -4m^2x^2 + 4(4m^2 - m) + (1 - 16m^2) = 0$

$\Rightarrow \quad 4m^2x^2 - 4(4m^2 - m)x - (1 - 16m^2) = 0$

Since it will provide us equal roots, so

$D = 0$

$\Rightarrow \quad 16(4m^2 - m)^2 + 16m^2(1 - 16m^2) = 0$

$\Rightarrow \quad (4m^2 - m)^2 + m^2(1 - 16m^2) = 0$

$\Rightarrow \quad 16m^4 - 8m^3 + m^2 + m^2 - 16m^4 = 0$

$\Rightarrow \quad -8m^3 + 2m^2 = 0$

$\Rightarrow \quad 4m^3 - m^2 = 0$

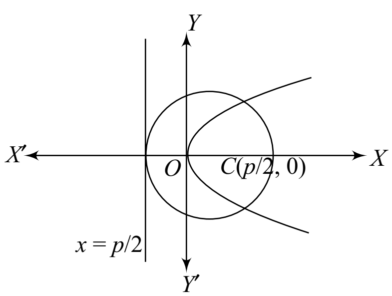
$\Rightarrow \quad m = 0, \frac{1}{4}$

Hence, the equation of the common tangents are

$y = 0$ and $y = 4(x - 1)$

15. Given parabola is $y^2 = 2px$.

So, the focus is $\left(\frac{p}{2}, 0\right)$.



Clearly, the equation of the circle is

$$\left(x - \frac{p}{2}\right)^2 + y^2 = p^2$$

$\Rightarrow \quad \left(x - \frac{p}{2}\right)^2 + 2px = p^2$

$\Rightarrow \quad x^2 - px + \frac{p^2}{4} + 2px = p^2$

$\Rightarrow \quad x^2 + px + \frac{p^2}{4} = p^2$

$\Rightarrow \quad \left(x + \frac{p}{2}\right)^2 = p^2$

$\Rightarrow \quad \left(x + \frac{p}{2}\right) = \pm p$

$\Rightarrow \quad x = -\frac{p}{2} \pm p = \frac{p}{2}, -\frac{3p}{2}$

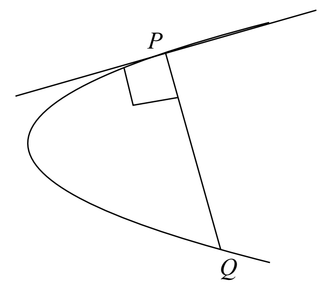
when $x = \frac{p}{2}$, then, $y = \pm p$

Thus the point of intersection are

$$\left(\frac{p}{2}, p\right) \text{ and } \left(\frac{p}{2}, -p\right)$$

16. Let the parabola be $y^2 = 4ax$ and the two points on the parabola are

$P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$



The equation of the normal to the parabola at

$P(at_1^2, 2at_1)$ is

$y = -t_1x + 2at_1 + at_1^3$

which meets the parabola again at $Q(at_2^2, 2at_2)$)

Thus, $2at_2 = -at_1t_2^2 + 2at_1 + at_1^3$

$\Rightarrow \quad 2a(t_2 - t_1) + at_1(t_2^2 - t_1^2) = 0$

$\Rightarrow \quad (t_2 - t_1)[2a + at_1(t_2 + t_1)] = 0$

$\Rightarrow \quad [2 + t_1(t_2 + t_1)] = 0$

$\Rightarrow \quad t_1^2 + t_1t_2 + 2 = 0$

Hence, the result.

17. Given parabola is $y^2 = x$.

So, $4a = 1$

$\Rightarrow \quad a = \frac{1}{4}$

The equation of any normal to the parabola

$$y = mx - \frac{1}{2}m - \frac{1}{4}m^3$$

which is passing through $(c, 0)$. So

$$mc - \frac{1}{2}m - \frac{1}{4}m^3 = 0$$

$\Rightarrow \quad 4mc - 2m - m^3 = 0$

$\Rightarrow \quad m^3 + 2(1 - 2c) = 0$

$\Rightarrow \quad m^2 + 2(1 - 2c) = 0$

$\Rightarrow \quad (1 - 2c) = -\frac{m^2}{2}$

$\Rightarrow \quad (1 - 2c) = -\frac{m^2}{2} < 0$

$\Rightarrow \quad c > \frac{1}{2}$

18. The equation of any tangent to the parabola $y^2 = 8x$ is $y = mx + \dfrac{2}{m}$

Clearly,

$$\tan(\pm 45°) = \frac{m-3}{1+3m}$$

$\Rightarrow \quad \dfrac{m-3}{1+3m} = \pm 1$

$\Rightarrow \quad \dfrac{m-3}{1+3m} = 1, \dfrac{m-3}{1+3m} = -1$

$\Rightarrow \quad m = -2, \dfrac{1}{2}$

Hence, the equations of tangents are

$$y = -2x - 1, y = \frac{x}{2} + 4$$

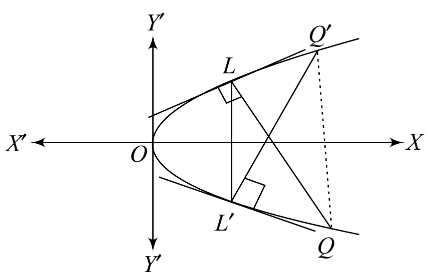
$\Rightarrow \quad y = -2x - 1, x - 2y + 8 = 0$

Solving $y^2 = 8x$ and $y = -2x - 1$ we get, the point of intersection is $\left(\dfrac{1}{2}, -2\right)$.

Again, solving $y^2 = 8x$ and $x - 2y + 8 = 0$ we get the point of intersection is (8, 8).

Thus, the point of contacts are $\left(\dfrac{1}{2}, -2\right)$ and (8, 8).

19. Clearly, the ends of a latus rectum are $L(a, 2a)$ and $L' = (a, -2a)$.



The equation of the normal at L is

$$y = mx - 2am - am^3$$

putting $m = -1$, we get

$$y = -x + 2a + a$$

$\Rightarrow \quad x + y = 3a$

The equation of the normal at L is

$$y = mx - 2am - am^3$$

putting $m = 1$, we get

$$y = x - 2a - a$$

$\Rightarrow \quad x - y = 3a$

On solving $x + y = 3a$ and $y^2 = 4ax$, we get $Q = (9a, -6a)$

Again, solving $x - y = 3a$ and $y^2 = 4ax$, we get $Q' = (9a, -6a)$

Thus, the length of $QQ' = \sqrt{(9a - 9a)^2 + (6a + 6a)^2}$

$= 12a$

Hence, the result.

20. Prove that from any point $P(at^2, 2at)$ on the parabola $y^2 = 4ax$, two normals can be drawn and their feet Q and R have the parameters satisfying the equation $\lambda^2 + \lambda t + 2 = 0$.

21. Given parabola is $x^2 = 8y$.

We have $4a = 8$

$\Rightarrow \quad a = 2$

The equation of the normal to the parabola $x^2 = 8y$ at $(-2am_1, am_1^2), (-2am_2, am_2^2)$ are

$$x = m_1 y - 2am_1 - am_1^3 \quad \text{...(ii)}$$

and $\quad x = m_2 y - 2am_2 - am_2^3 \quad$ ...(i)

Let (h, k) be the point of intersection.

Thus, $h = -a(m_1 + m_2) \quad$ ...(iii)

and $\quad k = 2a + a(m_1^2 + m_2^2 - 1) \quad$ ...(iv)

From Eqs (iii) and (iv), we get

$$\frac{x^2}{a^2} = \frac{y-2a}{a} - 1 = \frac{y - 2a - a}{a} = \frac{y - 3a}{a}$$

$\Rightarrow \quad x^2 = a(y - 3a)$

$\Rightarrow \quad x^2 = 2(y - 6)$

which is the required locus.

22. The equation of the tangent at $P(at_1^2, 2at_1)$ to the parabola $y^2 = 4ax$ is

$$yt_1 = x + at_1^2$$

$\Rightarrow \quad x - yt_1 + at_1^2 = 0$

$\Rightarrow \quad at_1^2 - yt_1 + x = 0 \quad$ ...(i)

Also, the equation of the tangent at $Q(2bt_2, bt_2^2)$ to the parabola $x^2 = 4by$ is

$$xt_2 = y + bt_2^2 \quad \text{...(ii)}$$

It is given that the tangents (i) and (ii) are perpendicular, so

$$\left(\frac{1}{t_1}\right) \cdot t_2 = -1$$

$\Rightarrow \quad t_2 = -t_1$

Equation (ii) reduces to

$$-xt_1 = y + bt_1^2$$

$\Rightarrow \quad xt_1 + y + bt_1^2 = 0$

$\Rightarrow \quad bt_1^2 + xt_1 + y = 0 \quad$ ...(iii)

Solving Eqs (i) and (iii), we get

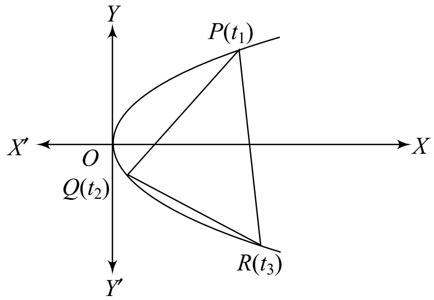
$$\frac{t_1^2}{-(x^2 + y^2)} = \frac{t_1}{bx - ay} = \frac{1}{ax + by}$$

Eliminating t_1, we get

$$(ax + by)(x^2 + y^2) + (bx - ay)^2 = 0$$

which is the required locus of the point of intersection of two tangents.

23. Let P, Q, and R be three points on the parabola.



Let the co-ordinates of the centroid be $G(\alpha, \beta)$.

Clearly, $\alpha = \dfrac{a(t_1^2 + t_2^2 + t_3^2)}{3}$

and $\beta = \dfrac{2a(t_1 + t_2 + t_3)}{3}$

Now

Slope of $PQ = \dfrac{2a(t_1 - t_2)}{a(t_1^2 - t_2^2)} = \dfrac{2}{(t_1 + t_2)}$

and the slope of $PR = \dfrac{2a(t_1 - t_3)}{a(t_1^2 - t_3^2)} = \dfrac{2}{(t_1 + t_3)}$

Clearly, $\angle QPR = 60°$

Thus, $\tan(60°) = \dfrac{\dfrac{2}{t_1 + t_2} - \dfrac{2}{t_1 + t_3}}{1 + \dfrac{2}{t_1 + t_2} \cdot \dfrac{2}{t_1 + t_3}}$

$\Rightarrow \quad \sqrt{3} = \dfrac{2(t_3 - t_2)}{(t_1 + t_2)(t_2 + t_3) + 4}$

$\sqrt{3}\,[(t_1 + t_2)(t_2 + t_3) + 4] = 2(t_2 - t_3)$ …(i)

Similarly, $\angle Q = 60°$ and $\angle R = 60°$

Thus, $\sqrt{3}\,[(t_1 + t_2)(t_1 + t_3) + 4] = 2(t_3 - t_1)$ …(ii)

and $\sqrt{3}\,[(t_1 + t_3)(t_2 + t_3) + 4] = 2(t_1 - t_2)$ …(iii)

Adding Eqs (i), (ii) and (iii), we get

$3(t_1t_2 + t_2t_3 + t_3t_1) + (t_1^2 + t_2^2 + t_3^2 + 12) = 0$

$\Rightarrow \quad 3(t_1t_2 + t_2t_3 + t_3t_1) + \dfrac{3\alpha}{a} + 12 = 0$

$\Rightarrow \quad (t_1t_2 + t_2t_3 + t_3t_1) + \dfrac{\alpha}{a} + 4 = 0$

$\Rightarrow \quad a(t_1t_2 + t_2t_3 + t_3t_1) + \alpha + 4a = 0$

$\Rightarrow \quad 2a(t_1t_2 + t_2t_3 + t_3t_1) + 2\alpha + 8a = 0$

$\Rightarrow \quad a\{(t_1 + t_2 + t_3)^2 - (t_1^2 + t_2^2 + t_3^2)\} + 2\alpha + 8a = 0$

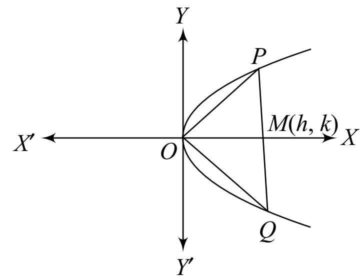
$\Rightarrow \quad a\left\{\dfrac{9\beta^2}{4a^2} - \dfrac{3\alpha}{a}\right\} + 2\alpha + 8a = 0$

$\Rightarrow \quad \dfrac{9\beta^2}{4a} - 3\alpha + 2\alpha + 8a = 0$

$\Rightarrow \quad 9\beta^2 = 4a(\alpha - 8a)$

Hence, the locus of $G(\alpha, \beta)$ is $9y^2 = 4a(x - 8a)$

24. Let $P = (at_1^2, 2at_1)$ and $Q = (at_2^2, 2at_2)$.



Slope of $PQ = \dfrac{2a(t_2 - t_1)}{a(t_2^2 - t_1^2)} = \dfrac{2}{(t_1 + t_2)}$

Equation of PQ is

$y - 2at_1 = \dfrac{2}{(t_1 + t_2)}(x - at_1^2)$

$\Rightarrow \quad 2x - (t_1 + t_2)y = -2at_1t_2$ …(i)

Now, $m_1 = m(OP) = \dfrac{2}{t_1}$

and $m_2 = m(OQ) = \dfrac{2}{t_2}$

As OP is perpendicular to OQ, so

$m_1m_2 = -1$

$\Rightarrow \quad \dfrac{4}{t_1t_2} = -1$

$\Rightarrow \quad t_1t_2 = -4$ …(ii)

From Eqs (i) and (ii), we get

$2(x - 4a) - (t_1 + t_2)\,y = 0$

Let $M(h, k)$ be the mid-point of PQ. So

$h = \dfrac{a}{2}(t_1^2 + t_2^2)$ and $k = \dfrac{a}{2}(2t_1 + 2t_2)$

$\Rightarrow \quad h = \dfrac{a}{2}(t_1^2 + t_2^2)$ and $k = a(t_1 + t_2)$

Now, $k^2 = a^2(t_1 + t_2)^2$

$\Rightarrow \quad k^2 = a^2\{(t_1^2 + t_2^2) + 2t_1t_2\}$

$\Rightarrow \quad k^2 = a^2(2h + 2(-4))$

Hence, the locus of $M(h, k)$ is

$y^2 = 2a^2(x - 4)$

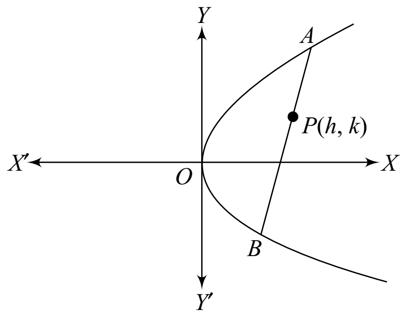
25. Let AB be a chord of a parabola, in which

$A = (t_1^2, 2t_1), B = (t_2^2, 2t_2)$

Slope of $AB = 2$

$\Rightarrow \quad \dfrac{2}{t_1 + t_2} = 2$

$\Rightarrow \quad t_1 + t_2 = 1$



Let P be a point which divides AB internally in the ratio 1 : 2

So, $h = \frac{2t_1^2 + t_2^2}{3}$ and $k = \frac{4t_1 + 2t_2}{3}$

$$3h = (2t_1^2 + t_2^2) \text{ and } 3k = (4t_1 + 2t_2)$$

Eliminating t_1 and t_2, we get

$$\left(k - \frac{8}{9}\right)^2 = \frac{4}{9}\left(h - \frac{2}{9}\right)$$

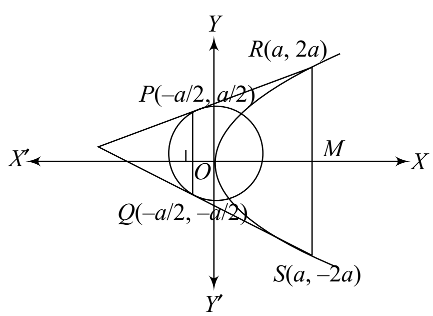
Thus, the locus of $P(h, k)$ is

$$\left(y - \frac{8}{9}\right)^2 = \frac{4}{9}\left(x - \frac{2}{9}\right)$$

Hence, the vertex is $\left(\frac{2}{9}, \frac{8}{9}\right)$.

26. The equation of any tangent to the parabola can be considered as

$$y = mx + \frac{a}{m}$$



i.e. $m^2x - my + a = 0$

As we know that the length of perpendicular from the centre to the tangent to the circle is equal to the radius of a circle.

Thus, $\frac{a}{\sqrt{m^4 + m^2}} = \frac{a}{\sqrt{2}}$

$\Rightarrow \quad m^4 + m^2 = 2$

$\Rightarrow \quad m^4 + m^2 - 2 = 0$

$\Rightarrow \quad (m^2 + 2)(m^2 - 1) = 0$

$\Rightarrow \quad m = \pm 1$

Hence, the equation of the tangents are

$$y = x + a,\ y = -x - a$$

Therefore, the points P, Q are $\left(-\frac{a}{2}, \frac{a}{2}\right)$, $\left(-\frac{a}{2}, -\frac{a}{2}\right)$ and R, S are $(a, 2a)$ and $(a, -2a)$ respectively.

Thus, the area of the equadrilateral $PQRS$

$$= \frac{1}{2}(PQ + RS) \times LM$$

$$= \frac{1}{2} \times (a + 4a) \times \left(\frac{a}{2} + a\right) = \frac{15a^2}{4}$$

27. Let the point P be (h, k).

The equation of any normal to the parabola $y^2 = 4x$ is $y = mx - 2m - m^3$ which is passing through P. So

$$k = mh - 2m - m^3$$

$\Rightarrow \quad m^3 + (2 - h)m + k = 0 \qquad \ldots$(i)

which is a cubic equation in m.

Let its roots are m_1, m_2, m_3.

Thus, $\quad m_1 + m_2 + m_3 = 0$

$$m_1m_2 + m_2m_3 + m_3m_1 = (2 - h)$$

$$m_1m_2m_3 = -k$$

It is given that $m_1m_2 = \alpha$, so

$$m_3 = -\frac{k}{\alpha}$$

Since m_3 is root of Eq. (i), so

$$m_3^3 + (2 - h)m_3 + k = 0$$

$\Rightarrow \quad -\frac{k^3}{\alpha^3} + (2 - h)\left(-\frac{k}{\alpha}\right) + k = 0$

$\Rightarrow \quad -\frac{k^2}{\alpha^3} + (2 - h)\left(-\frac{1}{\alpha}\right) + 1 = 0$

$\Rightarrow \quad -k^2 - (2 - h)\alpha^2 + \alpha^3 = 0$

$\Rightarrow \quad k^2 + (2 - h)\alpha^2 - \alpha^3 = 0$

Hence, the locus of P is

$$y^2 + (2 - x)\alpha^2 - \alpha^3 = 0$$

which represents a parabola.

clearly, $\alpha = 2$, since $\alpha = 2$ satisfies the given parabola $y^2 = 4x$.

28. Given parabola is

$$y^2 - 2y - 4x + 5 = 0$$

$\Rightarrow \quad (y - 1)^2 = 4x - 4 = 4(x - 1)$

$\Rightarrow \quad Y^2 = 4X$

where $X = (x - 1)$, $Y = (y - 1)$

So, the directrix is

$$X + a = 0$$

$\Rightarrow \quad (x - 1) + 1 = 0$

$\Rightarrow \quad x = 0$

Any point on the parabola is

$$P(1 + t^2, 2t + 1)$$

The equation of the tangent at P is

$$t(y - 1) = x - 1 + t^2$$

which meets the directrix $x = 0$ at

$$Q\left(0, 1+t-\frac{1}{t}\right)$$

Let the co-ordinates of R be (h, k).

Since it divides QP externally in the ratio $\frac{1}{2}:1$, so Q is the mid-point of R and P.

$\therefore \quad \frac{h+1+t^2}{2} = 0$ and $1+t-\frac{1}{t} = \frac{k+1+2t}{2}$

$\Rightarrow \quad t^2 = -(h+1)$ and $t = \frac{2}{1-k}$

Thus, $\frac{4}{(k-1)^2} + (h+1) = 0$

$\Rightarrow \quad (k-1)^2(h+1) + 4 = 0$

Hence, the locus of $R(h, k)$ is

$$(y-1)^2(x+1) + 4 = 0$$

29. Now, $f(x+1) = -\frac{(x+1)^2}{2} + x + 2$

$$= \frac{-x^2-2x-1+2x+4}{2} = \frac{-x^2+3}{2}$$

Also, $f(1-x) = \frac{-(1-x)^2}{2} + 1 - x + 1$

$$= \frac{-1+2x-x^2+4-2x}{2} = \frac{-x^2+3}{3}$$

Thus, $y = -\frac{x^2}{2} + x + 1$ is symmetric about the line $x = 1$

Also, given curve is $y = -\frac{x^2}{2} + x + 1$

$\Rightarrow \quad 2y = -x^2 + 2x + 2$

$\Rightarrow \quad x^2 - 2x = -2y + 2$

$\Rightarrow \quad (x-1)^2 = -2y + 3 = -2\left(y - \frac{3}{2}\right)$

$\Rightarrow \quad X^2 = -2Y$

$$X = (x-1), Y = \left(y - \frac{3}{2}\right)$$

Axis is $X = 0$

$\Rightarrow \quad x - 1 = 0$

$\Rightarrow \quad x = 1$

which is already proved that the given curve is symmetric about the line $x = 1$.

Hence, the result.

30. Given parabola is $y^2 - 16x - 8y = 0$

$\Rightarrow \quad y^2 - 8y = 16x$

$\Rightarrow \quad (y-4)^2 = 16(x+1)$

$\Rightarrow \quad Y^2 = 16X$

where $Y = y - 4$, $X = x + 1$

The equation of any normal to the parabola

$Y = mX - 2am - am^3$ at $(am^2, -2am)$

$y - 4 = m(x+1) - 8m - 4m^3$ at

$(am^2 - 1, 4 - 2am)$

i.e. $(4m^2 - 1, 4 - 8m)$

which is passing through (14, 7). So

$3 = 15m - 8m - 4m^3$

$\Rightarrow \quad 4m^3 - 7m + 3 = 0$

$\Rightarrow \quad 4m^3 - 4m^2 + 4m^2 - 4m - 3m + 3 = 0$

$\Rightarrow \quad 4m^2(m-1) + 4m(m-1) - 3(m-1) = 0$

$\Rightarrow \quad (m-1)(4m^2 + 4m - 3) = 0$

$\Rightarrow \quad (m-1) = 0, (4m^2 + 4m - 3) = 0$

$\Rightarrow \quad (m-1) = 0, (4m^2 + 6m - 2m - 3) = 0$

$\Rightarrow \quad (m-1) = 0, (2m-1)(2m+3) = 0$

$\Rightarrow \quad (m-1) = 0, (2m-1) = 0, (2m+3) = 0$

$\Rightarrow \quad m = 1, \frac{1}{2}, -\frac{3}{2}$

Hence, the feet of the normals are

(3, –4), (0, 0), (8, 16)

31. Let the parabola be $y^2 = 4ax$.

The equation of any tangent to the given parabola is

$$y = mx + \frac{a}{m} \quad \ldots\text{(i)}$$

Let the fixed point be (h, k).

The equation of any line passing through P and perpendicular to (i) is

$$y - k = -\frac{1}{m}(x - h) \quad \ldots\text{(ii)}$$

Eliminating m between Eqs (i) and (ii), we get

$$y = -\frac{(x-h)}{(y-k)} - \frac{a(y-k)}{(x-h)}$$

which is the required locus.

32. Let $P(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ be the extremities of the focal chord.

Thus, $t_1t_2 = -1$

Let the points of intersection of the normals at P and Q are $R(\alpha, \beta)$.

Then $\alpha = a(t_1^2 + t_2^2 + t_1t_2 + 2)$

$= a(t_1^2 + t_2^2 - 1 + 2)$

$= a(t_1^2 + t_2^2 + 1)$

$= a[(t_1 + t_2)^2 - 2t_1t_2 + 1]$

$= a[(t_1 + t_2)^2 + 3] \quad \ldots\text{(i)}$

and $b = -at_1t_2(t_1 + t_2)$

$= a(t_1 + t_2) \quad \ldots\text{(ii)}$

From Eqs (i) and (ii), we get,

$$\alpha = a\left(\frac{\beta^2}{a^2} + 3\right)$$

$$= \left(\frac{\beta^2}{a} + 3a\right)$$

$\Rightarrow \quad \beta^2 = a(\alpha - 3a)$

Hence, the locus of (α, β) is $y^2 = a(x - 3a)$

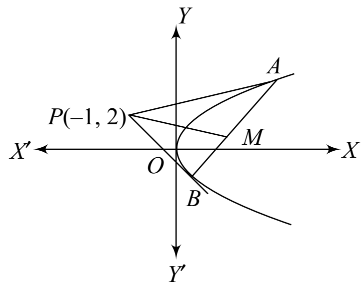
Level IV

1. The equation of the chord of contact is

$yy_1 - 2(x + x_1) = 0$

$\Rightarrow \quad 2y - 2(x - 1) = 0$

$\Rightarrow \quad y = (x - 1)$



On solving $y^2 = 4x$ and $y = x - 1$, we get, the co-ordinates of the points A and B.

Thus, $A = (3 - 2\sqrt{2}, 2 - 2\sqrt{2})$

and $\quad B = (3 + 2\sqrt{2}, 2 + 2\sqrt{2})$

Now, the length $AB = 8$

Therefore, the area of ΔPAB

$$= \frac{1}{2} \times PM \times AB$$

$$= \frac{1}{2} \times \left|\frac{-1 - 2 - 1}{\sqrt{2}}\right| \times 8$$

$$= \frac{1}{2} \times \frac{4}{\sqrt{2}} \times 8$$

$$= 8\sqrt{2} \text{ s.u.}$$

2. Given curve is $y^2 - 2x - 2y + 5 = 0$

$\Rightarrow \quad y^2 - 2y = 2x - 5$

$\Rightarrow \quad (y - 1)^2 = 2x - 5 + 1$

$\Rightarrow \quad (y - 1)^2 = 2x - 4 = 2(x - 2)$

$\Rightarrow \quad Y^2 = 2X$

where $X = x - 2$, $Y = y - 1$

which represents a parabola.

Now, the focus = $(a, 0)$

$\Rightarrow \quad X = a, Y = 0$

$$\Rightarrow \quad x - 2 = \frac{1}{2}, y - 1 = 0$$

$$\Rightarrow \quad x = \frac{5}{2}, y = 1$$

Hence, the focus is $\left(\frac{5}{2}, 1\right)$.

Also, the directrix: $X + a = 0$

$$\Rightarrow \quad x - 2 + \frac{1}{2} = 0$$

$$\Rightarrow \quad x = \frac{3}{2}$$

3.

4. The equation of the tangent to the parabola $y^2 = 4x$ at $(t^2, 2t)$ is $yy_1 = 2(x + x_1)$

$\Rightarrow \quad 2t \cdot y = 2(x + t^2)$

$\Rightarrow \quad t \cdot y = (x + t^2)$

$\Rightarrow \quad x - ty + t^2 = 0$ …(i)

and the equation of the normal to the ellipse is

$$\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$$

$$\Rightarrow \quad \frac{5x}{\sqrt{5}\cos\varphi} - \frac{4y}{2\sin\varphi} = 5 - 4 = 1$$

$$\Rightarrow \quad \frac{\sqrt{5}x}{\cos\varphi} - \frac{2y}{\sin\varphi} = 1$$

Equations (i) and (ii) are identical, so

$$\frac{1}{\frac{\sqrt{5}}{\cos\varphi}} = \frac{-t}{\frac{-2}{\sin\varphi}} = \frac{-t^2}{1}$$

$$\Rightarrow \quad \frac{\cos\varphi}{\sqrt{5}} = \frac{t\sin\varphi}{2} = -t^2$$

$$\Rightarrow \quad \cos\varphi = -\sqrt{5}t^2, \sin\varphi = -2t$$

Now, $\cos^2\varphi = 5t^4$

$\Rightarrow \quad 1 - \sin^2\varphi = 5t^4$

$\Rightarrow \quad 1 - 4t^2 = 5t^4$

$\Rightarrow \quad 5t^4 + 4t^2 - 1 = 0$

$\Rightarrow \quad (t^2 + 1)(5t^2 - 1) = 0$

$\Rightarrow \quad (5t^2 - 1) = 0$

$$\Rightarrow \quad t = \pm\frac{1}{\sqrt{5}}$$

Now, $\cos\varphi = -\sqrt{5} \times \frac{1}{5} = -\frac{1}{\sqrt{5}}$

$$\Rightarrow \quad \varphi = \cos^{-1}\left(-\frac{1}{\sqrt{5}}\right)$$

5. The equation of the normal to the parabola

$y^2 = 4ax$ is $y = mx - 2am - am^3$

which is passing through $(5a, 2a)$.

$\Rightarrow \quad 2a = 5am - 2am - am^3$

$\Rightarrow \quad 2 = 5m - 2m - m^3$

$\Rightarrow \quad m^3 - 3m + 2 = 0$

$\Rightarrow \quad m^3 - m^2 + m^2 - m - 2m + 2 = 0$

$\Rightarrow \quad m^2(m - 1) + m(m - 1) - 2(m - 1) = 0$

$\Rightarrow \quad (m - 1)(m^2 + m - 2) = 0$

$\Rightarrow \quad (m - 1) = 0, (m^2 + m - 2) = 0$

$\Rightarrow \quad (m - 1) = 0, (m - 1)(m + 2) = 0$

$\Rightarrow \quad m = 1, -2$

Hence, the equation of the normals are

$y = x - 2a - a = x - 3a$

and $\quad y = -2x + 4a + 8a = -2x + 12a$.

6. Let the point of intersection be $P(h, k)$.
The equation of any normal to the given parabola is
$$x = my - 2am - am^3$$
$\Rightarrow \quad x = my - 4m - 2m^3 \qquad (\because a = 2)$
which is passing through P.
$\Rightarrow \quad h = mk - 4m - 2m^3$
$\Rightarrow \quad 2m^3 + (4 - k)m + h = 0$
Let its roots be m_1, m_2, m_3.
$$m_1 + m_2 + m_3 = 0$$
$$m_1m_2 + m_1m_3 + m_2m_3 = \frac{4-k}{2}$$
and $\quad m_1m_2m_3 = -\frac{h}{2}$
Now, $m_1m_2m_3 = -\frac{h}{2}$
$\Rightarrow \quad m_3 = \frac{h}{2}, \qquad (\because m_1m_2 = -1)$
Also, $m_1 + m_2 = -m_3$
and $\quad m_1m_2 + m_1m_3 + m_2m_3 = \frac{4-k}{2}$
$\Rightarrow \quad -1 + (m_1 + m_2)m_3 = \frac{4-k}{2}$
$\Rightarrow \quad -1 - m_3^2 = \frac{4-k}{2}$
$\Rightarrow \quad -1 - \frac{h^2}{4} = \frac{4-k}{2}$
$\Rightarrow \quad \frac{4+h^2}{4} = \frac{k-4}{2}$
$\Rightarrow \quad 4 + h^2 = 2k - 8$
$\Rightarrow \quad h^2 = 2(k - 6)$
Hence, the locus of P is $x^2 = 2(y - 6)$.

7. Let the parabola be $y^2 = 4ax$ and two points on the parabola are $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$., where $a = 3$
It is given that
$$\frac{2at_1}{2at_2} = \frac{1}{2}$$
$\Rightarrow \quad \frac{t_1}{t_2} = \frac{1}{2}$
$\Rightarrow \quad 2t_1 = t_2$
The equation of the normal at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ are
$$y = -t_1x + 2at_1 + at_1^3 \qquad \ldots\text{(i)}$$
and
$$y = -t_2x + 2at_2 + at_2^3 \qquad \ldots\text{(ii)}$$
Solving Eqs (i) and (ii), we get
$$x = 2a + a(t_1^2 + t_2^2 + t_1t_2)$$
and $\quad y = -at_1t_2(t_1 + t_2)$
Therefore,
$$x = 2a + a(t_1^2 + 4t_1^2 + 2t_1^2) = 2a + 7at_1^2$$
$\Rightarrow \quad t_1^2 = \left(\frac{x-2a}{7a}\right)$
$\Rightarrow \quad t_1 = \left(\frac{x-2a}{7a}\right)^{1/2}$
and $\quad y = -at_1 \cdot 2t_1(t_1 + 2t_1) = -6at_1^3$
$\Rightarrow \quad y = -6a\left(\frac{x-2a}{7a}\right)^{3/2}$
$\Rightarrow \quad y = -18\left(\frac{x-6}{21}\right)^{3/2}$
$\Rightarrow \quad y + 18\left(\frac{x-6}{21}\right)^{3/2} = 0.$

8. Given circle is $x^2 + (y - 3)^2 = 5$.
The equation of any tangent to the parabola $y^2 = x$. can be considered as
$$y = mx + \frac{1}{4m}$$
$\Rightarrow \quad 4my = 4m^2x + 1$
$\Rightarrow \quad 4m^2x - 4my + 1 = 0$
Now, $OM = \sqrt{5}$
$\Rightarrow \quad \left|\frac{0 - 12m + 1}{\sqrt{16m^4 + 16m^2}}\right| = \sqrt{5}$
$\Rightarrow \quad (1 - 12m)^2 = 5(16m^4 + 16m^2)$
$\Rightarrow \quad (1 - 12m)^2 = 5(16m^4 + 16m^2)$
$\Rightarrow \quad 1 - 24m + 144m^2 = 80m^4 + 80m^2$
$\Rightarrow \quad 80m^4 - 64m^2 + 24m - 1 = 0$
$\Rightarrow \quad 80m^4 - 40m^3 + 40m^3 - 20m^2$
$\qquad -44m^2 + 22m + 2m - 1 = 0$
$\Rightarrow \quad 40m^3(2m - 1) + 20m^2(2m - 1)$
$\qquad -22m(2m - 1) + 1(2m - 1) = 0$
$\Rightarrow \quad (2m - 1)(40m^3 + 20m^2 - 22m + 1) = 0$
$\Rightarrow \quad m = \frac{1}{2}$
Hence, the equation of the common tangent be
$$y = \frac{x}{2} + \frac{1}{2}$$
$\Rightarrow \quad 2y = x + 1,$
$\Rightarrow \quad x - 2y + 1 = 0$

9. The equation of the normal to the given parabola $y^2 = 8(x - 1)$ is
$$y = m(x - 1) - 2am - am^3$$
$\Rightarrow \quad y = m(x - 1) - 4m - 2m^3$
$\Rightarrow \quad y = mx - 5m - 2m^3 \qquad \ldots\text{(i)}$
Let the co-ordinates of the point of intersection of the tangents be $P(h, k)$.

Thus, $yy_1 = 8\left(\frac{x+x_1}{2}\right) - 8$

$\Rightarrow \quad yk = 4(x+h) - 8 = 4(x+h-2)$

$\Rightarrow \quad y = \frac{4}{k}(x+h-2)$

$\Rightarrow \quad y = \frac{4}{k}x + \frac{4}{k}(h-2)$ ...(ii)

Comparing Eqs (i) and (ii), we get

$$m = \frac{4}{k}, \frac{4}{k}(h-2) = -(5m + 2m^3)$$

$\Rightarrow \quad \frac{4}{k}(h-2) = -\left(5\cdot\frac{4}{k} + 2\cdot\left(\frac{4}{k}\right)^3\right)$

$\Rightarrow \quad (h-2) = -\left(5. + 2.\left(\frac{4}{k}\right)^2\right)$

$\Rightarrow \quad (h-2)k^2 = -(5k^2 + 32)$

Hence, the locus of $P(h, k)$ is

$$(2-x)y^2 = (5y^2 + 32)$$

10. Let AB be a double ordinate, where

$$A = (at_1^2, 2at_1), B = (at_2^2, 2at_2)$$

Let $P(h, k)$ be the point of trisection. Then

$$3h = 2at^2 + at^2 \text{ and } 3k = 4at - 2at$$

$\Rightarrow \quad 3h = 3at^2$ and $3k = 2at$

$\Rightarrow \quad h = at^2$ and $3k = 2at$

Solving, we get

$$t^2 = \frac{h}{a} \text{ and } t = \frac{3k}{2a}$$

$\Rightarrow \quad \left(\frac{3k}{2a}\right)^2 = \frac{h}{a}$

$\Rightarrow \quad \frac{9k^2}{4a^2} = \frac{h}{a}$

$\Rightarrow \quad 9k^2 = 4ah$

Hence, the locus of $P(h, k)$ is

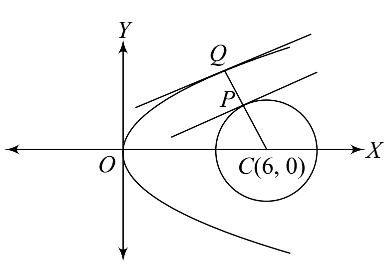
$$9y^2 = 4ax$$

11. Do yourself.

12. Given circle is $x^2 + y^2 - 12x + 31 = 0$

$\Rightarrow \quad (x-6)^2 + y^2 = 5$

The centre is (6, 0) and the radius is $\sqrt{5}$.



Given parabola is

$$y^2 = 4x$$

$\Rightarrow \quad 2y\frac{dy}{dx} = 4$

$\Rightarrow \quad \frac{dy}{dx} = \frac{2}{y}$

Also, given circle is

$$x^2 + y^2 - 12x + 31 = 0$$

$\Rightarrow \quad 2x + 2y\frac{dy}{dx} - 12 = 0$

$\Rightarrow \quad x + y\frac{dy}{dx} - 6 = 0$

$\Rightarrow \quad \frac{dy}{dx} = \frac{6-x}{y}$

Since the tangents are parallel, so their slopes are the same.

Thus, $\frac{2}{y} = \frac{6-x}{y}$

$\Rightarrow \quad x = 4$

When $x = 4$, then $y^2 = 16$

$\Rightarrow \quad y = \pm 4$

Thus, the point Q is (4, 4).

Therefore, the shortest distance,

$$\begin{aligned} PQ &= CQ - CP \\ &= \sqrt{(6-4)^2 + (4-0)^2} - 1 \\ &= \sqrt{20} - 1 \\ &= (2\sqrt{5} - 1) \end{aligned}$$

13. Let the parabola be $y^2 = 4ax$

The equation of the tangent to the parabola at $(a, 2a)$ is

$$y \cdot 2a = 2a(x+a)$$

$\Rightarrow \quad y = x + a$

$\Rightarrow \quad x - y - a = 0$

The equation of a circle touching the parabola at $(a, 2a)$ is

$$(x-a)^2 + (y-2a)^2 + \lambda(x - y - a) = 0$$

which is passing through (0, 0). So

$$a^2 + 4a^2 - \lambda a = 0$$

$\Rightarrow \quad \lambda = 5a$

Thus, the required circle is

$$x^2 + y^2 - 7ax + ay = 0$$

Hence, the radius $= \sqrt{\left(\frac{7a}{2}\right)^2 + \left(\frac{a}{2}\right)^2} = \sqrt{\frac{50a^2}{4}} = \frac{5a}{\sqrt{2}}$

14. The tangent to $y^2 = 4x$ in terms of m is

$$y = mx + \frac{1}{m}$$

and the normal to $x^2 = 4by$ in terms of m is

$$y = mx + 2b + \frac{b}{m^2}$$

If these are the same line, then

$$\frac{1}{m} = 2b + \frac{b}{m^2}$$

$\Rightarrow \quad 2bm^2 - m + b = 0$

For two different tangents, we get

$D > 0$

$\Rightarrow \quad 1 - 8b^2 > 0$

$\Rightarrow \quad 8b^2 < 1$

$\Rightarrow \quad b^2 < \frac{1}{8}$

$\Rightarrow \quad |b| < \frac{1}{2\sqrt{2}}$

which is the required condition.

15. Given parabola is $y^2 = 8x$

Extremities of the latus rectum are (2, 4) and (2, –4).

Since any circle is drawn with any focal chord as its diameter touches the directrix, the equation of the required circle is

$(x - 2)(x - 2) + (y - 4)(y + 4) = 0$

$\Rightarrow \quad x^2 + y^2 - 4x - 12 = 0$

Hence, the radius = $\sqrt{4 + 12} = 4$.

Integer Type Questions

1. A circle and a parabola can meet at most in four points. Thus, the maximum number of common chords is ${}^4C_2 = \frac{4 \times 3}{2} = 6$.

2. Clearly, both the lines pass through $(-a, b)$ which a point lying on the directrix of the parabola.

Thus, $m_1 m_2 = -1$, since tangents are drawn from any point on the directrix always mutually perpendicular. Hence, the value of $(m_1 m_2 + 4)$ is 3.

3. Let AB be a normal chord, where

$$A = (at_1^2, 2at_1),\ B = (at_2^2, 2at_2)$$

Now, the normal at A meets the parabola again at B, so

$$t_2 = -t_1 - \frac{2}{t_1} \text{ and } t_1 t_2 = -4$$

Solving, we get

$t_1^2 = 2$

Thus, $m = m(AB) = \frac{2}{t_1 + t_2}$

$\Rightarrow \quad m = \frac{2}{(-2/t_1)} = -t_1 = \mp\sqrt{2}$

$\Rightarrow \quad (m^2 + 3) = 2 + 3 = 5$

4. Let AB be a normal chord, where

$$A = (at_1^2, 2at_1),\ B = (at_2^2, 2at_2)$$

Clearly,

$4(t_1 + t_2) = 4$

$\Rightarrow \quad (t_1 + t_2) = 1$

Now, $m = m(AB) = \frac{2}{t_1 + t_2} = \frac{2}{1} = 2$

Hence, the slope of the normal chord is 2.

5. Normals to $y^2 = 4ax$ and $x^2 = 4by$ in terms of m are

$y = mx - 2am - am^3$

and $\quad y = mx + 2b + \frac{b}{m^2}$

For a common normal,

$$2b + \frac{b}{m^2} = -2am - am^3$$

$\Rightarrow \quad 2bm^2 + b + 2am^3 + am^5 = 0$

$\Rightarrow \quad am^5 + 2am^3 + 2b^2 + b = 0$

Thus, the number of common normals is 5.

6. We have, $x + y = 2(t^2 + 1)$

and $\quad x - y = 2t$

Eliminating t, we get,

$$x + y = 2\left(\left(\frac{x - y}{2}\right)^2 + 1\right)$$

$\Rightarrow \quad x + y = \frac{(x - y)^2}{2} + 2$

$\Rightarrow \quad 2(x + y) = (x - y)^2 + 4$

$\Rightarrow \quad (x - y)^2 - 2(x + y) + 4 = 0$

$\Rightarrow \quad (x - y)^2 = 2(x + y - 2)$

Comparing with $y^2 = 4ax$, we get

$4a = 2$

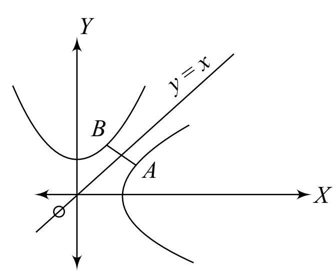
Thus, the length of latus rectum is 2.

7. Both the given curves are symmetrical about the line $y = x$.

If the line AB is the shortest distance then at A and B the slopes of the curve should be equal to 1.

For $y^2 = x - 1$, $\frac{dy}{dx} = \frac{1}{2y} = 1$

$\Rightarrow \quad y = \frac{1}{2}$



Then $x = \frac{1}{4} + 1 = \frac{5}{4}$.

Therefore, $A = \left(\frac{5}{4}, \frac{1}{2}\right)$ and $B = \left(\frac{1}{2}, \frac{5}{4}\right)$

Hence, the shortest distance,

$$d = \sqrt{\left(\frac{5}{4} - \frac{1}{2}\right)^2 + \left(\frac{1}{2} - \frac{5}{4}\right)^2}$$

$$= \sqrt{\left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^2} = \sqrt{\frac{18}{16}} = \frac{3\sqrt{2}}{4}$$

Hence, the value of

$(8d^2 - 3)$

$= 9 - 3 = 6$

8. The equation of any tangent to the parabola $y^2 = 4x$ is

$$y = mx + \frac{a}{m} = mx + \frac{1}{m}$$

which is passing through (2, 3). So

$$3 = 2m + \frac{1}{m}$$

$\Rightarrow \quad 2m^2 - 3m + 1 = 0$

$\Rightarrow \quad 2m^2 - 2m - m + 1 = 0$

$\Rightarrow \quad 2m(m - 1) - 1(m - 1) = 0$

$\Rightarrow \quad (m - 1)(2m - 1) = 0$

$\Rightarrow \quad m = 1$ and $1/2$

$\Rightarrow \quad m_1 = 1, m_2 = \frac{1}{2}$

Hence, the value of $\left(\frac{1}{m_1} + \frac{1}{m_2} + 2\right) = 1 + 2 + 2 = 5.$

9. Let the co-ordinates of the point R be $(at_3^2, 2at_3)$.

The normal at P meets the parabola again at R, so

$$t_3 = -t_1 - \frac{2}{t_1} \quad \ldots\text{(i)}$$

and the normal at Q meets the parabola again at R, so

$$t_3 = -t_2 - \frac{2}{t_2} \quad \ldots\text{(ii)}$$

From Eqs (i) and (ii), we get

$$-t_1 - \frac{2}{t_1} = -t_2 - \frac{2}{t_2}$$

$\Rightarrow \quad t_2 - t_1 = \frac{2}{t_1} - \frac{2}{t_2} = \frac{2(t_2 - t_1)}{t_1 t_2}$

$\Rightarrow \quad t_2 - t_1 = \frac{2(t_2 - t_1)}{t_1 t_2}$

$\Rightarrow \quad t_1 t_2 = 2$

Hence, the value of $(t_1 t_2 + 3) = 5$

10. Solving, we get,

$y^2 = 4(1 - y)$

$\Rightarrow \quad y^2 = 4 - 4y$

$\Rightarrow \quad y^2 + 4y - 4 = 0$

$\Rightarrow \quad (y + 2)^2 = 8$

$\Rightarrow \quad y = -2 \pm 2\sqrt{2}$

when $y = -2 + 2\sqrt{2}$, then

$$x = 1 - (-2 + 2\sqrt{2}) = 3 - 2\sqrt{2}$$

when $y = -2 - 2\sqrt{2}$, then

$$x = 1 - (-2 - 2\sqrt{2}) = 3 + 2\sqrt{2}$$

Let the chord be AB,

where $A = (3 - 2\sqrt{2}, -2 + 2\sqrt{2})$,

and $B = (3 + 2\sqrt{2}, -2 - 2\sqrt{2})$

Hence, the length of the chord AB

$$= \sqrt{(4\sqrt{2})^2 + (4\sqrt{2})^2} = \sqrt{32 + 32} = 8$$

11. Given $L: y = -x + k$

Here, $m = -1$ and $c = k$

The line L will be a normal to the parabola, if

$c = -2am - am^3$

$\Rightarrow \quad k = -2 \times 3 \times (-1) - 3 \times (-1)^3$

$\Rightarrow \quad k = 6 + 3 = 9$

Hence, the value of k is 9.

12. Given parabola is

$y^2 - 4x - 2y - 3 = 0$

$\Rightarrow \quad y^2 - 2y = 4x + 3$

$\Rightarrow \quad (y - 1)^2 = 4(x + 1)$

The equation of any normal to the given parabola is

$(y - 1) = m(x + 1) - 2am - am^3$

$\Rightarrow \quad (y - 1) = m(x + 1) - 2m - m^3$

which is passing through (–2, 1), so

$(1 - 1) = m(-2 + 1) - 2m - m^3$

$\Rightarrow \quad 0 = -m - 2m - m^3$

$\Rightarrow \quad m^3 + 3m = 0$

Hence, the number of distinct normals is 3.

Previous Years' JEE-Advanced Examinations

1. The equation of the normal to the parabola $y^2 = 4ax$ is $y = mx - 2am - am^3$ which is passing through $(h, 0)$. So

$mh - 2am - am^3 = k$

$\Rightarrow \quad am^3 + (2a - h)m + k = 0$

Let its roots are m_1, m_2 and m_3.

Thus, $m_1 + m_2 + m_3 = 0$

$$m_1 m_2 + m_2 m_3 + m_3 m_1 = \frac{2a - h}{a}$$

and $\quad m_1 m_2 m_3 = -\frac{k}{a}$

For any real values of m_1, m_2, m_3,

$m_1^2 + m_2^2 + m_3^2 > 0$

$\Rightarrow \quad (m_1 + m_2 + m_3)^2 - 2(m_1 m_2 + m_2 m_3 + m_3 m_1) > 0$

$\Rightarrow \quad 0 - 2\left(\frac{2a - h}{a}\right) > 0$

$\Rightarrow \quad h - 2a > 0$

$\Rightarrow \quad h > 2a$

2. Let the point A be $(at^2, 2at)$.

The equation of the normal at A is
$y = -tx + 2at + at^3$
and the equation of the pair of lines OA and OB is
$$y^2 = 4ax\left(\frac{y + tx}{2at + at^3}\right)$$
$\Rightarrow \quad (t^3 + 2t)y^2 = 4tx^2 + 4xy$

As $\angle AOB = \frac{\pi}{2}$, we get

co-efficient of x^2 + co-efficient of $y^2 = 0$
$\Rightarrow \quad -4t + t^3 + 2t = 0$
$\Rightarrow \quad t^3 - 2t = 0$
$\Rightarrow \quad t(t^2 - 2) = 0$
$\Rightarrow \quad t = 0, t = \pm\sqrt{2}$

Clearly, $t \neq 0$, so $t = \pm\sqrt{2}$

Thus, slope of normal $= -t = \pm\sqrt{2}$.

3. The equation of the normal to the given parabola is
$x = my - 2am - am^3$
$\Rightarrow \quad x = my - 2m - m^3$
which is passing through (1, 2)
$1 = 2m - 2m - m^3$
$\Rightarrow \quad m^3 = -1$
$\Rightarrow \quad m = -1$

Hence, the equation of the normal is
$x = -y + 2 + 1$
$\Rightarrow \quad x + y = 3$

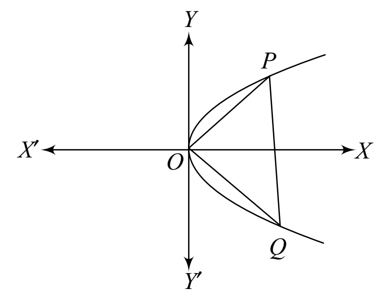
4. The equation of the normal to the given parabola is
$y = mx - 2am - am^3$
$$\Rightarrow \quad y = mx - 2\cdot\frac{1}{4}m - \frac{1}{4}m^3$$
$$\Rightarrow \quad y = mx - \frac{1}{2}m - \frac{1}{4}m^3$$
$\Rightarrow \quad 4y = 4mx - 2m - m^3$
which is passing through $(c, 0)$, so
$0 = 4mc - 2m - m^3$
$\Rightarrow \quad m^3 + 2m - 4mc = 0$
$\Rightarrow \quad m^3 + 2(1 - 2c)m = 0$
$\Rightarrow \quad m = 0, m^2 + 2(1 - 2c) = 0$
$\Rightarrow \quad m = 0, m^2 = 2(2c - 1)$
$\Rightarrow \quad m = 0, m = \pm\sqrt{2(2c-1)}$

So, one normal is always the x-axis.

Let $m_1 = \sqrt{2(2c-1)}$ and $m_2 = -\sqrt{2(2c-1)}$

It is given that,
$m_1 m_2 = -1$
$\Rightarrow \quad 2(2c - 1) = 1$
$$\Rightarrow \quad (2c - 1) = \frac{1}{2}$$
$$\Rightarrow \quad 2c = 1 + \frac{1}{2} = \frac{3}{2}$$
$$\Rightarrow \quad c = \frac{3}{4}$$

5. Let $P = (t_1^2, 2t)$ and $Q = (t_2^2, 2t_2)$



The equation of PQ is
$2x - (t_1 + t_2)y = -2t_1t_2$ ...(i)

Now, $m_1 = m(OP) = \frac{2}{t_1}$

and $\quad m_2 = m(OQ) = \frac{2}{t_2}$

As OP is perpendicular to OQ, so
$m_1 m_2 = -1$
$$\Rightarrow \quad \frac{4}{t_1t_2} = -1$$
$\Rightarrow \quad t_1t_2 = -4$ ...(ii)

From Eqs (i) and (ii), we get
$2(x - 4) - (t_1 + t_2)y = 0$
which is passing through a fixed point (4, 0) on the axis of the parabola.

Let $M(h, k)$ be the mid-point of PQ.

Thus, $h = \frac{1}{2}(t_1^2 + t_2^2)$ and $k = \frac{1}{2}(2t_1 + 2t_2)$

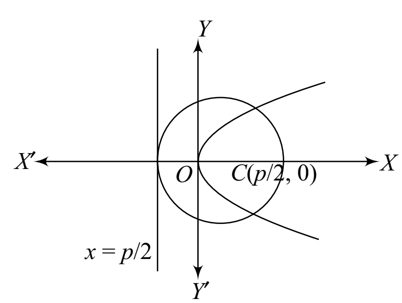
$\Rightarrow \quad h = \frac{1}{2}(t_1^2 + t_2^2)$ and $k = (t_1 + t_2)$

Now, $k^2 = (t_1 + t_2)^2$
$\Rightarrow \quad k^2 = (t_1^2 + t_2^2) + 2t_1t_2$
$\Rightarrow \quad k^2 = 2h + 2(-4)$

Hence, the locus of $M(h, k)$ is
$y^2 = 2(x - 4) = 2x - 8$

6. Given parabola is $y^2 = 2px$. So

the focus is $\left(\frac{p}{2}, 0\right)$.



Clearly, the equation of the circle is

$$\left(x-\frac{p}{2}\right)^2+y^2=p^2$$

$$\Rightarrow \quad \left(x-\frac{p}{2}\right)^2+2px=p^2$$

$$\Rightarrow \quad x^2-px+\frac{p^2}{4}+2px=p^2$$

$$\Rightarrow \quad x^2+px+\frac{p^2}{4}=p^2$$

$$\Rightarrow \quad \left(x+\frac{p}{2}\right)^2=p^2$$

$$\Rightarrow \quad \left(x+\frac{p}{2}\right)=\pm p$$

$$\Rightarrow \quad x=-\frac{p}{2}\pm p=\frac{p}{2},-\frac{3p}{2}$$

when $x=\frac{p}{2}$, then, $y=\pm p$

Thus the point of intersection are

$$\left(\frac{p}{2},p\right) \text{ and } \left(\frac{p}{2},-p\right)$$

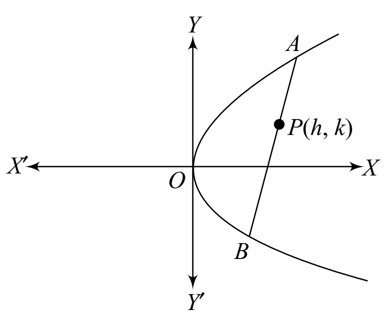
7. Let AB be a chord of a parabola, in which

$A=(t_1^2,2t_1), B=(t_2^2,2t_2)$

Slope of $AB=2$

$$\Rightarrow \quad \frac{2}{t_1+t_2}=2$$

$$\Rightarrow \quad t_1+t_2=1$$



Let P be a point which divides AB internally in the ratio 1 : 2. So

$$h=\frac{2t_1^2+t_2^2}{3} \text{ and } k=\frac{4t_1+2t_2}{3}$$

$$\Rightarrow \quad 3h=(2t_1^2+t_2^2) \text{ and } 3k=(4t_1+2t_2)$$

Eliminating t_1 and t_2, we get

$$\left(k-\frac{8}{9}\right)^2=\frac{4}{9}\left(h-\frac{2}{9}\right)$$

Thus, the locus of $P(h, k)$ is

$$\left(y-\frac{8}{9}\right)^2=\frac{4}{9}\left(x-\frac{2}{9}\right)$$

Hence, the vertex is $\left(\frac{2}{9},\frac{8}{9}\right)$.

8. Let the three points of the parabola be

$P(at_1^2,2at_1), Q(at_2^2,2at_2)$ and $R(at_3^2,2at_3)$,

and the points of intersections of the tangents at these points are $A[t_2t_3, a(t_2+t_3)]$, $B[t_1t_3, a(t_1+t_3)]$ and $A[t_1t_2, a(t_1+t_2)]$

Now,

$$ar(\Delta PQR)=\frac{1}{2}\begin{vmatrix} at_1^2 & 2at_1 & 1\\ at_2^2 & 2at_2 & 1\\ at_3^2 & 2at_3 & 1\end{vmatrix}$$

$$=a_2(t_1-t_2)(t_1-t_3)(t_3-t_1)$$

Also,

$$ar(\Delta ABC)=\frac{1}{2}\begin{vmatrix} at_2t_3 & a(t_2+t_3) & 1\\ at_3t_1 & a(t_3+t_1) & 1\\ at_1t_2 & a(t_1+t_2) & 1\end{vmatrix}$$

$$=\frac{1}{2}a^2(t_1-t_2)(t_2-t_3)(t_3-t_1)$$

$$\therefore \quad \frac{\Delta PQR}{\Delta ABC}=2$$

9.

10. The equation of the normal to the parabola $y^2=12x$ is $\quad y=mx-2am-am^3$

$\Rightarrow \quad y=mx-6m-3m^3$ …(i)

Given normal is $y=-x+k$ …(ii)

Equations (i) and (ii) are identical, so

$m=-1$

and $\quad k=-6m-3m^3=6+3=9$

Hence, the value of k is 9.

11 Given parabola is $y^2=kx-8$

$$\Rightarrow \quad y^2=k\left(x-\frac{8}{k}\right)$$

Here, $4a=k$

$$\Rightarrow \quad a=\frac{k}{4}$$

So, the directrix is $\left(x-\frac{8}{k}\right)+\frac{k}{4}=0$

Given directrix is $x-1=0$.

Thus, $\frac{8}{k}-\frac{k}{4}=1$

$\Rightarrow \quad 32-k^2=4k$

$\Rightarrow \quad k^2+4k-32=0$

$\Rightarrow \quad (k+8)(k-4)=0$

$\Rightarrow \quad k=4,-8$

12. Given parabola is

$$y^2 + 4y + 4x + 2 = 0$$

$\Rightarrow \quad (y + 2)^2 = -4x - 2 + 4$

$\Rightarrow \quad (y+2)^2 = -4x + 2 = -4\left(x - \frac{1}{2}\right)$

So, the directrix is $x - \frac{1}{2} = a = 1$

$\Rightarrow \quad x = \frac{3}{2}$

13. Any tangent to the parabola $y^2 = 4x$ is

$$y = mx + \frac{a}{m}$$

$\Rightarrow \quad y = mx + \frac{1}{m}$

$\Rightarrow \quad m^2x - my + 1 = 0 \quad \ldots\text{(i)}$

If it is a tangent to the circle $x^2 + (y - 3)^2 = 9$ the length of the perpendicular from the centre to the tangent is equal to the radius of the circle. So

$$\left|\frac{3m^2+1}{\sqrt{m^4+m^2}}\right| = 3$$

$\Rightarrow \quad (3m^2 + 1)^2 = 9(m^4 + m^2)$

$\Rightarrow \quad (9m^4 + 6m^2 + 1) = 9(m^4 + m^2)$

$\Rightarrow \quad 3m^2 = 1$

$\Rightarrow \quad m = \pm\left(\frac{1}{\sqrt{3}}\right)$

Since, the tangent touches the parabola above x-axis, so it will make an acute angle with x-axis, so that m is positive.

Thus $m = \frac{1}{\sqrt{3}}$

Hence, the common tangent is $x - \sqrt{3}y + 3 = 0$.

14. Ans. (c)

If (h, k) be the mid point of line joining the focus $(a, 0)$ and $Q(at^2, 2at)$ on the parabola, then $h = \frac{a + at^2}{2}$, $k = at$.

Eliminating 't', we get,

$$2h = a + a\left(\frac{k^2}{a^2}\right)$$

$\Rightarrow \quad k^2 = 2a\left(h - \frac{a}{2}\right)$

Now, directrix: $\left(x - \frac{a}{2}\right) = -\frac{a}{2}$

$\Rightarrow \quad x = 0.$

15. Let the equation of the tangent to the parabola $y^2 = 8x$ is

$$y = mx + \frac{2}{m} \quad \ldots\text{(i)}$$

If it is a tangent to the curve $xy = -1$, then

$$x\left(mx + \frac{2}{m}\right) = -1$$

$\Rightarrow \quad m^2x^2 + 2x + m = 0$

It has equal roots. So,

$D = 0$

$\Rightarrow \quad 4 - 4m^3 = 0$

$\Rightarrow \quad m^3 = 1$

$\Rightarrow \quad m = 1$

Hence, the equation of the common tangent is $y = x + 2$.

16. Ans. (a)

For the parabola, $y^2 = 16x$, focus = (4, 0)

Let m be the slope of the focal chord.

So, its equation is $y = m(x - 4) \quad \ldots\text{(i)}$

which is a tangent to the circle

$$(x - 6)^2 + y^2 = 2$$

where centre = (6, 0) and radius = $\sqrt{2}$.

Length of perpendicular from (6, 0) to (i) is equal to r

$$\left|\frac{6m - 4m}{m^2+1}\right| = \sqrt{2}$$

$$\left|\frac{2m}{m^2+1}\right| = \sqrt{2}$$

$4m^2 = 2(m^2 + 1)$

$2m^2 = (m^2 + 1)$

$m^2 = 1$

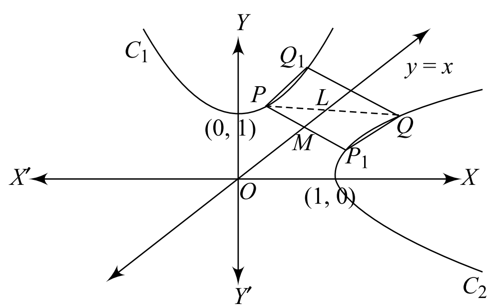
$m = \pm 1$

17 Given that

$C_1: x^2 = y - 1$

$C_2: y^2 = x - 1$

Let $P(x_1, x_1^2 + 1)$ on C_1 and $Q(y_2^2 + 1, y_2)$ on C_2.



Now, the reflection of the point P in the line $y = x$ can be obtained by interchanging the values of abscissa and the ordinate.

Thus, the reflection of the point $P(x_1, x_1^2 + 1)$ is $P_1(x_1^2 + 1, x_1)$

and the reflection of the point $Q(y_2^2 + 1, y_2)$ is $Q_1(y_2, y_2^2 + 1)$

It can be seen clearly that, P_1 lies on C_2 and Q_1 on C_1

Now, PP_1 and QQ_1 both are perpendicular to the mirror line $y = x$.

Also, M is the mid point of PP_1

Thus, $PM = \frac{1}{2}PP_1$

In triangle PML, $PL > PM$

$$PL > \frac{1}{2}PP_1 \quad \text{...(i)}$$

Similarly, $LQ > \frac{1}{2}QQ_1$ ...(ii)

Adding (i) and (ii), we get,

$$PL + LQ > \frac{1}{2}(PP_1 + QQ_1)$$

$$PQ > \frac{1}{2}(PP_1 + QQ_1)$$

PQ is more than the mean of PP_1 and QQ_1

$PQ \geq \min(PP_1, QQ_1)$

Let $\min(PP_1, QQ_1) = PP_1$

then $PQ^2 \geq PP_1^2$

$$= (x_1^2 + 1 - x_1)^2 + (x_1^2 + 1 - x_1)^2$$

$$= 2(x_1^2 + 1 - x_1)^2 = f(x_1)$$

Now, $f'(x_1) = 4(x_1^2 + 1 - x_1)(2x_1 - 1)$

$$= 4\left(\left(x_1 - \frac{1}{2}\right)^2 + \frac{3}{4}\right)(2x_1 - 1)$$

$f'(x_1) = 0$ gives $x_1 = \frac{1}{2}$

Also, $f'(x_1) < 0$ if $x_1 < \frac{1}{2}$

and $f'(x_1) > 0$ if $x_1 > \frac{1}{2}$

Thus, $f(x_1)$ is minimum when $x_1 = \frac{1}{2}$

Thus, if at $x_1 = \frac{1}{2}$ at P is P_0 on C_1

So, $P_0 = \left(\frac{1}{2}, \left(\frac{1}{2}\right)^2 + 1\right) = \left(\frac{1}{2}, \frac{5}{4}\right)$

Similarly Q_0 on C_2 will be image of P_0 with respect to the line $y = x$

So, $Q_0 = \left(\frac{5}{4}, \frac{1}{2}\right)$

18. Let the point P be (h, k).

The equation of any normal to the given parabola is

$y = mx - 2am - am^3$

$\Rightarrow$ $y = mx - 2m - m^3$, since $a = 1$

which is passing through P. So

$k = mh - 2m - m^3$

$\Rightarrow$ $m^3 + (2 - h)m + k = 0$ …(i)

Let its roots be m_1, m_2, m_3.

So, $m_1 + m_2 + m_3 = 0$

$m_1m_2 + m_2m_3 + m_3m_1 = (2 - h)$

and $m_1m_2m_3 = -k$

It is given that $m_1m_2 = \alpha$. So

$$m_3 = -\frac{k}{\alpha}$$

Since m_3 is the roots of (i), so

$$m_3^3 + (2 - h)m_3 + k = 0$$

$$\Rightarrow \left(-\frac{k}{\alpha}\right)^3 + (2 - h)\left(-\frac{k}{\alpha}\right) + k = 0$$

$$\Rightarrow -\left(\frac{k}{a}\right)^3 - (2 - h)\left(\frac{k}{\alpha}\right) + k = 0$$

$$\Rightarrow \frac{k^2}{\alpha^3} + \frac{(2 - h)}{\alpha} - 1 = 0$$

$\Rightarrow$ $k^2 + (2 - h)\alpha^2 - \alpha^3 = 0$

Hence, the locus of $P(h, k)$ is

$y^2 + (2 - x)\alpha^2 - \alpha^3 = 0$

As this locus is a part of the parabola $y^2 = 4x$ so, $\alpha^2 = 4$ and $-2\alpha^2 + \alpha^3 = 0$

Thus, $\alpha = 2$.

19. The equation of any tangent to the given parabola can be considered as

$$y = mx + \frac{a}{m} = mx + \frac{1}{m}$$

which is passing through $(1, 4)$. So

$$4 = m + \frac{1}{m}$$

$\Rightarrow$ $m^2 - 4m + 1 = 0$

Let its roots are m_1, m_2.

$\therefore$ $m_1 + m_2 = 4$ and $m_1m_2 = 1$

Let θ be the angle between them. Then

$$\tan\theta = \left|\frac{m_2 - \mathrm{m}_1}{1 + m_1m_2}\right|$$

$$= \left|\frac{\sqrt{(m_2 + m_1)^2 - 4m_1m_2}}{1 + m_1m_2}\right|$$

$$= \left|\frac{\sqrt{16 - 4}}{1 + 1}\right| = \frac{2\sqrt{3}}{2}$$

$$= \sqrt{3} = \frac{\pi}{3}$$

$$\Rightarrow \theta = \frac{\pi}{3}$$

20. Given parabola is

$y^2 - 2y - 4x + 5 = 0$

$\Rightarrow$ $(y - 1)^2 = 4x - 4 = 4(x - 1)$

$\Rightarrow$ $Y^2 = 4X$

where $X = (x - 1)$, $Y = (y - 1)$

So, the directrix is

$X + a = 0$

$\Rightarrow \quad (x-1)+1=0$

$\Rightarrow \quad x=0$

Any point on the parabola is

$$P(1+t^2, 2t+1)$$

The equation of the tangent at P is

$$t(y-1)=x-1+t^2$$

which meets the directrix $x=0$ at

$$Q\left(0, 1+t-\frac{1}{t}\right)$$

Let the co-ordinates of R be (h, k).

Since it divides QP externally in the ratio $\frac{1}{2}:1$, so Q is the mid-point of R and P. Thus

$$\frac{h+1+t^2}{2}=0 \text{ and } 1+t-\frac{1}{t}=\frac{k+1+2t}{2}$$

$$\Rightarrow \quad t^2=-(h+1) \text{ and } t=\frac{2}{1-k}$$

Thus, $\frac{4}{(k-1)^2}+(h+1)=0$

$\Rightarrow \quad (k-1)^2(h+1)+4=0$

Hence, the locus of $R(h, k)$ is

$$(y-1)^2(x+1)+4=0$$

21. Clearly, the vertex is $(1, 1)$ and the focus is $S(2, 2)$ and the directrix is $x+y=0$.

Let P be (x, y)

Now, $SP=PM$

$\Rightarrow \quad SP^2=PM^2$

$$\Rightarrow \quad (x-2)^2+(y-2)^2=\left(\frac{x+y}{\sqrt{2}}\right)^2$$

$\Rightarrow \quad 2[(x-2)^2+(y-2)^2]=(x+y)^2$

$\Rightarrow \quad 2(x^2+y^2-4x-4y+8)=x^2+y^2+2xy$

$\Rightarrow \quad (x^2+y^2-2xy)=8(x+y+2)$

$\Rightarrow \quad (x-y)^2=8(x+y+2)$

22. Any point on the parabola $y=x^2$ is (t, t^2).

Now tangent at (t, t^2) is

$$xx_1=\frac{1}{2}(y+y_1)$$

$$\Rightarrow \quad tx=\frac{1}{2}(y+t^2)$$

$\Rightarrow \quad 2tx-y-t^2=0$

If it is a tangent to the parabola, $y=-(x-2)^2$, then

$$2tx-t^2=-(x-2)^2$$

$\Rightarrow \quad 2tx-t^2=-x^2+4x-4$

$\Rightarrow \quad x^2+2(2-t)x+(t^2-4)=0$

Since it has equal roots, so

$\Rightarrow \quad D=0$

$$4(2-t)^2-4(t^2-4)=0$$

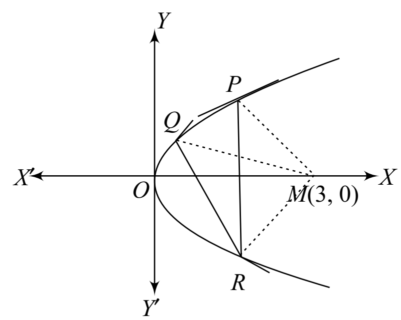
$\Rightarrow \quad (2-t)^2-(t^2-4)=0$

$\Rightarrow \quad t=2, 0$

Hence, the equation of the common tangent is

$$y=4x-4, \; y=0$$

23.



Equation of any normal to the given parabola is

$$y=mx-2am-am^3 \quad \ldots\text{(i)}$$

Let $P=(am_1^2, -2am_1), Q=(am_2^2, -2am_2)$

and $R=(am_3^2, -2am_3)$

Equation (i) passing through $(3, 0)$

So, $\quad 0=3m-2am-am^3$

$m^3-m=0, \quad (\because a=1)$

$m(m+1)(m-1)=0$

$m=-1, 0, 1$

Thus, $m_1=-1, m_2=0, m_3=1$

Now, $P=(m_1^2, 2m_1)=(1, -2)$

$Q=(m_2^2, 2m_2)=(0, 0)$

and $\quad R=(m_3^2, 2m_3)=(1, 2)$

(i) Area of ΔPQR

$$=\frac{1}{2}\begin{vmatrix}1 & -2 & 1\\ 0 & 0 & 1\\ 1 & 2 & 1\end{vmatrix}=\frac{1}{2}\times 4\times 1-2$$

(ii) Radius of circumcircle of $\Delta PQR=2$

(iii) Centroid of ΔPQR

$$=\left(\frac{m_1^2+m_2^2+m_3^2}{3}, \frac{2(m_1+m_2+m_3)}{3}\right)$$

$$=\left(\frac{2}{3}, 0\right)$$

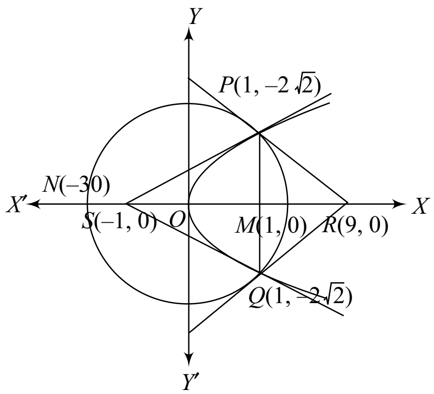
(iv) Clearly, ΔPQR is a right-angled triangle and right angle at Q.

Thus, Circumcentre of ΔPQR

= Mid-point of the hypotenuse PR

$=(1, 0)$.

24.



(i) Co-ordinates of P and Q are

$$P=(1, 2\sqrt{2}), Q=(1, -2\sqrt{2})$$

Area of $\Delta PQR=\frac{1}{2}\times 4\sqrt{2}\times 8=16\sqrt{2}$

Area of $\Delta PQS=\frac{1}{2}\times 4\sqrt{2}\times 2=4\sqrt{2}$

Thus, $\frac{\text{ar}(\Delta PQS)}{\text{ar}(\Delta PQR)}=\frac{4\sqrt{2}}{16\sqrt{2}}=\frac{1}{4}$

(ii) $R=\frac{abc}{4\Delta}=\frac{2\sqrt{3}\times 6\sqrt{2}\times 10}{4\cdot\left(\frac{1}{2}\times 10\times 2\sqrt{2}\right)}=3\sqrt{3}$

(iii) $r=\frac{\Delta}{s}$

$$=\frac{\frac{1}{2}\times 4\sqrt{2}\times 8}{\frac{1}{2}\times(6\sqrt{2}+6\sqrt{2}+4\sqrt{2})}$$

$$=\frac{32\sqrt{2}}{16\sqrt{2}}$$

$$=2$$

25. Given curve is

$$y=-\frac{x^2}{2}+x+1$$

$$\Rightarrow\quad y=-\frac{1}{2}\left(x^2-\frac{1}{2}x-\frac{1}{2}\right)$$

$$\Rightarrow\quad y=-\frac{1}{2}(x-1)^2+\frac{3}{2}$$

$$\Rightarrow\quad \left(y-\frac{3}{2}\right)=-\frac{1}{2}(x-1)^2$$

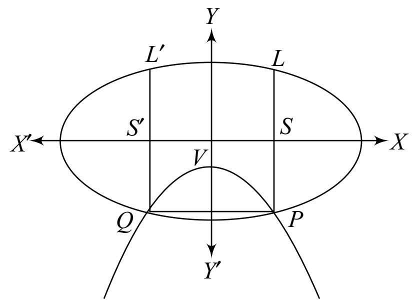
which is symmetric about the line $x=1$.

> ***Note:*** A function $f(x)$ is symmetric about the line $x=1$ then, $f(1-x)=f(x+1)$

26. Given ellipse is

$$x^2+4y^2=4$$

$$\Rightarrow\quad \frac{x^2}{4}+\frac{y^2}{1}=1$$



Thus, $e=\sqrt{1-\frac{b^2}{a^2}}=\sqrt{1-\frac{1}{4}}=\frac{\sqrt{3}}{2}$

Foci: $S=(ae, 0)=(\sqrt{3}, 0)$

and $S'=(-ae, 0)=(-\sqrt{3}, 0)$

End-points of latus recta:

and $L=\left(ae, \frac{b^2}{a}\right)=\left(\sqrt{3}, \frac{1}{2}\right)$

$$L'=\left(-ae, \frac{b^2}{a}\right)=\left(-\sqrt{3}, \frac{1}{2}\right)$$

Thus, $P=\left(\sqrt{3}, -\frac{1}{2}\right)$ and $Q=\left(-\sqrt{3}, -\frac{1}{2}\right)$

As we know that, the focus is the mid-point of the P and Q.

Thus, the focus of a parabola is $\left(0, -\frac{1}{2}\right)$.

The length of $PQ=2\sqrt{3}$

Now, $4a=2\sqrt{3}\Rightarrow a=\frac{\sqrt{3}}{2}$

Thus, the vertices of a desired parabola

$$=\left(0, -\frac{1}{2}\pm a\right)=\left(0, -\frac{1}{2}\pm\frac{\sqrt{3}}{2}\right)$$

Therefore, two desired parabolas are

$$\Rightarrow\quad x^2=\pm 4a\left(y-\left(-\frac{1}{2}\pm\frac{\sqrt{3}}{2}\right)\right)$$

$$\Rightarrow\quad x^2=2\sqrt{3}\left(y+\frac{1}{2}+\frac{\sqrt{3}}{2}\right)$$

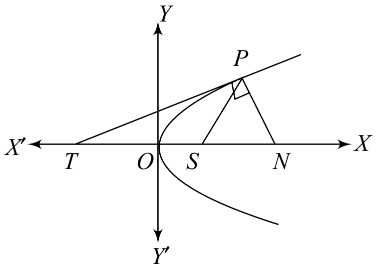
or

$$x^2=-2\sqrt{3}\left(y+\frac{1}{2}-\frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow\quad x^2=2\sqrt{3}y+(3+\sqrt{3})$$

or $\quad x^2=-2\sqrt{3}y+(3-\sqrt{3})$

27. Let the co-ordinates P be $(at^2, 2at)$.



The equation of PT is $yt = x + at^2$
So, T is $(-at^2, 0)$.
and the equation of PN is $y = -tx + 2at + at^3$
So, N is $(2a + at^2, 0)$.
Let the centroid be $G(h, k)$.

Thus, $h = \dfrac{at^2 - at^2 + 2a + at^2}{3}$ and $k = \dfrac{2at}{3}$

$\Rightarrow \quad h = \dfrac{2a + at^2}{3}$ and $k = \dfrac{2at}{3}$

$$\Rightarrow \quad \left(\frac{3h - 2a}{a}\right) = \left(\frac{3k}{2a}\right)^2$$

$$\Rightarrow \quad 3\left(h - \frac{2a}{3}\right) = \frac{9k^2}{4a}$$

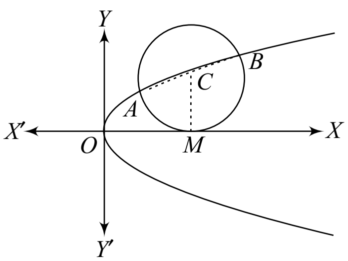
$$\Rightarrow \quad k^2 = \frac{4a}{3}\left(h - \frac{2a}{3}\right)$$

Hence, the locus of $G(h, k)$ is

$$y^2 = \frac{4a}{3}\left(x - \frac{2a}{3}\right)$$

So the vertex is $\left(\dfrac{2a}{3}, 0\right)$ and focus is $(a, 0)$.

28. Let $A = (t_1^2, 2t_1), B = (t_2^2, 2t_2)$



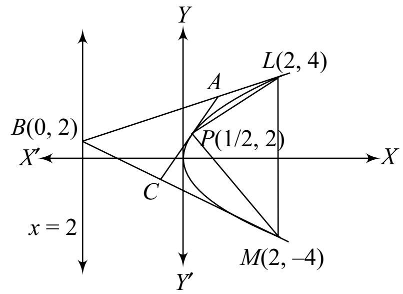
Then $C = \left(\dfrac{t_1^2 + t_2^2}{2}, t_1 + t_2\right)$

Clearly, $|t_1 + t_2| = r$
$\Rightarrow \quad (t_1 + t_2) = \pm r$

Now, $m(AB) = \dfrac{2(t_2 - t_1)}{(t_2^2 - t_1^2)} = \dfrac{2}{(t_1 + t_2)} = \pm\dfrac{2}{r}$

29. Here $a = 1$
The equation of normal to the parabola $y^2 = 4x$
is $\quad y = mx - 2am - am^3$
$\Rightarrow \quad y = mx - 2m - m^3$
which is passing through (9, 6).
$\Rightarrow \quad 6 = 9m - 2m - m^3$
$\Rightarrow \quad m^3 - 7m + 6 = 0$
$\Rightarrow \quad m^3 - m^2 + m^2 - m - 6m + 6 = 0$
$\Rightarrow \quad m^2(m - 1) + m(m - 1) - 6(m - 1) = 0$
$\Rightarrow \quad (m - 1)(m^2 + m - 6) = 0$
$\Rightarrow \quad (m - 1)(m - 2)(m + 3) = 0$
$\Rightarrow \quad m = 1, 2, -3$
Thus, the equation of normal can be
$y = x - 3, y = 2x - 12, y + 3x - 33 = 0$.

30.



Clearly, $\dfrac{\Delta LPM}{\Delta ABC} = 2$

$$\Rightarrow \quad \frac{\Delta_1}{\Delta_2} = 2$$

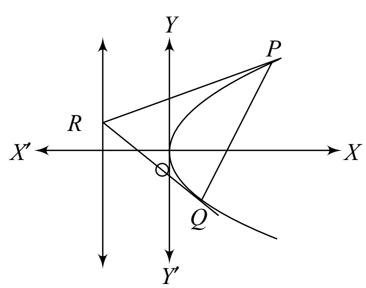
31. The parabola is $x = 2t^2, y = 4t$
Solving it with the circle, we get
$4t^4 + 16t^2 - 4t^2 - 16t = 0$
$\Rightarrow \quad t^4 + 3t^2 - 4t = 0$
$\Rightarrow \quad t(t^3 + 3t - 4) = 0$
$\Rightarrow \quad t = 0, 1$
So, the points P and Q are (0, 0) and (2, 4), respectively which are also diametrically opposite points on the circle.
The focus is $S = (2, 0)$

$$\text{The area of } \Delta PQS = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 2 & 4 & 1 \\ 2 & 0 & 1 \end{vmatrix}$$

$$= \frac{1}{2} \times 2 \times 4 = 4$$

32. Let $P = (at^2, 2at), Q = \left(\dfrac{a}{t^2}, -\dfrac{2a}{t}\right)$

and $R = \left(-a, a\left(t - \dfrac{1}{t}\right)\right)$



Since R lies on $y = 2x + a$, so

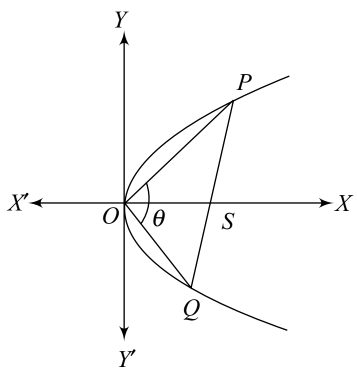
$$a\left(t - \frac{1}{t}\right) = -a$$

$$\Rightarrow \quad \left(t - \frac{1}{t}\right) = -1$$

$$\Rightarrow \quad \left(t + \frac{1}{t}\right)^2 = \left(t - \frac{1}{t}\right)^2 + 4 = 1 + 4 = 5$$

Thus, $PQ = a\left(t + \frac{1}{t}\right)^2 = 5a$

33. Here, $P = (at^2, 2at), Q = \left(\frac{a}{t^2}, -\frac{2a}{t}\right)$



Now, $t - \frac{1}{t} = -1$

$$\Rightarrow \quad \left(t + \frac{1}{t}\right)^2 = 1 + 4 = 5$$

$$\Rightarrow \quad \left(t + \frac{1}{t}\right) = \sqrt{5}$$

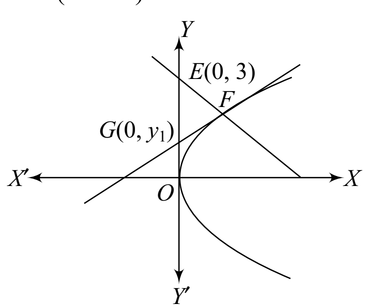
$$\tan\theta = \left(\frac{\frac{2}{t} + 2t}{1 - 4}\right) = \frac{2\left(t + \frac{1}{t}\right)}{-3} = -\frac{2\sqrt{5}}{3}$$

34. The tangent at $F(4t^2, 8t)$, is

$yy_1 = 8(x + x_1)$

$\Rightarrow \quad y \cdot 8t = 8(x + 4t^2)$

$\Rightarrow \quad y \cdot t = (x + 4t^2)$



Put $x = 0$, then $y = 4t$

Thus, *pt* G is $(0, 4t)$

Now,

$$ar(\Delta EFG) = \frac{1}{2}\begin{vmatrix} 0 & 3 & 1 \\ 0 & 4t & 1 \\ 4t^2 & 8t & 1 \end{vmatrix}$$

$$= \frac{1}{2}[4t^2(3 - 4t)]$$

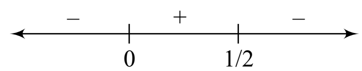
$$= 2t^2(3 - 4t)$$

$$= (6t^2 - 8t^3)$$

$$\Rightarrow \quad \frac{dA}{dt} = 12t - 24t^2 = 12t(1 - 2t)$$

For maximum or minimum,

$$\frac{dA}{dt} = 0 \text{ gives } t = 0, 1/2$$



So, $t = 1/2$ has a point of local maxima.

Thus, $G = (0, 4t) = (0, 2) \Rightarrow y_1 = 2$

$F = (x_0, y_0) = (4t^2, 8t) = (1, 4) \Rightarrow y_0 = 4$

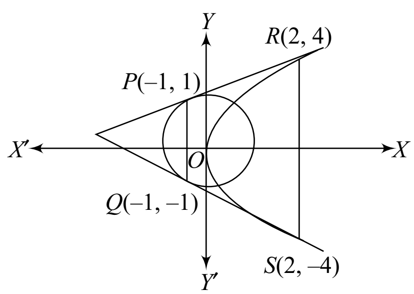
$$\text{Area} = 2\left(\frac{3}{4} - \frac{1}{2}\right) = 2 \times \frac{1}{4} = \frac{1}{2}$$

So, $y = mx + 3$ passes through $(1, 4)$.

Thus, $m = 1$.

35. The equation of any tangent to the parabola can be considered as

$$y = mx + \frac{a}{m} = mx + \frac{2}{m}$$



i.e. $\quad m^2x - my + 2 = 0$

As we know that the length of the perpendicular drawn from the centre to the tangent to the circle is equal to the radius of a circle.

Thus, $\dfrac{2}{\sqrt{m^4 + m^2}} = \sqrt{2}$

$\Rightarrow \quad m^4 + m^2 = 2$

$\Rightarrow \quad m^4 + m^2 - 2 = 0$

$\Rightarrow \quad (m^2 + 2)(m^2 - 1) = 0$

$\Rightarrow \quad m = \pm 1$

Hence, the equation of the tangents are

$y = x + 2, y = -x - 2$

Therefore, the points P, Q are $(-1, 1), (-1, -1)$ and R, S are $(2, 4)$ and $(2, -4)$ respectively.

Thus, the area of the equadrilateral $PQRS$

$$= \frac{1}{2} \times (2 + 8) \times 3 = 15$$

36. Given $P = (at^2, 2at)$

Since PQ is a focal chord, so the co-ordinates of Q are $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$.

Also, $R = (ar^2, 2ar)$, $S = (as^2, 2as)$

and $K = (2a, 0)$

It is given that,

$m(PK) = m(QR)$

$$\Rightarrow \quad \frac{0 - 2at}{2a - at^2} = \frac{2ar + \frac{2a}{t}}{ar^2 - \frac{a}{t^2}}$$

$$\Rightarrow \quad \frac{t}{t^2 - 2} = \frac{r + \frac{1}{t}}{r^2 - \frac{1}{t^2}}$$

$$\Rightarrow \quad \frac{t}{t^2 - 2} = \frac{r + \frac{1}{t}}{\left(r + \frac{1}{t}\right)\left(r - \frac{1}{t}\right)}$$

$$\Rightarrow \quad \frac{t}{t^2 - 2} = \frac{1}{\left(r - \frac{1}{t}\right)}$$

$$\Rightarrow \quad \left(r - \frac{1}{t}\right)t = (t^2 - 2)$$

$$\Rightarrow \quad rt - 1 = (t^2 - 2)$$

$$\Rightarrow \quad r = \left(\frac{t^2 - 1}{t}\right) = \left(t - \frac{1}{t}\right)$$

37. Now, $S = (as^2, 2as) = \left(\frac{a}{t^2}, \frac{2a}{t}\right)$

Tangent at P, $y \cdot t = x + at^2$ ...(i)

Tangent at S, $yy_1 = 2a(x + x_1)$

$$\Rightarrow \quad y \cdot \frac{2a}{t} = 2a\left(x + \frac{a}{t^2}\right)$$

$$\Rightarrow \quad y \cdot = t\left(x + \frac{a}{t^2}\right)$$

Normal at S, $y - \frac{2a}{t} = -\frac{1}{t}\left(x - \frac{a}{t^2}\right)$

$$\Rightarrow \quad y \cdot t - 2a = -\left(x - \frac{a}{t^2}\right)$$

$$\Rightarrow \quad y \cdot t = -\left(x - \frac{a}{t^2}\right) + 2a \qquad \text{...(ii)}$$

Solving (i) and (ii), we get

$$\Rightarrow \quad yt - at^2 = \frac{a}{t^2} + 2a - yt$$

$$\Rightarrow \quad 2yt = at^2 + \frac{a}{t^2} + 2a$$

$$\Rightarrow \quad 2yt = a\left(t^2 + \frac{1}{t^2} + 2\right)$$

$$\Rightarrow \quad 2yt = a\left(t + \frac{1}{t}\right)^2$$

$$\Rightarrow \quad y = \frac{a}{2t}\left(t + \frac{1}{t}\right)^2 = \frac{a(t^2 + 1)^2}{2t^3}$$

38. Image of $y = -5$ about the line $x + y + 4 = 0$ is $x = 1$

Hence, the required distance $AB = 4$

39. Equation of normals are

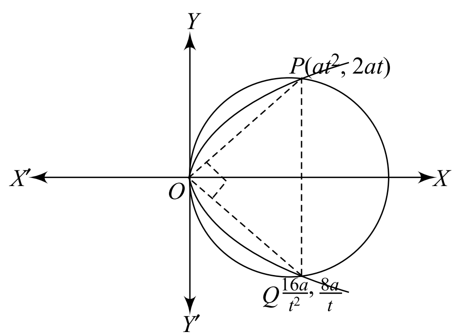
$x + y = 3$ and $x - y = 3$

Hence, the distance from $(3, -2)$ on both the normals is r

Thus, $\left|\frac{3 - 2 - 3}{\sqrt{2}}\right| = r$

$$\Rightarrow \quad r^2 = 2$$

40.



Clearly, $P = (at^2, 2at)$ and $Q = \left(\frac{16a}{t^2}, -\frac{8a}{t}\right)$

Area of the triangle $OPQ = 3\sqrt{2}$

$$\frac{1}{2} \cdot OP \cdot OQ = 3\sqrt{2}$$

$$\frac{1}{2}\left|at\sqrt{t^2 + 4} \times \frac{-4a}{t}\sqrt{\frac{16}{t^2} + 4}\right| = 3\sqrt{2}$$

$$t^2 - 3\sqrt{2}\,t + 4 = 0$$

$$P = (at^2, 2at) = \left(\frac{t^2}{2}, t\right)$$

when $t = \sqrt{2}$, $P = (1, \sqrt{2})$

when $t = 2\sqrt{2}$, $P = (4, 2\sqrt{2})$

41. Equation of tangent at $P(\sqrt{2}, 1)$ is

$\sqrt{2}\,x + y = 3$

If centre of $C2$ at $(0, \alpha)$ and the radius equal to $2\sqrt{3}$

$$\Rightarrow \quad 2\sqrt{3} = \left|\frac{\alpha - 3}{\sqrt{3}}\right|$$

$$\Rightarrow \quad \alpha = -3, 9$$

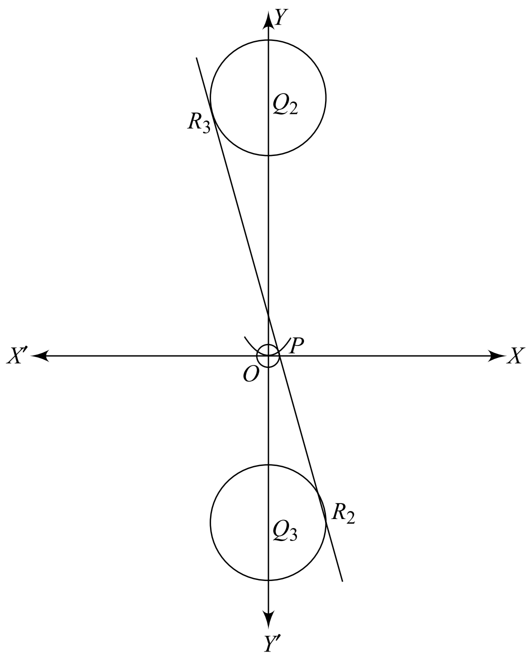
(a) $Q_2Q_3 = 12$

(b) R_2R_3 = length of the transverse common tangent

$$= \sqrt{(Q_2Q_3)^2 - (r_1 + r_2)^2}$$

$$= \sqrt{(12)^2 - (2\sqrt{3} + 2\sqrt{3})^2}$$

$$= 4\sqrt{6}$$



(c) Area of ΔOR_2R_3

$$= \frac{1}{2} \times R_2R_3 \times \perp^r \text{ distance from } O \text{ to the line}$$

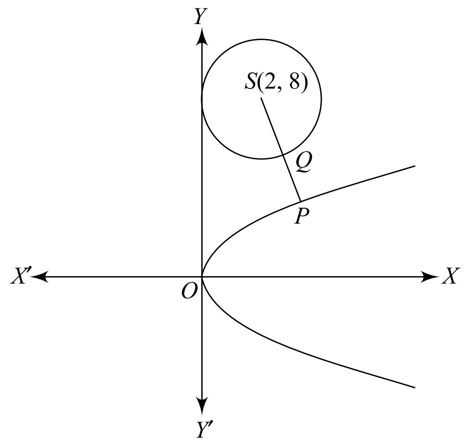
$$= \frac{1}{2} \times 4\sqrt{6} \times \sqrt{3} = 6\sqrt{2}$$

(d) $ar(\Delta PQ_2Q_3)$

$$= \frac{1}{2} \times 12 \times \sqrt{2} = 6\sqrt{2}$$

42. Equation of normal of parabola is

$$y + tx = 2t + t^3$$



Normal passes through $S(2, 8)$

$$8 + 2t = 2t + t^3$$

$$t^3 = 8$$

$$t = 2$$

Hence, $P = (4, 4)$ and SQ = radius = 2

CHAPTER

5 Ellipse

CONCEPT BOOSTER

1. INTRODUCTION

An oval is generally regarded as any ovum (egg)-shaped smooth, convex closed curve. The word convex means any chord connecting two points of the curve lies completely within the curve, and smooth means that the curvature does not change rapidly at any point. The ellipse is a typical oval, but a very particular one with a shape that is regular and can be exactly specified.

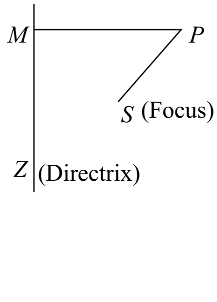
It has two diameters at right angles that are lines of symmetry. It is best to reserve the word ellipse for real ellipses, and to call others ovals. A diameter is any chord through the centre of the ellipse. The diameters that are lines of symmetry are called the major axis ($2a$), and the minor axis ($2b$), where $a > b$. If $a = b$, we have the very special ellipse, the circle, which has enough special properties that it should be distinguished from an ellipse, though, of course, it has all the properties of an ellipse in addition to its own remarkable properties.

A vertex of a curve is a point of maximum or minimum radius of curvature. An ellipse has vertices at the ends of the major axis (minimum) and at the ends of the minor axis (maximum).

2. MATHEMATICAL DEFINITIONS

Definition 1

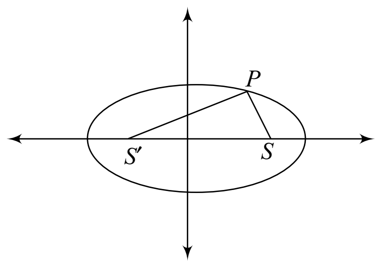
It is the locus of a point which moves in a plane in such a way that its distance from a fixed point is a constant ratio from a fixed straight line. This ratio is always less than 1. This fixed point is called the focus and the fixed straight line is called the directrix. The constant ratio is called the eccentricity.



Definition 2

An ellipse is the set of all points in a plane, the sum of whose distances from two fixed points is a constant, i.e.

$$SP + S'P = 3a$$



Definition 3

A conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents an ellipse if

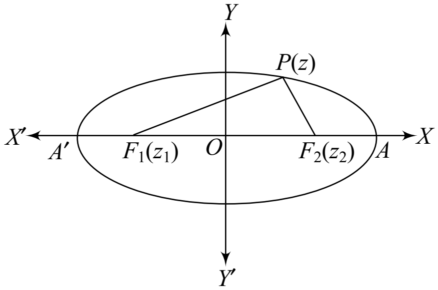
(i) $\Delta \neq 0$ and

(ii) $h^2 - ab < 0$, where

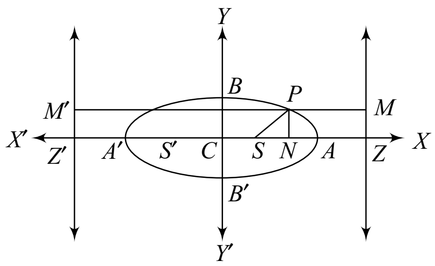
$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Definition 4

Let z, z_1 and z_3 be three complex numbers such that $|z - z_1| + |z - z_2| = k$, where $k > |z_1 - z_2|$ and k be a positive real number, the locus of z is an ellipse.



3. Equation of an Ellipse



Let S be the focus and ZM be the directrix of the ellipse. Draw $SZ \perp ZM$. Divide SZ internally and externally in the ratio $e : 1$ $(e < 1)$ and let A and A' be the internal and external point of division.

Then $SA = e\,AZ$ …(i)

and $SA' = e\,A'Z$ …(ii)

Clearly A and A' will lie on the ellipse.

Let $AA' = 2a$ and take C be the mid-point of AA' as origin.

Thus, $CA = CA' = a$ …(iii)

Let $P(x, y)$ be any point on the ellipse referred to CA and CB as co-ordinate axes.

Adding Eqs (i) and (ii), we get

$$SA + SA' = e(AZ + A'Z)$$

$$\Rightarrow \quad AA' = e(CZ - CA + CA' + CZ)$$

$$\Rightarrow \quad AA' = e(2CZ)$$

$$\Rightarrow \quad 2a = 2eCZ$$

$$\Rightarrow \quad CZ = \frac{a}{e}$$

Thus the directrix ZM is $x = CZ = \frac{a}{e}$.

Again subtracting Eqs (i) from (ii), we get

$$SA - SA' = e(A'Z - AZ)$$

$$\Rightarrow \quad (CA' + CS) - (CA - CS) = e(AA')$$

$$\Rightarrow \quad 2CS = e(AA')$$

$$\Rightarrow \quad 2CS = e(2a)$$

$$\Rightarrow \quad CS = ae$$

Thus the focus is $S(CS, 0) = S(ae, 0)$

Now draw $PM \perp ZM$,

$$\frac{SP}{PM} = e$$

$$\Rightarrow \quad SP^2 = e^2 PM^2$$

$$\Rightarrow \quad (x - ae)^2 + y^2 = e^2\left(\frac{a}{e} - x\right)^2$$

$$\Rightarrow \quad x^2 + a^2e^2 - 2aex + y^2 = a^2 - 2aex + e^2x^2$$

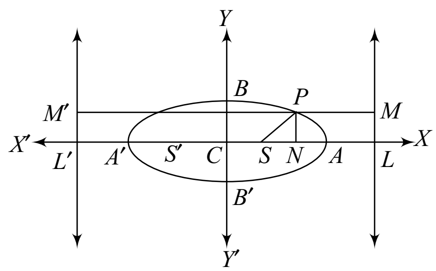
$$\Rightarrow \quad x^2(1 - e^2) + y^2 = a^2(1 - e^2)$$

$$\Rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

$$\Rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad b^2 = a^2(1 - e^2), \ldots$$

This is the standard equation of an ellipse.

4. Properties of an Ellipse



Centre

A point inside the ellipse which is the mid-point of the line segment linking the two foci, i.e. the intersection of the major and minor axes. Here $C = (0, 0)$.

Major/minor axis

The longest and the shortest diameters of an ellipse are known as the major axis and the minor axis respectively. The length of the major axis is equal to the sum of the two generator lines.

Here, major axis $= AA' = 2a$,

and minor axis $= BB' = 2b$

Semi-major/Half the major axis

The distance from the centre to the farthest point on the ellipse is known as the semi-major axis.

Semi-minor axis/Half the minor axis

The distance from the centre to the closest point on the ellipse is known as the semi-minor axis.

Directrices

LM and $L'M'$ are two directrices of the ellipse.

Thus LM: $x = \frac{a}{e}$ and $L'M'$: $x = -\frac{a}{e}$.

The distance between two directrices: $LL' = \frac{2a}{e}$

Foci (Focus points)

The two points that define the ellipse is known as the foci.

Here $S = (ae, 0)$ and $S' = (-ae, 0)$

Distance between two foci: $SS' = 2ae$

Perimeter

The perimeter is the distance around the ellipse, i.e.

Perimeter $= \pi \times \left[\frac{3}{2}(a + b) + \sqrt{ab}\right]$.

It is not easy to calculate.

Area

The number of square units it takes to fill the region inside an ellipse is called the area of an ellipse.

If the equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then its area is πab.

If the equation of ellipse is

$$Ax^2 + Bxy + Cy^2 = 1, \text{ then its area is } \frac{2\pi}{\sqrt{4AC - B^2}}.$$

Chord

A line segment linking any two points on an ellipse is known as the chord of the ellipse.

Focal chord

A chord of the ellipse passing through its focus is called the focal chord.

Focal distances

Let $P(x, y)$ be any point on the ellipse.

Here, $SP = ePM = e\left(\frac{a}{e} - x\right) = a - ex$

$$S'P = ePM' = e\left(\frac{a}{e} + x\right) = a + ex$$

Now, $SP + S'P = a - ex + a + ex = 2a$ = constant.

Thus the sum of the focal distances of a point on the ellipse is constant.

> ***Notes***
>
> Focal distances are also known as *focal radii* of the ellipse.

Vertices

The vertices of the ellipse are the points where the ellipse meets its major axis.

Here, $A = (a, 0)$ and $A' = (-a, 0)$ are the vertices of the ellipse.

Co-vertices

The co-vertices of the ellipse are the points where the ellipse meets its minor axis.

Here, $B = (0, b)$ and $B' = (0, -b)$ are the co-vertices of the ellipse.

Double ordinate

It is a chord perpendicular to the major axis and intersects the curve in two distinct points.

Latus rectum

It is a double ordinate, perpendicular to the major axis and passes through the foci. Here LSL' and L_1SL_1' are two latus recta.

Length of the LR

Let the co-ordinates of L and L' be (ae, y_1) and $(ae, -y_1)$.

Since L lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, so we have

$$\frac{a^2e^2}{a^2}+\frac{y_1^2}{b^2}=1$$

$\Rightarrow$ $y_1^2 = b^2(1-e^2) = b^2\left(\frac{b^2}{a^2}\right) = \frac{b^4}{a^2}$

$\Rightarrow$ $y_1 = b^2/a$

Thus the co-ordinates of L and L' are

$$\left(ae, \frac{b^2}{a}\right) \text{ and } \left(ae, -\frac{b^2}{a}\right).$$

Hence the length of the latus rectum,

$$LL' = \frac{2b^2}{a}.$$

Relation amongst a, b, and e:

$$b^2 = a^2(1 - e^2)$$

Eccentricity (e)

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{\frac{a^2 - b^2}{a^2}}$$

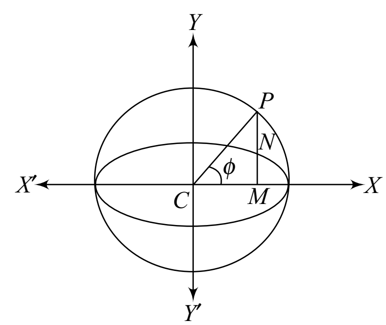
Auxiliary circle

The circle described on the major axis of an ellipse as diameter is called an auxilliary circle.

Relation to a circle

A circle is actually a special case of an ellipse. In an ellipse, if you make the major and minor axes of the same length, the result is a circle, with both foci at the center C.

5. Parametric Equation of an Ellipse



Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The equation of its auxilliary circle is $x^2 + y^2 = a^2$.

Let Q be a point on the auxilliary circle $x^2 + y^2 = a^2$ such that QP produced is perpendicular to the x-axis.

Thus P and Q are the corresponding points on the ellipse and the auxilliary circle.

Let $\angle QCA = \varphi$, where $0 \le \varphi < 2\pi$

Let $Q = (a \cos \varphi, a \sin \varphi)$ and $P = (a \cos \varphi, y)$.

Since P lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, so we can write

$$\frac{a^2\cos^2\varphi}{a^2}+\frac{y^2}{b^2}=1$$

$\Rightarrow$ $y^2 = b^2 \sin^2 \varphi$

$\Rightarrow$ $y = b \sin \varphi$

since P lies in the 1st quadrant.

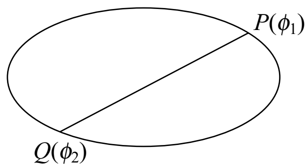
Thus the parametric equations of the ellipse are

$$x = a \cos \varphi, y = b \sin \varphi.$$

> ***Notes***
>
> Any point, say P, on the ellipse can be considered as $(a \cos \varphi, b \sin \varphi)$. Since the point is known when ϕ is given, then it is often called 'the point ϕ' or $P(\varphi)$..

6. Important Properties Related to Chord and Focal Chord

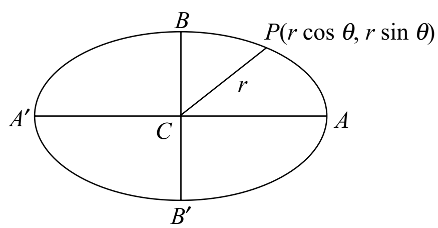
(i) Equation of the chord joining the points $P(\varphi_1)$ and $Q(\varphi_2)$



The equation of the chord joining the points $P(a \cos \varphi_1, b \sin \varphi_1)$ and $Q(a \cos \varphi_2, b \sin \varphi_2)$ is

$$\frac{x}{a}\cos\left(\frac{\varphi_1+\varphi_2}{2}\right)+\frac{y}{b}\sin\left(\frac{\varphi_1+\varphi_2}{2}\right)$$
$$=\cos\left(\frac{\varphi_1-\varphi_2}{2}\right)$$

(ii) The length of a radius vector from the centre drawn in a given direction



As we know that the equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

Put $x = r \cos \theta$, $y = r \sin \theta$, we have

$$\frac{r^2\cos^2\theta}{a^2}+\frac{r^2\sin^2\theta}{b^2}=1.$$

$$\Rightarrow \quad r^2=\frac{a^2b^2}{b^2\cos^2\theta+a^2\sin^2\theta}$$

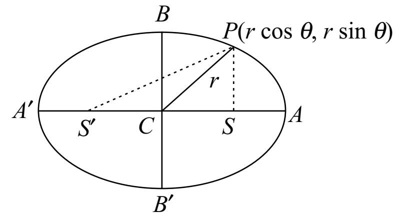
$$\Rightarrow \quad r=\frac{ab}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}$$

which is the required distance from the centre of the point $P(\theta)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

(iii) Product of the focal radii of an ellipse from any point $P(\theta)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$a^2 \sin^2 \theta + b^2 \cos^2 \theta.$$

Proof



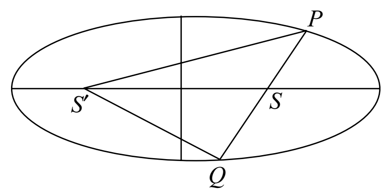
We have, $SP = a - ex$ and $S'P = a + ex$.

Thus,

$$SP.\, S'P = (a^2 - e^2x^2) = (a^2 - a^2e^2 \cos^2 \theta)$$
$$= a^2 + (b^2 - a^2) \cos^2 \theta$$
$$= a^2(1 - \cos^2 \theta) + b^2 \cos^2 \theta$$
$$= a^2 \sin^2 \theta + b^2 \cos^2 \theta$$

(iv) If PQ be a focal chord and S, S' are the foci of an ellipse, the perimeter of the triangle described by $\Delta S'PQ$ is $4a$.

Proof

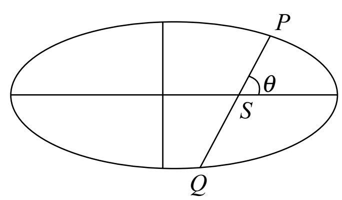


We have, perimeter of the D$S'PQ$

$$= S'P + S'Q + PQ$$
$$= (S'P + S'Q) + (SP + SQ)$$
$$= (S'P + SP) + (S'Q + SQ)$$
$$= 2a + 2a = 4a$$

(v) The length of the focal chord of an ellipse which makes an angle θ with the major axis is $\frac{2ab^2}{a^2\sin^2\theta+b^2\cos^2\theta}$.

Proof



Let the chord be PQ, where $P = (x_1, y_1)$, $Q = (x_2, y_2)$ and $S(ae, 0)$ be the focus.

The chord PQ be

$$(y - 0) = \tan \theta\, (x - ae) \qquad \ldots\text{(i)}$$

Now $PQ = SP + SQ$

$$= a - ex_1 + a - ex_2 = 2a = e(x_1 + x_2)$$

Let the equation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \ldots\text{(ii)}$$

Now $PQ = SP + SQ$

$$= a - ex_1 + a - ex_2$$
$$= 2a - e(x_1 + x_2)$$

From Eqs (i) and (ii), we get

$$\frac{x^2}{a^2}+\frac{\tan^2\theta(x-ae)^2}{b^2}=1$$

$$\Rightarrow \quad b^2x^2 + a^2\tan^2\theta(x-ae)^2 = a^2b^2$$

$$\Rightarrow \quad b^2x^2 + a^2\tan^2\theta(x^2-2aex+a^2e^2) = a^2b^2$$

$$\Rightarrow \quad (b^2+a^2\tan^2\theta)x^2 - 2a^3e\tan^2\theta\, x + a^2(a^2e^2\tan^2\theta - b^2) = 0$$

Let its roots are x_1 and x_2.

Then $x_1+x_2=\dfrac{2a^3e\tan^2\theta}{b^2+a^2\tan^2\theta}$

Therefore, $PQ = 2a - e(x_1+x_2)$

$$= 2a - e\left(\frac{2a^3e\tan^2\theta}{b^2+a^2\tan^2\theta}\right)$$

$$= 2a\left(\frac{b^2+a^2\tan^2\theta-a^2e^2\tan^2\theta}{b^2+a^2\tan^2\theta}\right)$$

$$= 2a\left(\frac{b^2+a^2(1-e^2)\tan^2\theta}{b^2+a^2\tan^2\theta}\right)$$

$$= 2a\left(\frac{b^2+b^2\tan^2\theta}{b^2+a^2\tan^2\theta}\right)$$

$$= 2a\left(\frac{b^2(1+\tan^2\theta)}{b^2+a^2\tan^2\theta}\right)$$

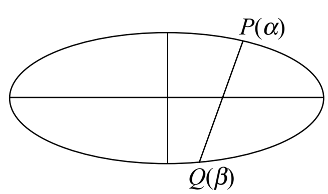
$$= \frac{2ab^2}{a^2\sin^2\theta+b^2\cos^2\theta}$$

Hence the result.

(vi) If $P(\alpha)$ and $P(\beta)$ are the extremities of a focal chord, then

$$\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)=\frac{e-1}{e+1}.$$

Proof



Let the equation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Equation of the chord PQ is

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$$

which is passing through the focus $(ae, 0)$, then

$$e\cos\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\Rightarrow \quad \frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}=e$$

$$\Rightarrow \quad \frac{\cos\left(\frac{\alpha-\beta}{2}\right)+\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)-\cos\left(\frac{\alpha+\beta}{2}\right)}=\frac{e-1}{e+1}$$

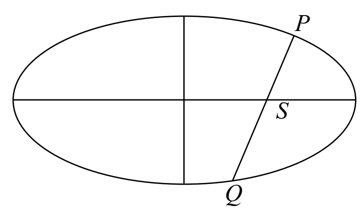
$$\Rightarrow \quad \tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)=\frac{e-1}{e+1}$$

Hence, the result.

(vii) If α and β are the eccentric angles of the extremities of a focal chord of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, then the eccentricity of the ellipse is

$$\frac{\sin\alpha+\sin\beta}{\sin(\alpha+\beta)}$$

Proof



The equation of the chord joining the points $P(\alpha)$ and $P(\beta)$ is

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$$

which is passing through $(ae, 0)$,

We have,

$$e\cos\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\Rightarrow \quad e\times 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\Rightarrow \quad e\times\sin(\alpha+\beta)=\sin\alpha+\sin\beta$$

$$\Rightarrow \quad e=\frac{\sin\alpha+\sin\beta}{\sin(\alpha+\beta)}$$

Hence, the result.

7. Position of a Point with Respect to an Ellipse

The point (x_1, y_1) lies outside, on or inside the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ according as

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 >, =, < 0$$

8. Intersection of a Line and an Ellipse

The line $y = mx + c$ intersects the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ into two

(i) real and distinct points if $x^2 < a^2m^2 + b^2$.

(ii) coincident points if $c^2 = a^2m^2 + b^2$.

(iii) imaginary points if $c^2 > a^2m^2 + b^2$.

Also

(iv) The line $y = mx + c$ will be a tangent to the given ellipse if

$$c^2 = a^2m^2 + b^2.$$

(v) The co-ordinates of the point of contact is $\left(\pm\frac{a^2m}{c}, \pm\frac{b^2}{c}\right)$ which is also known as the m-point on the ellipse.

(vi) The equation of any tangent to the ellipse can be considered as $y = mx + \sqrt{a^2m^2 + b^2}$.

(vii) The line $lx + my + n = 0$ will be a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if $a^2l^2 + b^2m^2 = n^2$

9. The Length of the Chord Intercepted by the Ellipse on the Line $y = mx + c$.

Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow \quad \frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1$$

$$\Rightarrow \quad (a^2m^2 + b^2)x^2 + 2a^2mcx + a^2(c^2 - b^2) = 0$$

Let its roots are x_1, x_2.

Then $x_1 + x_2 = -\frac{2a^2mc}{a^2m^2 + b^2}$ and $x_1 \cdot x_2 = \frac{a^2(c^2 - b^2)}{a^2m^2 + b^2}$.

Thus, $x_1 - x_2 = \frac{2ab\sqrt{a^2m^2 + b^2 - c^2}}{a^2m^2 + b^2}$.

Hence the length of the chord

$$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$= (x_1 - x_2)\sqrt{1 + m^2}$$

$$= \frac{2ab \times \sqrt{1 + m^2}\sqrt{a^2m^2 + b^2 - c^2}}{a^2m^2 + b^2}$$

10. Various Forms of Tangents

(i) Point form

The equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

(ii) Parametric form

The equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $(a \cos \theta, b \sin \theta)$ is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

(iii) Slope form

The equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in terms of the slope m is

$$y = mx + \sqrt{a^2m^2 + b^2}$$

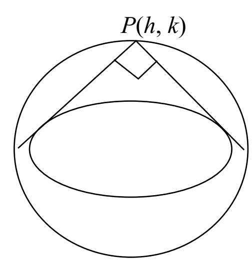
(iv) The co-ordinates of the points of contact are

$$\left(\mp\frac{a^2m}{\sqrt{a^2m^2 + b^2}}, \pm\frac{b^2}{\sqrt{a^2m^2 + b^2}}\right).$$

(v) The point of intersection of the tangents at $P(\theta)$ and $Q(\varphi)$ is

$$\left(\frac{a\cos\left(\frac{\theta+\varphi}{2}\right)}{\cos\left(\frac{\theta-\varphi}{2}\right)}, \frac{b\sin\left(\frac{\theta+\varphi}{2}\right)}{\cos\left(\frac{\theta-\varphi}{2}\right)}\right)$$

11. Director Circle



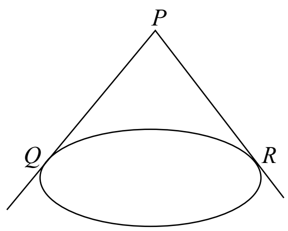
The locus of the point of intersection of two perpendicular tangents to an ellipse is known as the director circle.

The equation of the director circle to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $x^2 + y^2 = a^2 + b^2$.

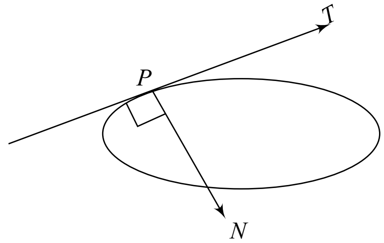
12. Pair of Tangents

Equation of a pair of tangents from a point (x_1, y_1) to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right)^2$$



13. Various Forms of Normals



Here, PT be a tangent and PN be a normal.
The angle between the tangent and the normal is 90°.

(i) **Point form**

The equation of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1, y_1) is

$$\frac{a^2 x}{x_1}-\frac{b^2 y}{y_1}=a^2-b^2 .$$

(ii) **Parametric form**

The equation of the normal to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $(a\cos\theta, b\sin\theta)$ is

$$ax\sec\theta - by\operatorname{cosec}\theta = a^2-b^2.$$

(iii) **Slope form**

The equation of the normal in terms of slope is

$$y = mx \mp \frac{m(a^2-b^2)}{\sqrt{a^2+b^2m^2}}$$

(iv) The line $y = mx + c$ is a normal to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if

$$c^2=\frac{m^2(a^2-b^2)^2}{(a^2+b^2m^2)}.$$

(v) The straight line $lx + my + n = 0$ is a normal to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if

$$\frac{a^2}{l^2}+\frac{b^2}{m^2}=\frac{(a^2-b^2)^2}{n^2}$$

14. Number of Normals are Drawn to an Ellipse From a Point to its Plane

The equation of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $P(\theta)$ is

$$ax\sec\theta - by\operatorname{cosec}\theta = a^2-b^2 \quad \ldots\text{(i)}$$

Let $Q(\alpha, \beta)$ be any point in the xy-plane.
Equation (i) passes through $Q(\alpha, \beta)$, so we have

$$a\alpha\sec\theta - b\beta\operatorname{cosec}\theta = a^2-b^2$$

$$\Rightarrow \quad a\alpha\left(\frac{1+\tan^2\left(\frac{\theta}{2}\right)}{1-\tan^2\left(\frac{\theta}{2}\right)}\right)-b\beta\left(\frac{1+\tan^2\left(\frac{\theta}{2}\right)}{2\tan\left(\frac{\theta}{2}\right)}\right)=a^2-b^2$$

$$\Rightarrow \quad 2a\alpha\left(1+\tan^2\left(\frac{\theta}{2}\right)\right)\tan\left(\frac{\theta}{2}\right)-b\beta\left(1-\tan^4\left(\frac{\theta}{2}\right)\right)$$

$$=2(a^2-b^2)\tan\left(\frac{\theta}{2}\right)\left(1-\tan^2\left(\frac{\theta}{2}\right)\right)$$

$$\Rightarrow \quad b\beta t^4+2(a^2-b^2+a\alpha)t^3-2(a^2-b^2-a\alpha)t-b\beta=0 \quad \ldots\text{(ii)}$$

where $t=\tan\left(\frac{\theta}{2}\right)$

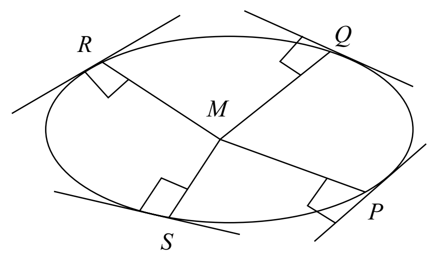
The above equation will give four values of t say, t_1, t_2, t_3, t_4.

Corresponding to these four values of t, we will get 4 points, say A, B, C, D on the ellipse, the normals to which pass through the point $Q(\alpha, \beta)$.

Hence, in general four normals can be drawn from any point to an ellipse.

(i) **Co-normal points**

Let P, Q, R, S are four points on the ellipse. If the normals at these points meet at a point, say M, then these four points are known as co-normal points.



(ii) If $\alpha, \beta, \gamma, \delta$ be the eccentric angles of the four points on the ellipse such that the normals at these points are concurrent, then $\alpha+\beta+\gamma+\delta=(2n+1)\pi$, $n \in 1$.

As we know that if four normals are concurrent at a point, say, $M(\alpha, \beta)$, then

$$b\beta t^2+2(a^2-b^2+a\alpha)t^3-2(a^2-b^2-a\alpha)t=b\beta=0,$$

where $t=\tan\left(\frac{\theta}{2}\right)$

Now, $\tan\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}+\frac{\delta}{2}\right)=\frac{S_1-S_3}{1-S_2+S_4}$

$$=\frac{\frac{-2[a\alpha+(a^2-b^2)]}{b\beta}+\frac{2[ab-(a^2-b^2)]}{b\beta}}{1-0-1}$$

$$=\infty$$

$$\Rightarrow \quad \cot\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}+\frac{\delta}{2}\right)=0$$

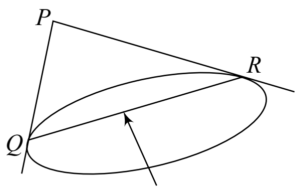
$$\Rightarrow \quad \cot\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}+\frac{\delta}{2}\right)=0=\cot\left(\frac{2n+1}{2}\right)\pi$$

$$\Rightarrow \quad \left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}+\frac{\delta}{2}\right)=\left(\frac{2n+1}{2}\right)\pi$$

$$\Rightarrow \quad \alpha+\beta+\gamma+\delta=(2n+1)\pi.$$

Hence, the result.

15. Chord of Contact

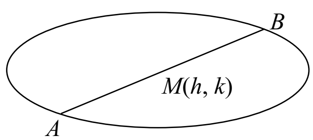


Chord of contact

The equation of the chord of contact of tangents drawn from a point (x_1, y_1) to an ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ is } \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$$

16. Chord Bisected at a Given Point



The equation of the chord bisected at a point (x_1, y_1) to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$$

i.e. $\quad T = S_1$

17. Pole and Polar

The equation of the polar of an ellipse

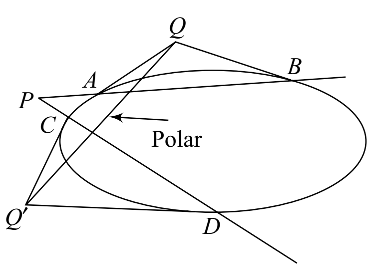
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

from a point (x_1, y_1) is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1,$$

where

(x_1, y_1) is the pole of polar.



Properties related to pole and polar

(i) The polar of the focus is the directrix.

(ii) Any tangent is the polar of the point of contact.

(iii) The pole of a line $lx + my + n = 0$ with respect to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\left(-\frac{a^2 l}{n}, -\frac{b^2 m}{n}\right)$.

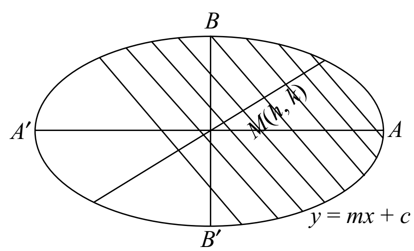
(iv) The pole of a given line is the same as the point of intersection of tangents at its extremities.

(v) If the polar of $P(x_1, y_1)$ passes through $Q(x_2, y_2)$, the polar of $Q(x_2, y_2)$ goes through $P(x_1, y_1)$ and such points are said to be **conjugate points**.

(vi) If the pole of a line $lx + my + n = 0$ lies on the another line $l'x + m'y + n' = 0$, the pole of the second line will lie on the first and such lines are said to be **conjugate lines**.

18. Diameter

The locus of the mid-points of a system of parallel chords of an ellipse is called a diameter and the point where the diameter intersects the ellipse is called the vertex of the diameter.



Let (h, k) be the mid-point of the chord, then $y = mx + c$ of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

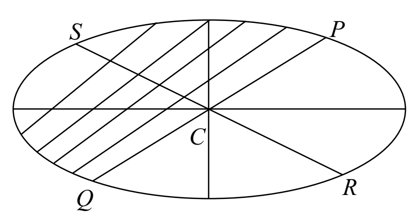
Then $\quad T = S_1$

$$\Rightarrow \quad \frac{xh}{a^2}+\frac{yk}{b^2}=\frac{h^2}{a^2}+\frac{k^2}{b^2}$$

$$\Rightarrow \quad k=-\frac{b^2 h}{a^2 m}$$

Hence, the locus of the mid-point is $y=-\frac{b^2 x}{a^2 m}$.

19. Conjugate Diameters



Two diameters are said to be conjugate when each bisects all chords parallel to the other.

If $y = m_1 x$ and $y = m_2 x$ be two conjugate diameters of an ellipse, then $m_1 m_2 = -\frac{b^2}{a^2}$.

Let PQ and RS be two conjugate diameters.

Then the co-ordinates of the four extremities of two conjugate diameters are

$P(a \cos \varphi, b \sin \varphi)$,

$Q(-a \cos \varphi, -b \sin \varphi)$,

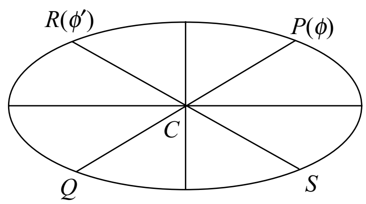
$S(-a \sin \varphi, b \cos \varphi)$,

and $R(a \sin \varphi, -b \cos \varphi)$

Properties of conjugate diameters

(i) Prove that the eccentric angles of the ends of a pair of conjugate diameters of an ellipse differ by a right angle.

Proof



Let PCQ and RCS be two conjugate diameters of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Then the co-ordinates are $P(a \cos \varphi, b \sin \varphi)$ and

$R(a \cos \varphi', a \sin \varphi')$.

Now, $m_1 = \text{Slope of } CP = \frac{b}{a}\tan\varphi$ and

$m_2 = \text{Slope of } CR = \frac{b}{a}\tan\varphi'$

Since the diameters PCQ and RCS are conjugate diameters, then

$$m_1 \cdot m_2 = -\frac{b^2}{a^2}$$

$$\Rightarrow \quad \frac{b^2}{a^2}\tan\varphi\tan\varphi' = -\frac{b^2}{a^2}$$

$$\Rightarrow \quad \tan\varphi\tan\varphi' = -1$$

$$\Rightarrow \quad \tan\varphi = -\cot\varphi' = \tan\left(\frac{\pi}{2}+\varphi'\right)$$

$$\Rightarrow \quad \varphi = \frac{\pi}{2}+\varphi'$$

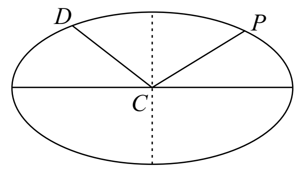
$$\Rightarrow \quad \varphi - \varphi' = \frac{\pi}{2}$$

Hence, the result.

(ii) Prove that the sum of the squares of any two conjugate semi-diameters of an ellipse is constant and equal to the sum of the squares of the semi-axes of the ellipse, i.e.

$$CP^2 + CD^2 = a^2 + b^2.$$

Proof



Let CP and CD be two conjugate semi-diameters of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the eccentric angle of P is ϕ.

Thus the eccentric angle of D is $\frac{\pi}{2}+\varphi$.

Therefore the co-ordinates of P and D are $(a \cos \varphi, b \sin \varphi)$ and

$$\left(a\cos\left(\frac{\pi}{2}+\varphi\right), b\sin\left(\frac{\pi}{2}+\varphi\right)\right)$$

i.e. $(-a \sin \varphi, b \cos \varphi)$.

Thus $CP^2 + CD^2$

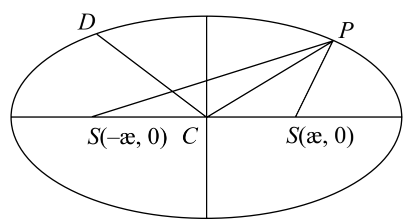
$= (a^2 \cos^2 \varphi + b^2 \sin^2 \varphi) + (a^2 \sin^2 \varphi + b^2 \cos^2 \varphi)$

$= a^2 + b^2$

Hence, the result

(iii) Prove that the product of the focal distances of a point on an ellipse is equal to the square of the semi-diameter, which is conjugate to the diameter through the point.

Proof



Let CP and CD be the conjugate diameters of the ellipse.

Let $P = (a \cos \varphi, b \sin \varphi)$, then the co-ordinates of D is $(-a \sin \varphi, b \cos \varphi)$.

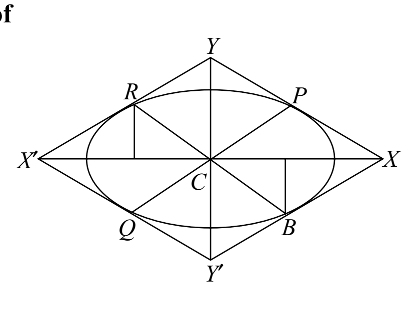
Thus,

$$\begin{aligned} SP \cdot S'P &= (a - ae\cos\varphi)\cdot(a + ae\cos\varphi) \\ &= a^2 - a^2e^2\cos^2\varphi \\ &= a^2 - (a^2 - b^2)\cos^2\varphi \\ &= a^2\sin^2\varphi + b^2\cos^2\varphi \\ &= CD^2 \end{aligned}$$

Hence, the result.

(iv) Prove that the tangent at the extremities of a pair of conjugate diameters form a parallelogram whose area is constant and equal to the product of the axes.

Proof



Let PCQ and RCS be two conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Then the co-ordinates of P, Q, R, and S are
$P(a\cos\varphi, b\sin\varphi)$, $Q(-a\cos\varphi, -b\sin\varphi)$,
$R(-a\sin\varphi, b\cos\varphi)$ and $S(a\sin\varphi, -b\cos\varphi)$ respectively.

Equations of tangents at P, R, Q and S are

$$\frac{x}{a}\cos\varphi+\frac{y}{b}\sin\varphi=1,$$

$$-\frac{x}{a}\sin\varphi+\frac{y}{b}\cos\varphi=1,$$

$$-\frac{x}{a}\cos\varphi-\frac{y}{b}\sin\varphi=1$$

and $$\frac{x}{a}\sin\varphi-\frac{y}{b}\cos\varphi=1$$

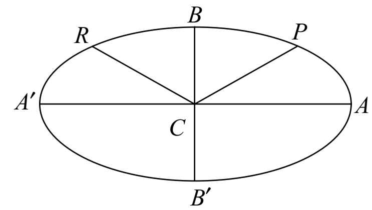
Thus, the tangents at P and Q are parallel.
Also the tangents at R and S are are parallel.
Hence, the tangents at P, R, Q, S form a parallelogram.
Area of the parallelogram = $MNM'N'$
= 4(the area of the parallelogram $CPMR$)

$$= 4\times\sqrt{a^2\cos^2\varphi+b^2\sin^2\varphi}\times\frac{ab}{\sqrt{a^2\cos^2\varphi+b^2\sin^2\varphi}}$$

$= 4ab$
= constant

Hence, the result.

(v) Equi-conjugate diameters

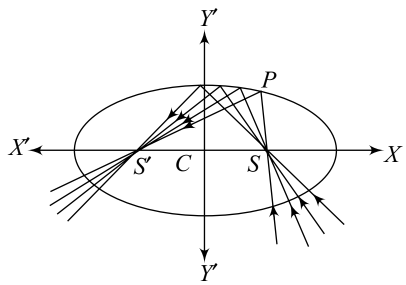


Two conjugate diameters are said to be equi-conjugate diameters if their lengths are equal.

i.e. $CP = CR = \sqrt{\frac{a^2+b^2}{2}}$

20. Reflection Property of an Ellipse

If an incoming light ray passes through one focus (S) strikes the concave side of the ellipse, it will get reflected towards other focus.



Exercises

Level I

(Problems based on Fundamentals)

ABC OF AN ELLIPSE

1. Find the centre, vertices, co-vertices, lengths of major and minor axes, eccentricity, lengths of latus rectum, equation of directrices and the end-points of a latus recta.
   (i) $9x^2 + 16y^2 = 144$
   (ii) $2x^2 + 3y^2 - 4x - 12y + 8 = 0$
2. Find the sum of the focal distances of any point on the ellipse $16x^2 + 25y^2 = 400$.
3. If the equation $\frac{x^2}{10-a}+\frac{y^2}{a-4}=1$ represents of an ellipse such that the length of the interval, where a lies, is m, find m.
4. If (5, 12) and (24, 7) are the foci of an ellipse passing through the origin, find the eccentricity of the ellipse.
5. Find the equation of the ellipse whose axes are co-ordinate axes and foci are ($\pm$2, 0) and the eccentricity is 1/2.
6. If the distance between the foci of an ellipse is equal to the length of its latus rectum, the eccentricity is $\frac{\sqrt{5}-1}{2}$.
7. Find the eccentricity of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose latus rectum is half of its major axis.
8. Find the eccentric angle of a point on the ellipse $\frac{x^2}{6}+\frac{y^2}{2}=1$ whose distance from the centre of the ellipse is $\sqrt{5}$.
9. Find the area of the greatest rectangle that can be inscribed in the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$.
10. Find the locus of a point whose co-ordinates are given by $x = 3 + 4\cos\theta\ cy = 2 + 3\sin\theta$.
11. If PSQ is a focal chord of an ellipse $16x^2 + 25y^2 = 400$ such that $SP = 8$, find the length of SQ.

12. Find the area bounded by the curve $\frac{x^2}{16}+\frac{y^2}{9}\le 1$ and the line $\frac{x}{4}+\frac{y}{3}\ge 1$.

POSITION OF A POINT W.R.T. AN ELLIPSE

13. Find the location of the point (2, 3) with respect to the ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$.
14. If $(\lambda, -\lambda)$ be an interior point of an ellipse $4x^2 + 5y^2 = 1$ such that the length of the interval, where λ lies, is m, where $m \in Q^+$, find the value of $(3m - 2)^{2013} + 2013$.

TANGENT AND TANGENCY

15. Find the number of tangents drawn from a point (2, 3) to an ellipse $4x^2 + 3y^2 = 12$.
16. Find the equations of the tangents drawn from the point (2, 3) to the ellipse $9x^2 + 16y^2 = 144$.
17. If the line $3x + 4y = 5$ touches the ellipse $9x^2 + 16y^2 = 144$, find the points of contact.
18. For what value of λ does the line $y = x + \lambda$ touches the ellipse $9x^2 + 16y^2 = 144$?
19. Find the equations of the tangents to the ellipse $\frac{x^2}{3}+\frac{y^2}{4}=1$ having slope 2.
20. Prove that the locus of the feet of perpendicular drawn from the centre upon any tangent to the given ellipse is $r^2 = a^2 \cos^2\theta + b^2 \sin^2\theta$.
21. A circle of radius r is concentric with the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Prove that the common tangent is inclined to the major axis at an angle of $\tan^{-1}\sqrt{\left(\frac{r^2-b^2}{a^2-r^2}\right)}$.
22. Prove that the tangents at the extremities of latus rectum of an ellipse intersect on the corresponding directrix.
23. Prove that the locus of the mid-points of the portion of the tangents to the given ellipse intercepted between the axes is $4r^2 = a^2 \sin^2\theta + b^2 \cos^2\theta$.
24. A tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the ellipse $\frac{x^2}{a}+\frac{y^2}{b}=a+b$ in the points P and Q, prove that the tangents at P and Q are at right angles.
25. Prove that the product of the perpendiculars drawn from the foci upon any tangent to an ellipse is constant, i.e. b^2.
26. If an ellipse slides between two perpendicular straight lines, prove that the locus of its centre is a circle.
27. Prove that the locus of the feet of the perpendiculars from foci upon any tangent to an ellipse is an auxiliary circle.
28. If p be the length of perpendicular from the focus S of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ on the tangent at P, prove that $\frac{b^2}{p^2}=\frac{2a}{SP}-1$.
29. If p be the perpendicular from the centre of an ellipse upon the tangent at any point P on it and r be the distance of P from the centre, prove that $\frac{a^2b^2}{p^2}=a^2+b^2-r^2$.
30. Prove that the locus of the mid-points of the portion of the tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ intercepted between the axes is $\frac{a^2}{x^2}+\frac{b^2}{y^2}=4$.
31. Prove that the portion of the tangent to the ellipse intercepted between the curve and the directrix subtends a right angle at the corresponding focus.
32. Prove that in an ellipse, the perpendicular from a focus upon any tangent and the line joining the centre of the ellipse to the point of contact meet on the corresponding directrix.
33. Prove that the co-ordinates of those points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ tangents at which make equal angles with the axes is $\left(\pm\frac{a^2}{\sqrt{a^2+b^2}}, \pm\frac{b^2}{\sqrt{a^2+b^2}}\right)$.
34. Find the locus of the point of intersection of two perpendicular tangents to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$.
35. Tangents are drawn from any point P on the parabola $(y-2)^2 = 4(x-1)$ to the ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$, which are mutually perpendicular to each other. Find the locus of the point P.
36. Find the equations of the pair of tangents to the ellipse $2x^2 + 3y^2 = 1$ from the point (1, 1).
37. If the tangents are drawn from a point (1, 2) to the ellipse $3x^2 + 2y^2 = 5$, find the angle between the tangents.

NORMAL AND NORMALCY

38. Find the equation of the normal to the ellipse $4x^2 + 9y^2 = 20$ at $\left(1, \frac{4}{3}\right)$.
39. Find the equation of the normal to the ellipse $5x^2 + 3y^2 = 137$ at the point whose ordinate is 2.
40. Find the equation of the normal to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ at the negative end of the latus rectum.

41. If the normal at the point $P(\theta)$ to the ellipse $5x^2 + 14y^2 = 70$ intersects it again at the point $Q(2\theta)$, prove that $3\cos\theta + 2 = 0$.

42. The normal at an end of a latus rectum of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ passes through one extremity of the minor axis, then prove that $e^4 + e^2 - 1 = 0$.

43. Prove that the tangent and the normal at any point of an ellipse bisect the the external and internal angles between the focal distances of the point.

44. The normal at any point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the major and minor axes at G and G', respectively and CF is perpendicular upon the normal from the centre C of the ellipse, show that $PF \cdot PG = b^2$ and $PF \times PG' = a^2$.

45. The normal at a point $P(\theta)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the axes of x and y at M and N, respectively, show that $x^2 + y^2 = (a + b)^2$.

46. An ordinate PN of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the auxiliary circle in Q. Show that the locus of the point of intersection of the normals at P and Q is the circle $x^2 + y^2 = (a + b)^2$.

47. Prove that the tangent of the angle between CP and the normal at $P(\theta)$ is $\left(\frac{a^2-b^2}{2ab}\right)\times \sin 2\theta$ and its greatest value is $\left(\frac{a^2-b^2}{2ab}\right)$.

48. Prove that in an ellipse, the distance between the centre and any normal does not exceed the difference between the semi-axes of the curve.

49. The tangent and the normal at any point P of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ cut its major axis in points Q and R, respectively. If $QR = a$, show that the eccentric angle of the point P is satisfying the equation

$$e^2\cos^2\varphi + \cos\varphi - 1 = 0$$

50. If the normals at $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are concurrent, show that

$$\begin{vmatrix} x_1 & y_1 & x_1y_1 \\ x_2 & y_2 & x_2y_2 \\ x_3 & y_3 & x_3y_3 \end{vmatrix} = 0$$

and if points $P(\alpha)$, $Q(\beta)$ and $R(\gamma)$, prove that

$$\begin{vmatrix} \sec\alpha & \operatorname{cosec}\alpha & 1 \\ \sec\beta & \operatorname{cosec}\beta & 1 \\ \sec\gamma & \operatorname{cosec}\gamma & 1 \end{vmatrix} = 0$$

CHORD OF CONTACT/CHORD BISECTED AT A POINT

51. Prove that the locus of the point, the chord of contact of tangents from which to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ subtends a right angle at the centre of the ellipse is $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2}+\frac{1}{b^2}$.

52. Prove that the locus of the point, from which the chord of contact of tangents are to be drawn to the ellipse touches the circle $x^2 + y^2 = c^2$ is $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{c^2}$.

53. The perpendicular tangents are drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Prove that the locus of the mid-point of the chord of contact is $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\left(\frac{x^2+y^2}{a^2+b^2}\right)$.

54. Tangents are drawn from any point on the circle $x^2 + y^2 = c^2$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Prove that the locus of the mid-points of the chord of contact is

$$\left(\frac{x^2}{c^2}+\frac{y^2}{b^2}\right)^2=\left(\frac{x^2+y^2}{c^2}\right)$$

55. Tangents PA and PB are drawn from a point P to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The area of the triangle formed by the chord of contact AB and axes of co-ordinates are constant. Prove that the locus of P is a hyperbola.

56. Prove that the locus of the mid-points of the normal chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$\left(\frac{a^6}{x^2}+\frac{b^6}{y^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=(a^2-b^2)^2$$

57. Prove that the locus of the mid-points of the chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which touch the auxiliary circle $\mathbf{x^2 + y^2 = a^2}$ is

$$a^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2$$

58. Prove that the locus of the mid-points of the chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which subtends a right angle at the centre of the ellipse is

$$\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$$

59 The eccentric angles of two points P and Q on the ellipse differ by $\pi/2$. Prove that the locus of the mid-point of PQ is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2$.

60. Prove that the locus of the mid-point of the chord of contact of the perpendicular tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\left(\frac{x^2+y^2}{a^2+b^2}\right)$.
61. Prove that the locus of the mid-points of the focal chords of the ellipse $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=1$ is
$$b^2x^2+a^2y^2=ab^2xe$$
62. Prove that the locus of the mid-points of the chords of the ellipse $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=1$ which are tangents to the ellipse $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$ is
$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\left(\frac{x^2\alpha^2}{a^4}+\frac{y^2\beta^2}{b^4}\right)$$

POLE AND POLAR

63. Prove that the polar of the focus of an ellipse is the directrix.
64. Find the pole of a given line $lx+my+n=0$ with respect to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.
65. Prove that the pole of a given line is the same as the point of intersection of tangents at its extremities.
66. Find the pole of the straight line $x+4y=4$ with respect to the ellipse $x^2+4y^2=4$.
67. Find the locus of the poles of the tangents to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with respect to the concentric ellipse $c^2x^2+d^2y^2=1$.
68. The perpendicular from the centre of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ on the polar of a point with respect to the ellipse is constant and equal to c. Prove that the locus of the point is the ellipse $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{c^2}$.
69. Show that the equation of the locus of the poles of normal chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is
$$(a^2-b^2)^2x^2y^2=a^6y^2+b^6x^2$$
70. If the polar with respect to $y^2=4x$ touches the ellipse $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$, find the locus of its pole.

CONJUGATE DIAMETERS

71. Prove that the equation of the diameter of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $y=-\frac{b^2x}{a^2m}$.
72. Find the co-ordinates of the four extremities of two conjugate diameters of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.
73. Prove that the sum of the squares of any two conjugate semi-diameters of an ellipse is constant and equal to the sum of the squares of the semi-axes of the ellipse, i.e. $CP^2+CD^2=a^2+b^2$.
74. Prove that the product of the focal distances of a point on an ellipse is equal to the square of the semi-diameter, which is conjugate to the diameter through the point. i.e. $SP\cdot S'P=CD^2$.
75. Prove that the locus of the poles of the line joining the eccentricities of two conjugate diameters is the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2$.
76. If P and D be the ends of the conjugate diameters of an ellipse, find the locus of the mid-point of PD.
77. For the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, find the equation of the diameter conjugate to $ax-by=0$.
78. If the point of intersection of the ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $\frac{x^2}{c^2}+\frac{y^2}{d^2}=1$ be at the extremities of the conjugate diameters of the former, prove that $\frac{a^2}{c^2}+\frac{b^2}{d^2}=1$.
79. If CP and CD are the conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, prove that the locus of the orthocentre of the ΔPCD is $2(b^2y^2+a^2x^2)^2=(a^2-b^2)^2(b^2y^2-a^2x^2)^2$.
80. Prove that the tangents at the extremities of a pair of conjugate diameters form a parallelogram whose area is constant and equal to the product of the axes.
81. Show that the tangents at the ends of conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ intersect on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2$.
82. Find the eccentricity of the ellipse if $y=x$ and $2x+3y=0$ are the equations of a pair of its conjugate diameters.

REFLECTION PROPERTY OF AN ELLIPSE

83. A ray is emanating from the point $(-3, 0)$ is incident on the ellipse $16x^2+25y^2=400$ at the point P with ordinate 4. Find the equation of the reflected ray after first reflection.
84. A ray is coming along the line $x-y+2=0$ on the ellipse $3x^2+4y^2=12$. After striking the elliptic mirror, it is then reflected. Find the equation of the line containing the reflected ray.

LEVEL II

(Mixed Problems)

1. Tangents are drawn from a point on the circle $x^2 + y^2 = 50$ to the ellipse $\frac{x^2}{30} + \frac{y^2}{20} = 1$, the tangents are at the angle is
   (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{8}$
2. The centre of the ellipse $4x^2 + 9y^2 - 8x - 36y + 4 = 0$ is
   (a) (2, 4) (b) (3, 2) (c) (1, 2) (d) (0, 1)
3. The co-ordinates of the foci of the ellipse $4x^2 + 9y^2 - 8x - 36y + 4 = 0$ is
   (a) $(1 \pm \sqrt{5}, 2)$ (b) $(2 \pm \sqrt{5}, 2)$
   (c) $(\pm\sqrt{5}, 2)$ (d) $(1 \pm \sqrt{5}, 3)$
4. The equation $\frac{x^2}{10 - a} + \frac{y^2}{4 - a} = 1$ represents an ellipse if
   (a) $a < 4$ (b) $a > 4$
   (c) $4 < a < 10$ (d) $a > 10$
5. Lep P be a variable point on the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ with foci at F and F'. If A be the area of the $\Delta PFF'$, the maximum value of A is
   (a) 12 s.u. (b) 24 s.u.
   (c) 36 s.u. (d) 48 s.u.
6. The eccentricity of the ellipse $(10x - 5)^2 + (10y - 5)^2 = (3x + 4y - 1)^2$ is
   (a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{1}{2}$ (d) $\frac{1}{4}$
7. If the line $y = x + \lambda$ touches the ellipse $9x^2 + 16y^2 = 144$, the value of λ is
   (a) ±5 (b) ±4 (c) ±7 (d) ±3
8. The equation of the tangents to the ellipse $3x^2 + 4y^2 = 12$ which are perpendicular to the line $y + 2x = 4$ is/are
   (a) $x - 2y + 4 = 0$ (b) $x - 2y + 7 = 0$
   (c) $x - 2y - 4 = 0$ (d) $x - 2y - 7 = 0$
9. The product of the perpendiculars from the foci of any tangent to an ellipse $b^2x^2 + a^2y^2 = a^2b^2$ is
   (a) a^2 (b) b^2 (c) $2b^2$ (d) $2a^2$
10. The number of tangents to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ from the point (4, 3) is
    (a) 0 (b) 1 (c) 2 (d) 3
11. If the normal at an end of a latus rectum of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through one extremity of the minor axis, then e^2 is
    (a) $\frac{\sqrt{3} - 1}{2}$ (b) $\frac{\sqrt{3} + 1}{2}$
    (c) $\frac{\sqrt{5} + 1}{2}$ (d) $\frac{\sqrt{5} - 1}{2}$
12. The equation of common tangent between the ellipses $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and $\frac{x^2}{2} + \frac{y^2}{1} = 1$ is
    (a) $x = 3$ (b) $y = 2$
    (c) $x = 1$ (d) not defined
13. If the equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{x}{a} + \frac{y}{b} = \sqrt{2}$, its eccentric angle is
    (a) 45° (b) 60° (c) 90° (d) 22.5°
14. The equations of the tangents to the ellipse $3x^2 + y^2 = 3$ making equal intercepts on the axes are
    (a) $y = \pm x \pm 2$ (b) $y = \pm x \pm 4$
    (c) $y = \pm x \pm 5$ (d) $y = \pm x \pm 7$
15. The number of real tangents can be drawn from (3, 5) to the ellipse $3x^2 + 5y^2 = 15$ is
    (a) 4 (b) 2 (c) 1 (d) 0
16. The number of normals that can be drawn from a point to a given ellipse is
    (a) 4 (b) 2 (c) 1 (d) 0
17. The set of possible values of m for which a line with the slope m is a common tangent to the ellipse $\frac{x^2}{b^2} + \frac{y^2}{c^2} = 1$ and the parabola $y^2 = 4ax$ is
    (a) (3, 5) (b) (2, 3) (c) (1, 3) (d) (0, 1)
18. The angle between the normals of the ellipse $4x^2 + y^2 = 5$, at the intersection of $2x + y = 3$ and ellipse is
    (a) $\tan^{-1}(3/5)$ (b) $\tan^{-1}(3/4)$
    (c) $\tan^{-1}(4/3)$ (d) $\tan^{-1}(4/5)$
19. If the latus rectum of an ellipse $x^2 \tan^2 \varphi + y^2 \sec^2 \varphi = 1$ is 1/2, then ϕ is
    (a) $\pi/2$ (b) $\pi/6$ (c) $\pi/3$ (d) $5\pi/12$
20. If pair of tangents are drawn to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ from a point P, so that the tangents are at right angles to each other, the possible co-ordinates of the point P are
    (a) $(3\sqrt{2}, \sqrt{7})$ (b) (5, 0)
    (c) (3, 4) (d) $(2\sqrt{5}, \sqrt{5})$
21. The eccentricity of the ellipse whose pair of a conjugate diameters are $y = x$ and $3y = -2x$ is
    (a) $1/\sqrt{3}$ (b) 2/3 (c) 1/3 (d) 1/5
22. The minimum length of intercept of any tangent of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ between the axes is
    (a) $2a$ (b) $2b$ (c) $a + b$ (d) $a - b$
23. The eccentric angle of a point on the ellipse $\frac{x^2}{6} + \frac{y^2}{2} = 1$ whose distance from the centre is 2, is
    (a) $\pi/4$ (b) $\pi/6$ (c) $\pi/2$ (d) $\pi/6$
24. If $\frac{x^2}{\lambda^2 - \lambda - 6} + \frac{y^2}{\lambda^2 - 6\lambda + 5} = 1$ represents an ellipse, then λ lies in

(a) $(-\infty, -2)$ (b) $(1, \infty)$
(c) $(3, \infty)$ (d) $(5, \infty)$

25. The eccentricity of the ellipse $ax^2 + by^2 + 2gx + 2fy + c = 0$, if its axis is parallel to x-axis, is
(a) $\sqrt{\frac{a+b}{4}}$ (b) $\sqrt{\frac{a-b}{2}}$
(c) $\sqrt{\frac{b}{a}-1}$ (d) $\sqrt{1-\frac{a}{b}}$

26. S and T are the foci of an ellipse and B is an end of the minor axis. If STB is an equilateral triangle, the eccentricity of the ellipse is
(a) 1/4 (b) 1/3 (c) 1/2 (d) 2/3

27. The centre of the ellipse
$$\frac{(x+y-1)^2}{9} + \frac{(x-y)^2}{16} = 1 \text{ is}$$
(a) (0, 0) (b) (1, 1) (c) (2, 1) (d) (1, 2)

28. Let P be a variable point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with foci F_1 and F_2. If A be the area of the ΔPF_1F_2, the maximum value of A is
(a) $\frac{ea}{b}$ (b) $\frac{ab}{e}$ (c) aeb (d) $\frac{e}{ab}$

29. If the angle between the lines joining the foci of an ellipse to an extremity of a minor axis is 90°, the eccentricity of the ellipse is
(a) 1/8 (b) 1/4 (c) $1/\sqrt{2}$ (d) $1/\sqrt{3}$

30. On the ellipse $4x^2 + 9y^2 = 1$, the points at which the tangents are parallel to the line $8x = 9y$ are
(a) $\left(\frac{2}{5}, \frac{1}{5}\right)$ (b) $\left(-\frac{2}{5}, \frac{1}{5}\right)$
(c) $\left(-\frac{2}{5}, -\frac{1}{5}\right)$ (d) $\left(\frac{2}{5}, -\frac{1}{5}\right)$

31. The equation of the largest circle with centre (1, 0) that can be inscribed in the ellipse $x^2 + 4y^2 = 36$ is
(a) $3x^2 + 3y^2 - 6x - 8 = 0$
(b) $3x^2 + 3y^2 + 6x - 8 = 0$
(c) $2x^2 + 2y^2 + 5x - 8 = 0$
(d) $2x^2 + 2y^2 - 5x - 8 = 0$

32. Let E be the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and C be the circle $x^2 + y^2 = 9$. Let P and Q be the points (1, 2) and (2, 1) respectively. Then
(a) Q lies inside C but outside E
(b) Q lies outside both C and E
(c) P lies inside both C and E
(d) P lies inside C but outside E

33. The radius of the circle passing through the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and having its centre at (0, 3) is
(a) 4 (b) 3 (c) 5 (d) 7

34. An ellipse has OB as semi-minor axis, F, F' are its foci and the angle FBF' is a right angle. The eccentricity of the ellipse is
(a) 1/2 (b) $1/\sqrt{2}$ (c) 1/3 (d) 1/4

35. The sum of the focal distances from any point on the ellipse $9x^2 + 16y^2 = 144$ is
(a) 32 (b) 18 (c) 16 (d) 8

36. If $P = (x, y)$, $F_1 = (3, 0)$, $F_2 = (-3, 0)$ and $16x^2 + 25y^2 = 400$, then $PF_1 + PF_2$ is equal to
(a) 8 (b) 6 (c) 10 (d) 12

37. The number of values of c such that the straight line $y = 4x + c$ touches the ellipse $\frac{x^2}{4} + y^2 = 1$ is
(a) 0 (b) 1 (c) 2 (d) infinite

38. The area of the quadrilateral formed by the tangents at the points of latus recta to the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$ is
(a) 27/4 s.u. (b) 9 s.u.
(c) 27/2 s.u. (d) 27 s.u.

39. Tangents are drawn to the ellipse $\frac{x^2}{27} + y^2 = 1$ at $(3\sqrt{3}\cos\theta, \sin\theta)$, where $0 < \theta < \pi/2$. The value of θ for which the sum of intercepts on the axes made by this tangent is minimum is
(a) $\pi/3$ (b) $\pi/4$ (c) $\pi/8$ (d) $\pi/6$

40. If tangents are drawn to the ellipse $x^2 + 2y^2 = 2$, the locus of the mid-point of the intercept made by the tangents between the co-ordinate axes is
(a) $\frac{1}{2x^2} + \frac{1}{4y^2} = 1$ (b) $\frac{1}{4x^2} + \frac{1}{2y^2} = 1$
(c) $\frac{x^2}{2} + \frac{y^2}{4} = 1$ (d) $\frac{x^2}{4} + \frac{y^2}{2} = 1$

41. The minimum area of the triangle formed by the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the co-ordinate axes is
(a) ab s.u. (b) $\frac{a^2+b^2}{2}$ s.u.
(c) $\frac{(a+b)^2}{2}$ s.u. (d) $\frac{a^2+ab+b^2}{2}$ s.u.

42. The point on the curve $x^2 + 2y^2 = 6$ whose distance from the line $x + y = 7$ is minimum, is
(a) (1, 2) (b) (1, 3) (c) (2, 1) (d) (3, 1)

43. The equation of the common tangent in 1st quadrant to the circle $x^2 + y^2 = 16$ and the ellipse $\frac{x^2}{25} + \frac{y^2}{4} = 1$ is
(a) $2x + y\sqrt{3} = 4\sqrt{7}$ (b) $2x - y\sqrt{3} = 4\sqrt{7}$
(c) $3x - y\sqrt{3} = 4\sqrt{7}$ (d) $3x + y\sqrt{3} = 4\sqrt{7}$

44. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$, $y_1 < 0$, be the end-points of the latus rectum of the ellipse $x^2 + 4y^2 = 4$. The equation of the parabolas with latus rectum PQ are
(a) $x^2+2\sqrt{3}y=3+\sqrt{3}$ (b) $x^2-2\sqrt{3}y=3+\sqrt{3}$
(c) $x^2+2\sqrt{3}y=3-\sqrt{3}$ (d) $x^2-2\sqrt{3}y=3-\sqrt{3}$

45. The line passing through the extremity A of the major axis and extremity B of the minor axis of the ellipse $x^2 + 9y^2 = 9$ meets the auxiliary circle at the point M. The area of the triangle with vertices at A, M and the origin is
(a) $\frac{31}{10}$ (b) $\frac{29}{10}$ (c) $\frac{21}{10}$ (d) $\frac{27}{10}$

46. The normal at a point P on the ellipse $x^2 + 4y^2 = 16$ meets the x-axis at Q. If M is the mid-point of the segment PQ, the locus of M intersects the latus rectum of the given ellipse at the points
(a) $\left(\pm\frac{3\sqrt{5}}{2}, \pm\frac{2}{7}\right)$ (b) $\left(\pm\frac{3\sqrt{5}}{2}, \pm\frac{\sqrt{19}}{4}\right)$
(c) $\left(\pm 2\sqrt{3}, \pm\frac{1}{7}\right)$ (d) $\left(\pm 2\sqrt{3}, \pm\frac{4\sqrt{3}}{7}\right)$

47. Tangents are drawn from the point $P(3, 4)$ to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ touching the ellipse at A and B. Then the co-ordinates of A and B are
(a) (3, 0) and (0, 2)
(b) $\left(-\frac{8}{5}, \frac{2\sqrt{161}}{15}\right)$ and $\left(-\frac{9}{5}, \frac{8}{5}\right)$
(c) $\left(-\frac{8}{5}, \frac{2\sqrt{161}}{15}\right)$ and (0, 2)
(d) (3, 0) and $\left(-\frac{9}{5}, \frac{8}{5}\right)$

48. The maximum area of an isosceles triangle inscribed in an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with its vertex at one end of the major axis is
(a) $\frac{3\sqrt{3}}{4}ab$ (b) $\frac{3\sqrt{2}}{4}ab$
(c) $\frac{2\sqrt{3}}{5}ab$ (d) $\frac{3\sqrt{3}}{5}ab$

49. The point (α, β) on the ellipse $4x^2 + 3y^2 = 12$, in the first quadrant, so that the area enclosed by the lines $y = x$ and $y = \beta$, $x = \alpha$ and the x-axis is maximum, is
(a) $\left(\frac{3}{2}, 1\right)$ (b) $\left(1, \frac{3}{2}\right)$ (c) $\left(2, \frac{3}{2}\right)$ (d) $\left(3, \frac{3}{2}\right)$

50. The number of points on the ellipse $\frac{x^2}{50}+\frac{y^2}{20}=1$ from which pair of perpendicular tangents are drawn to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ is
(a) 0 (b) 2 (c) 1 (d) 4

51. The eccentricity of the ellipse $(x-3)^2+(y-4)^2=\frac{y^2}{9}$ is
(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{3\sqrt{2}}$ (d) $\frac{1}{\sqrt{3}}$

52. For an ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ with vertices A and A', tangents are drawn at the point P in the first quadrant meets the y-axis in Q and the chord $A'P$ meets the y-axis in M. If O be the origin, then $OQ^2 - MQ^2$ is
(a) 9 (b) 13 (c) 4 (d) 5

53. The line $lx + my + n = 0$ will cut the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in points whose eccentric angles differ by $\pi/2$, if
(a) $a^2l^2 + b^2n^2 = 2m^2$ (b) $a^2m^2 + b^2l^2 = 2n^2$
(c) $a^2l^2 + b^2m^2 = 2n^2$ (d) $a^2n^2 + b^2m^2 = 2l^2$.

54. A circle has the same centre as an ellipse and passes through the foci F_1 and F_2 of the ellipse such that two curves intersect in 4 points. Let P be any of their points of intersection. If the major axis of the ellipse is 17 and the area of the triangle F_1F_2 is 30, the distance between the foci is
(a) 11 (b) 12 (c) 13 (d) None

55. The point O is the centre of the ellipse with major axis AB and minor axis CD and the point F is one focus of the ellipse. If $OF = 6$ and the diameter of the inscribed circle of ΔOCF is 2, the product of $(AB)(CD)$ is
(a) 65 (b) 52 (c) 78 (d) none

56. A tangent having slope –4/3 to the ellipse $\frac{x^2}{18}+\frac{y^2}{32}=1$ intersects the major and minor axes in points A and B, respectively. If C is the centre of the ellipse, the area of the triangle ABC is
(a) 12 s.u. (b) 24 s.u.
(c) 36 s.u. (d) 48 s.u.

57. The common tangent to the ellipse $\frac{x^2}{a^2+b^2}+\frac{y^2}{b^2}=1$ and $\frac{x^2}{a^2}+\frac{y^2}{a^2+b^2}=1$ is
(a) $ay = bx + \sqrt{a^4 - a^2b^2 + b^4}$
(b) $by = ax - \sqrt{a^4 + a^2b^2 + b^4}$

(c) $ay = bx - \sqrt{a^4 + a^2b^2 + b^4}$

(d) $by = ax + \sqrt{a^4 - a^2b^2 + b^4}$

58. The normal at a variable point P on an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ of eccentricity e meets the axes of the ellipse in Q and R, the locus of the mid-point of QR is a conic with an eccentricity e' such that
(a) e' is independent of e
(b) $e' = 1$ (c) $e' = e$ (d) $e' = 1/e$

59. An ellipse is drawn with major and minor axes of lengths 10 and 8, respectively. Using one focus as the centre, a circle is drawn that is the tangent to the ellipse, and no part of the circle being outside of the ellipse. The radius of the circle is
(a) $\sqrt{3}$ (b) 2 (c) $2\sqrt{2}$ (d) $\sqrt{5}$

60. A common tangent to $9x^2 + 16y^2 = 144$; $y^2 = x - 4$ and $x^2 = y^2 - 12x + 32 = 0$ is
(a) $y = 3$ (b) $x + 4 = 0$
(c) $x = 4$ (d) $y + 3 = 0$

61. The area of the rectangle formed by the perpendiculars from the centre of the standard ellipse to the tangent and the normal at its point whose eccentric angle is $\pi/4$ is
(a) $\left(\frac{a^2 - b^2}{a^2 + b^2}\right)ab$ (b) $\left(\frac{a^2 - b^2}{a^2 + b^2}\right)\frac{1}{ab}$
(c) $\left(\frac{a^2 + b^2}{a^2 - b^2}\right)\frac{1}{ab}$ (d) $\left(\frac{a^2 + b^2}{a^2 - b^2}\right)ab$

62. An ellipse having foci at (3, 3) and (–4, 4) and passing through the origin has eccentricity equal to
(a) 3/7 (b) 2/7 (c) 5/7 (d) 3/5

63. A bar of length 20 units moves with its ends on two fixed straight lines at right angles. A point P marked on the bar at a distance of 8 units from one and describes a conic whose eccentricity is
(a) 5/9 (b) $\frac{\sqrt{2}}{3}$ (c) $\frac{4}{9}$ (d) 3/5

64. If maximum distance of any point on the ellipse $x^2 + 2y^2 + 2xy = 1$ from its centre be r, the value of r is
(a) $3 + \sqrt{3}$ (b) $2 + \sqrt{2}$
(c) $\frac{\sqrt{2}}{\sqrt{3} - \sqrt{5}}$ (d) $\sqrt{2 - \sqrt{2}}$

65. If the ellipse $\frac{x^2}{a^2 - 7} + \frac{y^2}{13 - 5a} = 1$ is inscribed in a square of side length $a\sqrt{2}$, then a is
(a) $\frac{6}{5}$
(b) $(-\infty, -\sqrt{7}) \cup \left(\sqrt{7}, \frac{13}{5}\right)$
(c) $(-\infty, -\sqrt{7}) \cup \left(\sqrt{7}, \frac{12}{5}\right)$
(d) no such value of a exists

66. If the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is inscribed in a rectangle whose length : breadth is 2 : 1, the area of the rectangle is
(a) $\frac{4}{7}(a^2 + b^2)$ (b) $\frac{4}{3}(a^2 + b^2)$
(c) $\frac{12}{5}(a^2 + b^2)$ (d) $\frac{8}{5}(a^2 + b^2)$

67. The length of the side of the square which can be made by four perpendicular tangents to the ellipse $\frac{x^2}{7} + \frac{y^2}{11} = 1$ is
(a) 10 (b) 8 (c) 6 (d) 5

68. If the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ makes angles α and β with the major axis such that $\tan\alpha + \tan\beta = \lambda$, the locus of their points of intersection is
(a) $x^2 + y^2 = a^2$ (b) $x^2 + y^2 = b^2$
(c) $x^2 = 2\lambda\lambda x = a^2$ (d) $\lambda(x^2 - a^2) = 2xy$.

69. If $\alpha - \beta = c$, the locus of the points of intersection of tangents at
$P(a\cos\alpha, b\sin\alpha)$ and $Q(a\cos\beta, b\sin\beta)$
to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is
(a) a circle (b) a straight line
(c) an ellipse (d) a parabola

70. If the eccentricity of the ellipse $\frac{x^2}{a^2 + 1} + \frac{y^2}{a^2 + 2} = 1$ is $\frac{1}{\sqrt{6}}$, the latus rectum of the ellipse is
(a) $\frac{5}{\sqrt{6}}$ (b) $\frac{10}{\sqrt{6}}$ (c) $\frac{8}{\sqrt{6}}$ (d) $\frac{7}{\sqrt{6}}$

LEVEL III

(Problems for JEE Advanced)

1. Let P be a variable point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with foci F_1 and F_2. If A be the area of ΔPF_1F_2, find the maximum value of A.

2. Let d be the perpendicular distance from the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to the tangent drawn at a point P on the ellipse. If F_1 and F_2 be the two foci of the ellipse, prove that

$$(PF_1 - PF_2)^2 = 4a^2\left(1 - \frac{b^2}{d^2}\right)$$

3. Find the radius of the circle passing through the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and having its centre at (0, 3).
4. If a tangent drawn at a point $(t^2, 2t)$ on the parabola $y^2 = 4x$ is the same as the normal drawn at a point $(\sqrt{5}\cos\varphi, 2\sin\varphi)$ on the ellipse $4x^2 + 5y^2 = 20$. Find the values of t and φ.
5. An ellipse has OB as a minor axis. F and F' are its foci and the angle FBF' is a right angle. Find the eccentricity of the ellipse.
6. A tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipse $x^2 + 2y^2 = 6$ at P and Q. Prove that the tangents at P and Q of the ellipse $x^2 + 2y^2 = 6$ are at right angles.
7. Find the co-ordinates of the points P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, for which the area of the ΔPON is maximum, where O denotes the origin and N, the foot of the perpendicular from O to the tangent at P.
8. Find the angle between the pair of tangents drawn to the ellipse $3x^2 + 2y^2 = 5$ from the point (1, 2).
9. If the normal at $P(\theta)$ to the ellipse $\frac{x^2}{14} + \frac{y^2}{5} = 1$ intersects it again at $Q(2\theta)$, prove that $\cos\theta = -\frac{2}{3}$.
10. If $\left(\frac{1}{5}, \frac{2}{5}\right)$ be the mid-point of the chord of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, find its length.
11. Prove that, in an ellipse, the perpendicular from a focus upon any tangent and the line joining the centre of the ellipse to the point of contact meet on the corresponding directrix.
12. Find the area of the quadrilateral formed by the tangents at the end-points of latus rectum to the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$.
13. A tangent is drawn to the ellipse $\frac{x^2}{27} + y^2 = 1$ at $(3\sqrt{3}\cos\theta, \sin\theta)$, where $0 < \theta < \frac{\pi}{2}$.
    Find the value of θ such that the sum of the intercepts on axes made by the tangent is minimum.
14. If tangents are drawn to the ellipse $x^2 + 2y^2 = 2$, find the locus of the mid-point of the intercept made by the tangents between the co-ordinate axes.
15. Find the minimum area of the triangle formed by the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the co-ordinate axes.
16. Find the equation of the common tangent in first quadrant to the circle $x^2 + y^2 = 16$ and the ellipse $\frac{x^2}{25} + \frac{y^2}{4} = 1$. Also find the length of the intercept of the tangent between the co-ordinate axes.
17. P and Q are two points on the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ such that sum of their ordinates is 3. Find the locus of the points of intersection of tangents at P and Q.
18. From any point P lying in the first quadrant on the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, PN is drawn perpendicular to the major axis and produced at Q so that NQ equals to PS, where S is a focus. Prove that the locus of Q is $3x + 5y + 25 = 0$.
19. If a tangent of slope 2 of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is the normal to the circle $x^2 + y^2 + 4x + 1 = 0$, find the maximum value of ab.
20. If the maximum distance of any point on the ellipse $x^2 + 2xy + 2y^2 = 1$ from its centre be r, find r.
21. Prove that the locus of the mid-points of the chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which are of constant length $2c$ is $\left(\frac{b^2x^2 + a^2y^2}{a^4y^2 + b^4x^2}\right) = \frac{1}{c^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$.
22. Chords of the ellipse touch the parabola $ay^2 = -2b^2x$. Prove that the locus of their poles is the parabola $ay^2 = 2b^2x$.
23. An ellipse is rotated through a right angle in its own plane about its centre, which is fixed. Find the locus of the point of intersection of a tangent to the ellipse in its original position with the tangent at the same point of the curve in its new position.
24. The tangents drawn from a point P to the ellipse make angles θ_1 and θ_2 with the major axis. Find the locus of P for which $\theta_1 + \theta_2 = 2\alpha$.
25. Find the length of the sides of a square which can be made by four perpendicular tangents to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.
26. Find the locus of the point which divides the double ordinates of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ in the ratio 1 : 2 internally.
27. An ellipse has the points (1, –1)and (2, –1) as its foci and $x + y = 5$ as one of its tangent. Find the point where the line be a tangent to the ellipse from the origin.
28. An ellipse is sliding along the co-ordinate axes. If the foci of the ellipse are (1, 1) and (3, 3), find the area of the director circle of the ellipse.

29. What are the values of the parameter θ for points where the line $bx = ay$ intersects the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$?
30. Prove that the sum of the squares of the reciprocals of two perpendicular radius vectors of an ellipse is constant.
31. Find the eccentricity of the ellipse
$$4(x-2y+1)^2+9(2x+y+2)^2=25.$$
32. If two vertices of a rectangle lie on $y = 2x + c$ and other two vertices are (0, 4) and (–1, 2). Find c and other two vertices such that the area of the ellipse inscribed in the rectangle is $\frac{5\pi}{2}$.

Level IV

(Tougher Problems for JEE Advanced)

1. Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with its vertex at one end of the major axis. **[Roorkee, 1994]**
2. If a tangent drawn at a point $(t^2, 2t)$ on the parabola $y^2 = 4x$ is the same as the normal drawn at a point $(\sqrt{5}\cos\varphi, 2\sin\varphi)$ on the ellipse $4x^2 + 5y^2 = 20$. Find the values of $t = \varphi$.
3. Find the equation of the largest circle with centre (1, 0) that can be inscribed in the ellipse $x^2 + 4y^2 = 16$. **[Roorkee, 1999]**
4. Find the condition so that the line $px + qy = r$ intersects the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in points whose eccentric angles differ by $\frac{\pi}{4}$. **[Roorkee, 2001]**
5. A point moves so that the sum of the squares of its distances from two intersecting straight lines is constant. Prove that its locus is an ellipse.
6. Prove that the line joining the extremities of any pair of diameters of an ellipse which are at right angles, will touch a fixed circle.
7. If P be any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose ordinate is y', prove that the angle between the tangent at P and the focal distance of P is $\tan^{-1}\left(\frac{b^2}{aey'}\right)$.
8. Prove that the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the circle $x^2 + y^2 = ab$ intersect at an angle $\tan^{-1}\left(\frac{a-b}{\sqrt{ab}}\right)$.
9. If the eccentric angles of points P and Q on the ellipse be θ and $\frac{\pi}{2}+\theta$ and α be the angle between the normals at P and Q, prove that the eccentricity e is given by $\tan\alpha = \frac{2\sqrt{1-e^2}}{e^2(\sin^2 2\theta)}$.
10. The tangent and the normal at any point P of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ cut its major axis in points T and T' respectively such that $TT' = a$. Prove that the eccentric angle of the point P is given by $c^2\cos^2\varphi + \cos\varphi - 1 = 0$.
11. A variable point P on an ellipse of eccentricity e is joined to its foci S and S'. Prove that the locus of the in-centre of the $\Delta PSS'$ is an ellipse whose eccentricity is $\sqrt{\frac{2e}{1+e}}$.
12. The eccentric angle of any point P measured from the semi-major axis CA is ϕ. If S be the focus nearest to A and $\angle ASP = \theta$, prove that
$$\tan\left(\frac{\theta}{2}\right)=\sqrt{\frac{1+e}{1-e}}\tan\left(\frac{\varphi}{2}\right)$$
13. Prove that the locus of the centroid of an equilateral inscribed in an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is
$$\frac{(a^2+3b^2)^2}{a^2(a^2-b^2)^2}x^2+\frac{(3a^2+b^2)^2}{a^2(a^2-b^2)^2}y^2=1$$
14. The tangent at a point $P(a\cos\varphi, b\sin\varphi)$ of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets its auxiliary circle in two points, the chord joining which subtends a right angle at the centre. Prove that the eccentricity of the ellipse is $\frac{1}{\sqrt{1+\sin^2\theta}}$.
15. Prove that the locus of the point of intersection of tangents to an ellipse at two points whose eccentric angles differ by a constant α is an ellipse
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\sec^2\left(\frac{\alpha}{2}\right)$$
16. If two concentric ellipses be such that the foci of one be on the other end and e and e' be their eccentricities. Prove that the angle between their axes is $\cos^{-1}\left(\frac{\sqrt{e^2+e'^2-1}}{ee'}\right)$.

17. If the normals at the four points (x_1, y_1), (x_2, y_2), $(x_3 y_3)$ and (x_4, y_4) on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are concurrent, prove that

$$(x_1+x_2+x_3+x_4)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)=1$$

18. If θ be the difference of the eccentric angles of two points on an ellipse, the tangent at which are at right angles. Prove that $ab \sin \theta = d_1 d_2$, where d_1, d_2 are the semi-diameters parallel to the tangents at the points and a, b are the semi-axes of the ellipse.

19. From any point on the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=4$, tangents are drawn to the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Prove that the normals at the points of contact meet on the conic $a^2x^2+b^2y^2=\frac{1}{4}(a^2-b^2)^2$.

20. Show that the locus of the centre of the circle which cuts the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in a fixed point (h, k) and two other points at the extremities of a diameter is $2(a^2x^2 + b^2y^2) = (a^2 - b^2)(hx - ky)$.

Integer Type Questions

1. Find the minimum area of the triangle formed by the tangent to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$.
2. Let P be a variable point on the ellipse $\frac{x^2}{5}+\frac{y^2}{1}=1$ with foci F_1 and F_2. If A be the area of the triangle ΔPF_1F_2, find the maximum value of A.
3. If the tangent at $\left(4\cos\varphi, \frac{16}{\sqrt{11}}\sin\varphi\right)$ to the ellipse $16x^2 + 11y^2 = 256$ is also a tangent to the circle $x^2 + y^2 - 2x - 15 = 0$, find the number of values of ϕ.
4. Find the area of a parallelogram formed by the tangents at the extremities of a pair of conjugate diameters of an ellipse $\frac{x^2}{9}+16y^2=1$.
5. Find the number of integral values of a for which the equation $\frac{x^2}{a-10}+\frac{y^2}{4-a}=1$ represents an ellipse.
6. Find the area of the greatest rectangle that can be inscribed in an ellipse $x^2 + 4y^2 = 4$.
7. If $P(x, y)$, $F_1=(\sqrt{7},0)$, $F_2=(-\sqrt{7},0)$ and $9x^2 + 16x^2 = 144$, find the value of $(PF_1 + PF_2 - 2)$.
8. If F_1 and F_2 be the feet of perpendiculars from the foci S_1 and S_2 of an ellipse $\frac{x^2}{5}+\frac{y^2}{3}=1$ on the tangent at any point P on the ellipse, find the value of $(S_1F_1)\cdot(S_1F_2)$.
9. Find the minimum length of the intercept of any tangent to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ between the co-ordinate axes.
10. Find the number of distinct normals that can be drawn to the ellipse $\frac{x^2}{169}+\frac{y^2}{25}=1$ from the point (0, 6).

Comprehensive Link Passage

Passage I

Let C: $x^2 + y^2 = r^2$ and E: $\frac{x^2}{16}+\frac{y^2}{9}=1$ intersect at four distinct points A, B, C, D. Their common tangents form a parallelogram $EFGH$.

1. If $ABCD$ is a square, then r is
(a) $\frac{12\sqrt{2}}{5}$ (b) $\frac{12}{5}$ (c) $\frac{12}{5\sqrt{5}}$ (d) none
2. If $EFGH$ is a square, then r is
(a) $\sqrt{20}$ (b) $\sqrt{12}$ (c) $\sqrt{15}$ (d) none
3. If $EFGH$ is a square, the ratio of $\frac{\text{ar}(\text{Circle } C)}{\text{ar}(\text{circumcircle } \Delta EFG)}$ is
(a) $\frac{9}{16}$ (b) $\frac{3}{4}$ (c) $\frac{1}{2}$ (d) none

Passage II

A curve is represented by C:

$$21x^2 - 6xy + 29y^2 + 6x - 58y - 151 = 0$$

1. The eccentricity of the given curve is
(a) $\frac{1}{3}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{2}{3}$ (d) $\frac{2}{\sqrt{5}}$
2. The length of axes are
(a) $6, 2\sqrt{6}$ (b) $5, 2\sqrt{5}$ (c) $4, 4\sqrt{5}$ (d) None
3. The centre of the conic C is
(a) (1, 0) (b) (0, 0) (c) (0, 1) (d) None.

Passage III

Let F and F' be the foci of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose eccentricity is e, P is a variable point on the ellipse.

1. The locus of the incentre of the $\Delta PFF'$ is a/an
(a) ellipse (b) hyperbola
(c) parabola (d) circle
2. The eccentricity of the locus of P is
(a) $\sqrt{\frac{2e}{1-e}}$ (b) $\sqrt{\frac{2e}{1+e}}$ (c) 1 (d) none

3. The maximum area of the rectangle inscribed in the ellipse is

   (a) $\frac{2abe^2}{1+e}$ (b) $\frac{2abe}{1+e}$ (c) $\frac{abe}{1+e}$ (d) none

Passage IV

The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is such that it has the least area but contains the circle $(x-1)^2 + y^2 = 1$.

1. The eccentricity of the ellipse is

   (a) $\sqrt{\frac{2}{3}}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{1}{2}$ (d) none

2. The equation of the auxiliary circle of the ellipse is

   (a) $x^2 + y^2 = \frac{13}{2}$ (b) $x^2 + y^2 = 5$

   (c) $x^2 + y^2 = \frac{9}{2}$ (d) none

3. The length of the latus rectum of the ellipse is

   (a) 2 (b) 1 (c) 3 (d) 5/2

Passage V

The conic section is the locus of a point which moves in a plane in such a way that, the ratio of its distance from a fixed point to a fixed straight line is constant. The fixed point is called focus and the fixed straight line is called the directrix. The constant ratio is called the eccentricity. It is denoted by e.

If e is less than 1, the conic section is called an ellipse. A line joining two points on the ellipse is called the chord of the ellipse. If through a point C, any chord of an ellipse is bisected, the point C is called the centre of the ellipse.

Let the equation of the curve is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

It will represents an ellipse if $h^2 < ab$ and $\Delta \neq 0$, where

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

1. The length of the longest chord of the ellipse $x^2 + xy + y^2 = 1$ is

   (a) $\sqrt{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $2\sqrt{2}$ (d) 1

2. The length of the chord passing through the centre and perpendicular to the longest chord of the ellipse $x^2 + xy + y^2 = 1$ is

   (a) $\frac{1}{\sqrt{2}}$ (b) $\frac{\sqrt{3}}{2}$ (c) $2\sqrt{\frac{2}{3}}$ (d) $\frac{1}{\sqrt{3}}$

3. There are exactly n integral values of λ for which the equation $x^2 + \lambda xy + y^2 = 1$ represents an ellipse, then n must be

   (a) 0 (b) 1 (c) 2 (d) 3

4. The centre of the ellipse $x^2 + xy + 2y^2 = 1$ is

   (a) (0, 0) (b) (1, 1) (c) (1, 2) (d) (2, 1)

Passage VI

Tangents are drawn from the point $P(3, 4)$ to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ touching the ellipse at points A and B.

1. The co-ordinates of A and B are

   (a) (3, 0) and (0, 2)

   (b) $\left(-\frac{8}{5}, \frac{2\sqrt{161}}{15}\right)$ and $\left(-\frac{9}{5}, \frac{8}{5}\right)$

   (c) $\left(-\frac{8}{5}, \frac{2\sqrt{161}}{15}\right)$ and (0, 2)

   (d) (3, 0) and $\left(-\frac{9}{5}, \frac{8}{5}\right)$

2. The orthocentre of the ΔPAB is

   (a) $\left(5, \frac{8}{7}\right)$ (b) $\left(\frac{7}{5}, \frac{25}{8}\right)$

   (c) $\left(\frac{11}{5}, \frac{8}{5}\right)$ (d) $\left(\frac{8}{25}, \frac{7}{5}\right)$

3. The equation of the locus of the point whose distances from the point P and the line AB are equal is

   (a) $9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$

   (b) $9x^2 + y^2 + 6xy - 54x + 62y + 241 = 0$

   (c) $9x^2 + 9y^2 - 6xy - 54x - 62y + 241 = 0$

   (d) $x^2 + y^2 - 2xy - 27x - 31y + 120 = 0$

Matrix Match
(For JEE-Advanced Examination Only)

1. Match the following columns:

   Let the equation of the curve is $x^2 + 2y^2 + 4x + 12y = 0$.

Column I		Column II	
(A)	The centre is	(P)	$(-2-\sqrt{11}, -)$
(B)	The focus is	(Q)	$(-2, -3)$
(C)	One extremity of the major axis is	(R)	$(\sqrt{11}-2, -3)$
(D)	The latus rectum is	(S)	$\left(-2+\frac{\sqrt{11}}{2}, -2\right)$
		(T)	$\sqrt{22}$

2. Match the following columns:

   Let the equation of the ellipse is $\frac{(x-3)^2}{25}+\frac{(y-2)^2}{16}=1$.

Column I		Column II	
(A)	The centre is	(P)	(8, 2)
(B)	One extremity of major axis is	(Q)	(3, 2)
(C)	One of the foci is	(R)	(6, 2)
(D)	One extremity of minor axis is	(S)	(3, 6)
(E)	The length of latus rectum is	(T)	10
(F)	The focal distance is	(U)	6.4

3. Match the following columns:
For all real p, the line $2px + 2\sqrt{1-p^2} = 1$ touches a fixed ellipse.

Column I		Column II	
(A)	The eccentricity is	(P)	1/2
(B)	The latus rectum is	(Q)	$\sqrt{3}/2$
(C)	The focal distance is	(R)	2
(D)	One extremity of major axis is	(S)	(0, 1)
(E)	One of the foci is	(T)	$(0, \sqrt{3}/2)$

4. Match the following columns:
A tangent having slope –4/3 touches the ellipse $\frac{x^2}{18} + \frac{y^2}{32} = 1$ at point P and intersects the major and minor axes at A and B respectively, where O is the centre of the ellipse.

Column I		Column II	
(A)	The distance between the parallel tangents having slopes –4/3 is	(P)	24
(B)	Area of ΔAOB is	(Q)	7/24
(C)	If the tangent in first quadrant touches the ellipse at (h, k), the value of hk is	(R)	48/5
(D)	If the equation of the tangent intersecting positive axes is $lx + my = 1$, the value of $l + m$ is	(S)	12

5. Match the following columns:
Let $M(t, t + 1)$ is a point moving in a straight line and C_1, C_2, C_3 be three conics whose equations are $x^2 + y^2 = 2$, $y^2 = 8x$ and $x^2 + 2y^2 = 1$. Let M lies within the interior of C_1, C_2, C_3.

Column I		Column II	
(A)	C_1 is	(P)	$(3-2\sqrt{2}, 3+2\sqrt{2})$
(B)	C_2 is	(Q)	$\left(-1, -\frac{1}{3}\right)$
(C)	C_3 is	(R)	$\left(\frac{-1-\sqrt{3}}{2}, \frac{-1+\sqrt{3}}{2}\right)$

6. Match the following columns:
Tangents are drawn from (2, 3) to the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$.

Column I		Column II	
(A)	The equation of PQ is	(P)	$3 \times \sqrt{\frac{3}{2}}$
(B)	The length of PQ is	(Q)	$12\sqrt{3}$
(C)	Area of ΔTPQ is	(R)	$9\sqrt{3}$
(D)	Area of quadrilateral $OPTQ$ is	(S)	$x + 2y = 2$
		(T)	$2x + y = 1$

Questions asked in Previous Years' JEE-Advanced Examinations

1. The locus of a point whose distance from (–2, 0) is 2/3 times its distance from the line $x = -\frac{9}{2}$ is a/an
(a) ellipse (b) parabola
(c) hyperbola (d) none **[IIT-JEE, 1994]**

2. Let E be the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and C be the circle $x^2 + y^2 = 9$. Let P and Q be the points (1, 2) and (2, 1) respectively, then
(a) Q lies inside C but outside E
(b) Q lies outside both C and E
(c) P lies inside both C and E
(d) P lies inside C but outside E **[IIT-JEE, 1994]**

3. Let P be a variable point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with foci F_1 and F_2. If A is the area of ΔPF_1F_2, find the maximum value of A. **[IIT-JEE, 1994]**

4. The radius of the circle passing through the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$, and having its centre at (0, 3) is
(a) 4 (b) 3 (c) $\sqrt{\frac{1}{2}}$ (d) $\frac{7}{2}$
[IIT-JEE, 1995]

5. Let d be the perpendicular distance from the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to the tangent drawn at a point P on the ellipse.
If F_1 and F_2 are two foci of the ellipse, show that $(PF_1 - PF_2)^2 = 4a^2\left(1 - \frac{b^2}{d^2}\right)$. **[IIT-JEE, 1995]**

No questions asked in 1996.

6. A tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipse $x^2 + 2y^2 = 6$ at P and Q. Prove that the tangents at P and Q of the ellipse $x^2 + 2y^2 = 6$ are at right angles.
[IIT-JEE, 1997]

7. An ellipse has OB as a semi-minor axis. F and F' are its foci, and the ellipse FBF' is a right angle. Find the eccentricity of the ellipse. **[IIT-JEE, 1997]**

8. If $P = (x, y)$ be any point on $16x^2 + 25y^2 = 400$ with foci $F_1 = (3, 0)$ and $F_2 = (-3, 0)$, then $PF + PF_2$ is
(a) 8 (b) 6 (c) 10 (d) 12
[IIT-JEE, 1998]

9. Find the co-ordinates of all points P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, for which the area of ΔPON is maximum, where O denotes the origin and N, the foot of the perpendicular from O to the tangent at P. **[IIT-JEE, 1999]**

10. On the ellipse $4x^2 + 9y^2 = 1$, the points at which the tangents are parallel to the line $9y = 8x$ are
(a) $\left(\frac{2}{5},\frac{1}{5}\right)$ (b) $\left(-\frac{2}{5},\frac{1}{5}\right)$
(c) $\left(-\frac{2}{5},-\frac{1}{5}\right)$ (d) $\left(\frac{2}{5},-\frac{1}{5}\right)$
[IIT-JEE, 1999]

11. Let P be a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $0 < b < a$. Let the line parallel to y-axis passing through P meet the circle $x^2 + y^2 = a^2$ at the point Q such that P and Q are on the same side of x-axis. For two positive real numbers r and s, find the locus of the point R on PQ such that $PR : RQ = r : s$ and p varies over the two ellipses. **[IIT-JEE, 2001]**

12. Prove that, in an ellipse, the perpendicular from a focus upon any tangent and the line joining the centre of the ellipse to the point of contact meet on the corresponding directrix. **[IIT-JEE, 2002]**

13. The area of the quadrilateral formed by the tangents at the end-points of latus rectum to the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$, is
(a) 27/4 s.u. (b) 9 s.u.
(c) 27/2 s.u. (d) 27 s.u.
[IIT-JEE, 2003]

14. If tangents are drawn to the ellipse $x^2 + 2y^2 = 2$, the locus of the mid-point of the intercept made by the tangents between the co-ordinate axes is
(a) $\frac{1}{2x^2}+\frac{1}{4y^2}=1$ (b) $\frac{1}{4x^2}+\frac{1}{2y^2}=1$
(c) $\frac{x^2}{2}+\frac{y^2}{4}=1$ (d) $\frac{x^2}{4}+\frac{y^2}{2}=1$
[IIT-JEE, 2004]

15. Find the equation of the common tangent in first quadrant to the circle $x^2+y^2=16$ and the ellipse $\frac{x^2}{25}+\frac{y^2}{4}=1$. Also find the length of the intercept of the tangent between the co-ordinate axes. **[IIT-JEE, 2005]**

16. A triangle is formed by a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the co-ordinate axes. The area of the triangle cannot be less than
(a) $\frac{1}{2}(a^2+b^2)$ s.u. (b) $\frac{1}{3}(a^2+b^2+ab)$ s.u.
(c) $\frac{1}{2}(a+b)^2$ s.u. (d) ab s.u.
[IIT-JEE, 2005]

No questions asked in between 2006-2007.

17. Let $P(x_1, y_1)$, $Q(x_2, y_2)$, $y_1 < 0$, $y_2 < 0$, be the end-points of the latus rectum of the ellipse $x^2 + 4y^2 = 4$. The equation of the parabola with latus rectum PQ are
(a) $x^2+2\sqrt{3}y=3+\sqrt{3}$ (b) $x^2-2\sqrt{3}y=3+\sqrt{3}$
(c) $x^2+2\sqrt{3}y=3-\sqrt{3}$ (d) $x^2-2\sqrt{3}y=3-\sqrt{3}$
[IIT-JEE, 2008]

18. The line passing through the extremity A of the major axis and extremity B of the minor axis of the ellipse $x^2 + 9y^2 = 9$ meets its auxiliary circle at the point M. Then the area of the triangle with vertices at A, M and the origin O is
(a) 31/10 (b) 29/10
(c) 21/10 (d) 27/10
[IIT-JEE, 2009]

19. The normal at P on the ellipse $x^2 + 4y^2 = 16$ meets the x-axis at Q. If M is the mid-point of the line segment PQ, the locus of M intersects the latus rectums of the given ellipse at the points
(a) $(\pm 3\sqrt{5}/2, \pm 2/7)$ (b) $(\pm 3\sqrt{5}/2, \pm\sqrt{19}/7)$
(c) $(\pm 2\sqrt{3}, \pm 1/7)$ (d) $(\pm 2\sqrt{3}, \pm 4\sqrt{3}/7)$
[IIT-JEE, 2009]

Comprehension

Tangents are drawn from the point $P(3, 4)$ to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ touching the ellipse at points A and B.

20. The co-ordinates of A and B are
(a) (3, 0) and (0, 2)
(b) $\left(-\frac{8}{5},\frac{2\sqrt{16}}{15}\right)$ and $\left(-\frac{9}{5},\frac{8}{5}\right)$
(c) $\left(-\frac{8}{5},\frac{2\sqrt{16}}{15}\right)$ and (0, 2)
(d) (3, 0) and $\left(-\frac{9}{5},\frac{8}{5}\right)$

21. The orthocentre of ΔPAB is
(a) $\left(5,\frac{8}{7}\right)$ (b) $\left(\frac{7}{5},\frac{25}{8}\right)$
(c) $\left(\frac{11}{5},\frac{8}{5}\right)$ (d) $\left(\frac{8}{25},\frac{7}{5}\right)$
[IIT-JEE, 2010]

22. The equation of the locus of the point whose distances from the point P and the line AB are equal, is

(a) $9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$
(b) $x^2 + 9y^2 + 6xy - 54x + 62y - 241 = 0$
(c) $9x^2 + 9y^2 - 6xy - 54x - 62y - 241 = 0$
(d) $x^2 + y^2 - 2xy + 27x + 31y - 120 = 0$.

[IIT-JEE, 2011]

23. The ellipse $E_1 : \frac{x^2}{9} + \frac{y^2}{4} = 1$ is inscribed in a rectangle R whose sides are parallel to the co-ordinate axes. Another ellipse E_2: passing through the point (0, 4) circumscribes the rectangle R. The eccentricity of the ellipse E_2 is

(a) $\frac{\sqrt{2}}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{1}{2}$ (d) $\frac{3}{4}$

[IIT-JEE, 2012]

24. A vertical line passing through the point (h, 0) intersects the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ at the points P and Q. Let the tangents to the ellipse at P and Q meet at R.
If $\Delta(h)$ = area of ΔPQR,
$\Delta_1 = \max_{\frac{1}{2} \le h \le 1} \Delta(h)$
and $\Delta_2 = \min_{\frac{1}{2} \le h \le 1} \Delta(h)$
then $\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2$ =........

[IIT-JEE, 2013]

25. If the normal from the point $P(h, 1)$ on the ellipse $\frac{x^2}{6} + \frac{y^2}{3} = 1$ is perpendicular to the line $x + y = 3$, then the value of h is...

[IIT-JEE, 2014]

26. Suppose that the foci of the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$ are (f_1, 0) where (f_2, 0). Let P_1 and P_2 be two parabolas with a common vertex at (0, 0) and with foci at (f_1, 0) and ($2f_2$, 0), respectively. Let T_1 be a tangent to P_1 which passes through ($2f_2$, 0) and T_2 be a tangent to P_2 which passes through (f_1, 0).
The m_1 is the slope of T_1 and m_2 is the slope of T_2, then the value of $\left(\frac{1}{m_1^2} + m_2^2\right)$ is...

[IIT-JEE-2015]

27. Let E_1 and E_2 be two ellipses whose centers are at the origin. The major axes of E_1 and E_2 lie along the x-axis and the y-axis, respectively. Let S be the circle $x^2 + (y - 1)^2 = 2$. The straight line $x + y = 3$ touches the curves S, E_1 ad E_2 at P, Q and R, respectively. Suppose that $PQ = PR = \frac{2\sqrt{2}}{3}$. If e_1 and e_2 are the eccentricities of E_1 and E_2, respectively, then the correct expression(s) is(are)

(a) $e_1^2 + e_2^2 = \frac{43}{40}$ (b) $e_1 e_2 = \frac{\sqrt{7}}{2\sqrt{10}}$
(c) $|e_1^2 - e_2^2| = \frac{5}{8}$ (d) $e_1 e_2 = \frac{\sqrt{3}}{4}$

[IIT-JEE-2015]

28. **Comprehension**
Let $F_1(x_1, 0)$ and $F_2(x_2, 0)$, for $x_1 < 0$ and $x_2 > 0$, be the foci of the ellipse $\frac{x^2}{9} + \frac{y^2}{8} = 1$
Suppose a parabola having vertex at the origin and focus at F_2 intersects the ellipse at point M in the first quadrant and at point N in the fourth quadrant.
(i) The orthocentre of the triangle F_1MN is
(a) $\left(-\frac{9}{10}, 0\right)$ (b) $\left(\frac{2}{3}, 0\right)$
(c) $\left(\frac{9}{10}, 0\right)$ (d) $\left(\frac{2}{3}, \sqrt{6}\right)$
(ii) If the tangents to the ellipse at M and N meet at R and the normal to the parabola at M meets the x-axis at Q, then the ratio of area of the triangle MQR to area of the quadrilateral MF_1NF_2 is
(a) 3:4 (b) 4:5 (c) 5:8 (d) 2:3

[IIT-JEE-2016]

Answers

Level I

2. 10
3. $4 < a < 10$
4. $\frac{1}{\sqrt{3}}$
5. $\frac{x^2}{16} + \frac{y^2}{12} = 1$
6. $e = \frac{\sqrt{5} - 1}{2}$
7. 0
8. $\theta = \frac{\pi}{6}$
9. 24
10. $\frac{(x-3)^2}{16} + \frac{(y-2)^2}{9} = 1$

11. $SQ = \dfrac{40}{3}$
12. $(9\pi - 6)$ s.u.
13. $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$
14. 2013
15. 2
16. $y = -\dfrac{1}{2}x + \sqrt{13}$
17. $\left(\dfrac{\pm(-48)}{5}, \dfrac{\pm 36}{5}\right)$
18. 1 and ±5
19. $y = 2x \pm \sqrt{14}$
34. $x^2 + y^2 = 25$
35. $x^2 + y^2 = 25$
36. $4x^2 + 3y^2 - 12xy + 4x + 6y - 3 = 0$
50. $\begin{vmatrix} \text{cosec}\,\alpha & \sec\alpha & 1 \\ \text{cosec}\,\beta & \sec\beta & 1 \\ \text{cosec}\,\gamma & \sec\gamma & 1 \end{vmatrix} = 0$
51. $\left(\dfrac{x^2}{a^4} + \dfrac{y^2}{b^4}\right) = \dfrac{1}{a^2} + \dfrac{1}{b^2}$
52. $\dfrac{x^2}{a^4} + \dfrac{y^2}{b^4} = \dfrac{1}{c^2}$
53. $(a^2 + b^2)\left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}\right)^2 = (x^2 + y^2)$
54. $\left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}\right)^2 = \left(\dfrac{c^2}{x^2 + y^2}\right)$
55. $x \cdot y = \dfrac{a^2b^2}{\lambda}$
56. $\left(\dfrac{a^6}{x^2} + \dfrac{b^6}{y^2}\right)\left(\dfrac{x^2}{a^2} + \dfrac{\mathrm{y}^2}{\mathrm{b}^2}\right)^2 = (a^2 - b^2)^2$
57. $a^2\left(\dfrac{x^2}{a^4} + \dfrac{y^2}{b^4}\right) = \left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}\right)^2$
58. $\left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}\right)^2\left(\dfrac{1}{a^2} + \dfrac{1}{b^2}\right) = \left(\dfrac{x^2}{a^4} + \dfrac{y^2}{b^4}\right)$
59. $\left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}\right) = \dfrac{1}{2}$
60. $\left(\dfrac{x^2 + y^2}{a^2 + b^2}\right) = \left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}\right)^2$
61. $b^2x^2 + a^2y^2 = ab^2\,xc$
62. $\dfrac{\alpha^2 x^2}{a^4} + \dfrac{\beta^2 y^2}{b^4} = \left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}\right)^2$
63. $x = \dfrac{a}{e}$
64. $\left(-\dfrac{a^2 l}{n}, -\dfrac{b^2 m}{n}\right)$
66. (1, 1)
67. $a^2c^4\,x^2 + b^2d^4\,y^2 = 1$
68. $\dfrac{x^2}{a^4} + \dfrac{y^2}{b^4} = \dfrac{1}{c^2}$
69. $\dfrac{a^6}{x^2} + \dfrac{b^6}{y^2} = (a^2 - b^2)^2$
70. $4a^2x^2 = y^2\beta^2 + 4a^2\alpha^2$
71. $y = -\dfrac{b^2 x}{a^2 m}$
72. $P(a\cos\varphi, b\sin\varphi)$,
$Q(-a\cos\varphi, -b\sin\varphi)$,
$S(-a\sin\varphi, b\cos\varphi)$,
and $R(a\sin\varphi, -b\cos\varphi)$
75 $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 2$
76. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{1}{2}$
77. $a^3y + b^3x = 0$
78. $\dfrac{a^2}{c^2} + \dfrac{b^2}{d^2} = 2$
79. $2(a^2x^2 + b^2y^2)^3 = (a^2 - b^2)(a^2x^2 - b^2y^2)^2$
81. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 2$
82. $\dfrac{1}{\sqrt{3}}$
83. $4x + 3y = 12$
84. $x + y = 2$

Level II

1. (c)	2. (c)	3. (a)	4. (c)	5. (a)
6. (c)	7. (a)	8. (a)	9. (b)	10. (c)
11. (d)	12. (d)	13. (a)	14. (a)	15. (b)
16. (a, b)	17. (d)	18. (a)	19. (a)	20. (a,b,c,d)
21. (a)	22. (c)	23. (a)	24. (a)	25. (d)
26. (c)	27. (b)	28. (c)	29. (c)	30. (b)
31. (a)	32. (d)	33. (a)	34. (b)	35. (d)
36. (c)	37. (c)	38. (d)	39. (d)	40. (a)
41. (a)	42. (c)	43. (a)	44. (a)	45. (d)
46. (c)	47. (d)	48. (a)	49. (a)	50. (d)
51. (b)	52. (c)	53. (c)	54. (c)	55. (a)
56. (b)	57. (b)	58. (a)	59. (b)	60. (c)
61. (a)	62. (c)	63. (d)	64. (c)	65. (c)
66. (d)	67. (d)	68. (d)	69. (c)	70. (b)

Level III

1. $b\sqrt{(a^2 - b^2)}$
2. 4
4. $\varphi = \pi - \tan^2 2, t = -\frac{1}{\sqrt{5}}, \varphi = \pi + \tan^{-1}(2)$,

   $t = \frac{1}{\sqrt{5}}; \varphi = \pm\frac{\pi}{2}, t = 0$
5. $\frac{1}{\sqrt{2}}$
7. $\left(\frac{a^2}{\sqrt{(a^2 + b^2)}}, \frac{b^2}{\sqrt{(a^2 + b^2)}}\right)$
8. $(x - 1)^2 + y^2 = \frac{121}{9}$
9. $a^2p^2 + b^2q^2 = r^2\sec^2\left(\frac{\pi}{8}\right)$
12. 27 s.u.
13. $\frac{\pi}{6}$
14. $\frac{1}{2x^2} + \frac{1}{4y^2} = 1$
15. ab s.u.
16. $Y = -\frac{2}{\sqrt{3}}x + 4\sqrt{\frac{7}{3}}; \frac{14}{\sqrt{3}}$
17. $x^2 + y^2 = 1$
19. 4
20. $\frac{\sqrt{2}}{\sqrt{3 - \sqrt{5}}}$
21. $\left(\frac{b^2x^2 + a^2y^2}{a^4y^2 + b^4x^2}\right) = \frac{1}{c^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$
22. $y^2 = \left(\frac{2b^2}{a}\right)x$
23. $(x^2 + y^2)(x^2 + y^2 - a^2 - b^2 = 2xy(a^2 - b^2)$
24. $\{(x^2 - y^2) + (b^2 + a^2)\}\tan(2\alpha) + 2xy$
25. $5\sqrt{2}$
26. $\frac{x^2}{9} + \frac{9y^2}{4} = 1$
27.
28.
29. $\frac{\pi}{4}$ or $\frac{5\pi}{4}$
30. $\frac{1}{r_1^2} + \frac{1}{r_2^2} = \frac{1}{a^2} + \frac{1}{b^2}$
31. $\frac{\sqrt{5}}{3}$
32. $c = 14$ or -6, $A = (-4, 6)$, $D = (3, 0)$

Level IV

1. $\frac{3\sqrt{3}}{4}ab$
2. 1 s.u., $\left(\frac{3}{2}, 1\right)$
3. $(x - 1)^2 + y^2 = \frac{11}{3}$
4. $r^2 = \cos^2\left(\frac{\pi}{8}\right) \times (a^2p^2 + b^2q^2)$

INTEGER TYPE QUESTIONS

1. 6 s.u.
2. 2 s.u.
3. 2
4. 3 s.u.
5. 5
6. 4 s.u.
7. 6
8. 3
9. 7
10. 3

COMPREHENSIVE LINK PASSAGES

Passage I.:	1(a)	2(d)	3(c)	
Passage II:	1(b)	2(a)	3(c)	
Passage III:	1(a)	2(b)	3(a)	
Passage IV:	1(a)	2(c)	3(b)	
Passage V:	1(c)	2(c)	3(d)	4(a)
Passage VI:	1(a)	2(c)	3(a)	4(a)

MATRIX MATCH

1. (A) → (Q); (B) → (P, R); (C) → (S); (D) → (T)
2. (A) → (Q); (B) → (P); (C) → (R); (D) → (S); (E) → (U); (F) → (T)
3. (A) → (Q); (B) → (P); (C) → (R); (D) → (S); (E) → (T)
4. (A) → (R); (B) → (P); (C) → (S); (D) → (Q)
5. (A) → (R); (B) → (Q); (C) → (P)
6. (A) → (S); (B) → (P); (C) → (R); (D) → (Q)

Hints and Solutions

Level I

1. (i) The given equation of the ellipse is $9x^2 + 16y^2 = 144$

$$\Rightarrow \frac{x^2}{16} + \frac{y^2}{9} = 1$$

(a) Centre: $C(0, 0)$

(b) Vertices: $A(a, 0)\ A'(-a, 0) = A(4, 0)$ and $A'(-4, 0)$

(c) Co-vertices: $B = (0, b)$ and $B' = (0, -b)$
$\Rightarrow B = (0, 3)$ and $B' = (0, -3)$

(d) Length of the major axis: $2a = 8$

(e) Length of the minor axis: $2b = 6$

(f) Eccentricity $= e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{9}{16}} = \frac{\sqrt{7}}{4}$

(g) Lengths of the latus rectum $= \frac{2b^2}{a} = \frac{18}{4} = \frac{9}{2}$

(h) Equations of the directrices:

$$x = \pm\frac{a}{e}$$

$$\Rightarrow x = \pm\frac{4}{\frac{\sqrt{7}}{4}} = \pm\frac{16}{\sqrt{7}}$$

(i) End-points of the latus recta:

$$L\left(ae, \frac{b^2}{a}\right), L'\left(ae, -\frac{b^2}{a}\right)$$

$$= L\left(\sqrt{7}, \frac{9}{4}\right), L'\left(\sqrt{7}, -\frac{9}{4}\right)$$

(ii) The given equation of the ellipse is

$$2x^2 + 3y^2 - 4x - 12y + 8 = 0$$

$$\Rightarrow 2(x^2 - 2x + 1) + 3(y^2 - 4y + 4) = 6$$

$$\Rightarrow 2(x - 1)^2 + 3(y - 2)^2 = 6$$

$$\Rightarrow \frac{(x-1)^2}{3} + \frac{(y-2)^2}{2} = 1$$

$$\Rightarrow \frac{X^2}{3} + \frac{Y^2}{2} = 1, \text{ where } X = x - 1,\ Y = y - 2$$

(a) Centre: (0, 0)
$\Rightarrow X = 0, Y = 0$
$\Rightarrow x - 1 = 0$ and $y - 2 = 0$
$\Rightarrow x = 1$ and $y = 2$
Hence, the centre is (1, 2).

(b) Vertices : $(\pm a, 0)$
$\Rightarrow X = \pm a, Y = 0$
$\Rightarrow x - 1 = \pm 3, y - 2 = 0$
$\Rightarrow x = 1 \pm 3, y = 2$
$\Rightarrow x = 4, -2, y = 2$
Hence, the vertices are (4, 2) and (–2, 2)

(c) Co-vertices: $(0, \pm b)$
$\Rightarrow X = 0, Y = \pm 3$
$\Rightarrow x - 1 = 0, y - 2 = \pm 3$
$\Rightarrow x = 1, y = 2 \pm 3$
$\Rightarrow x = 1, y = 5, -1$
Hence, the co-vertices are (1, 5), (1, –1).

(d) The length of the major axis $= 2a = 6$

(e) The length of the minor axis $= 2b = 4$

(f) The length of the latus rectum $= \frac{2b^2}{a} = \frac{18}{4} = \frac{9}{2}$

(g) Eccentricity,

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{2}{3}} = \frac{1}{\sqrt{3}}$$

(h) Equations of the directrices:

$$X = \pm\frac{a}{e} = \pm\frac{3}{1/\sqrt{3}} = \pm 3\sqrt{3}$$

$$\Rightarrow x = 1 \pm 3\sqrt{3}$$

(i) End-points of the latus recta:

$$L\left(ae, \frac{b^2}{a}\right); L'\left(ae, -\frac{b^2}{a}\right)$$

$$= L\left(\sqrt{3}, \frac{9}{2}\right); L'\left(\sqrt{3}, -\frac{9}{2}\right)$$

2. The given equation of the ellipse is

$$16x^2 + 25y^2 = 400$$

$$\Rightarrow \frac{x^2}{25} + \frac{y^2}{16} = 1$$

Thus, the sum of the focal distances
$= 2a = 10$ units

3. The given equation of an ellipse is $\frac{x^2}{10 - a} + \frac{y^2}{a - 4} = 1$

$\Rightarrow (a - 4)x^2 + (10 - a)y^2 - (a - 4)(10 - a) = 0$

Since, the given equation represents an ellipse,
$h^2 - ab < 0$
$\Rightarrow 0 - (a - 4)(10 - a) < 0$
$\Rightarrow (a - 4)(a - 10) < 0$
$\Rightarrow 4 < a < 10$
Hence, the length of the interval = 10 – 4 = 6.
Thus, the value of m is 6.

4. Let the foci of the ellipse are S and S' respectively.
Then $S = (5, 12)$ and $S' = (24, 7)$.
Thus, the centre of the ellipse is $C\left(\frac{29}{2}, \frac{19}{2}\right)$.

Now $OC = \frac{1}{2} \times \sqrt{29^2 + 19^2} = \frac{1}{2} \times \sqrt{1202}$

We have,
$SS' = \sqrt{386}$

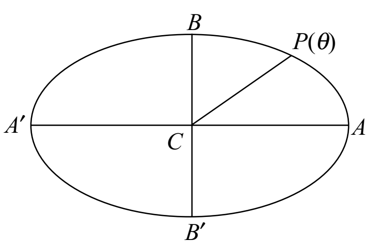
$$\Rightarrow e = \frac{\sqrt{386}}{2a} = \frac{\sqrt{386}}{\sqrt{1202}} = \sqrt{\frac{386}{1202}} = \frac{1}{\sqrt{3}}$$

5. Let $S(2, 0)$ and $S'(-2, 0)$ are two foci of the ellipse and $C(0, 0)$ be the centre of the ellipse.
We have,
$SS' = 4$
$\Rightarrow \quad 2ae = 4$
$\Rightarrow \quad 2a \times \frac{1}{2} = 4 \qquad \Rightarrow a = 4$
Also, $b^2 = a^2(1 - e^2) = 16\left(1 - \frac{1}{4}\right) = 12$
$\Rightarrow \quad b = \sqrt{12} = 2\sqrt{3}$
Hence, the equation of the ellipse is
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
$$\Rightarrow \quad \frac{x^2}{16} + \frac{y^2}{12} = 1$$

6. Given $2ae = \frac{2b^2}{a}$
$\Rightarrow \quad b^2 = a^2 e$
$\Rightarrow \quad a^2(1 - c^2) = a^2 c$
$\Rightarrow \quad (1 - c^2) = c$
$\Rightarrow \quad c^2 + c - 1 = 0$
$\Rightarrow \quad e = \frac{\sqrt{5} - 1}{2}$
Hence, the result

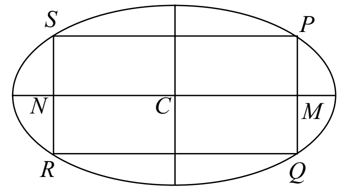
7. The equation of the given ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
It is given that the length of the latus rectum = $\frac{1}{2} \times$ the length of the major axis.
$\Rightarrow \quad \frac{2b^2}{a} = a$
$\Rightarrow \quad b = a$
As we know that,
$$\text{eccentricity} = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - 1} = 0$$

8. Let any point on the ellipse $\frac{x^2}{6} + \frac{y^2}{2} = 1$ is $P(\sqrt{6}\cos\theta, \sqrt{2}\sin\theta)$ and C be the centre of the ellipse.



Therefore,
$CP = \sqrt{5}$
$\Rightarrow \quad 6\cos^2\theta + 2\sin^2\theta = 5$
$\Rightarrow \quad 4\cos^2\theta = 3$
$\Rightarrow \quad \cos\theta = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$
$\Rightarrow \quad \theta = \frac{\pi}{6}$
Hence, the eccentric angle is $\frac{\pi}{6}$.

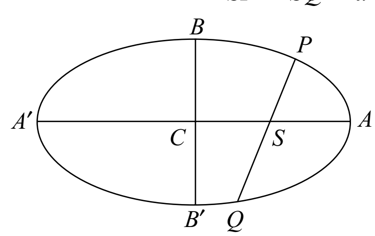
9. Let Q be any point on the given ellipse, whose co-ordinates are $(4\cos^2\theta, 3\sin^2\theta)$.



Thus, $PQ = 8\cos\theta$
and $QR = 6\sin\theta$
Thus, the area of the rectangle $PQRS$
$= PQ \times QR$
$= 48\sin\theta\cos\theta$
$= 24\sin 2\theta$
Hence, the area of the greatest rectangle
$= 24$ sq.u. at $\theta = \frac{\pi}{4}$

10. The co-ordinates of the given point are
$x = 3 + 4\cos\theta, y = 2 + 3\sin\theta$
$\Rightarrow \quad \frac{(x - 3)}{4} = \cos\theta, \frac{(y - 2)}{3} = \sin\theta$
$\Rightarrow \quad \frac{(x - 3)^2}{4^2} + \frac{(y - 2)^2}{3^2} = \cos^2\theta + \sin^2\theta = 1$
$\Rightarrow \quad \frac{(x - 3)^2}{16} + \frac{(y - 2)^2}{9} = 1$
which is the required locus.

11. The equation of the ellipse is
$16x^2 + 25y^2 = 400$
$\Rightarrow \quad \frac{x^2}{25} + \frac{y^2}{16} = 1$
As we know that, if SP and SQ are the focal segments of a focal chord PSQ, then $\frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a}$

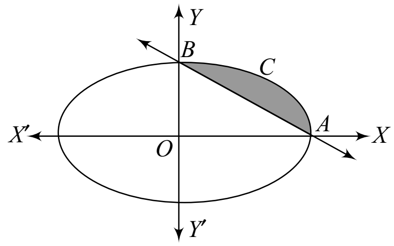


$\Rightarrow \quad \frac{1}{8} + \frac{1}{SQ} = \frac{1}{5}$
$\Rightarrow \quad \frac{1}{SQ} = \frac{1}{5} - \frac{1}{8} = \frac{8 - 5}{40} = \frac{3}{40}$

$\Rightarrow \quad SQ = \dfrac{40}{3}$

Hence, the length of SQ is 40/3.

12. As we know that the area of an ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is πab.



Thus, the area of $OACBO$

$= \left(\dfrac{3}{4} \times \pi \cdot 4 \cdot 3 - \dfrac{1}{2} \cdot 4 \cdot 3\right)$

$= (9\pi - 6)$ s.u.

13. Let $\quad N = \dfrac{x_1^2}{4} + \dfrac{y_1^2}{3} - 1$

Then, $N = \dfrac{4}{4} + \dfrac{9}{3} - 1 = 1 + 3 - 1 = 3$

Since, the value of N is positive at (2, 3), so the point (2, 3) lies outside of the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$.

14. Since the point $(\lambda, -\lambda)$ be an interior point of an ellipse $4x^2 + 5y^2 = 1$, then

$4\lambda^2 + 5\lambda^2 - 1 < 0$

$\Rightarrow \quad 9\lambda^2 - 1 < 0$

$\Rightarrow \quad (3\lambda + 1)(3\lambda - 1) < 0$

$\Rightarrow \quad -\dfrac{1}{3} < \lambda < \dfrac{1}{3}$

Therefore, $m = \dfrac{2}{3}$

$\Rightarrow \quad (3m - 2)^{2013} + 2013 = 0 + 2013 = 2013$

15. We have,

$4.4 + 3.3 - 12 = 16 + 9 - 12 = 13 > 0$

Thus, the point (2, 3) lies outside of the ellipse.

Thus, the number of tangents = 2

16. The given ellipse is

$9x^2 + 16y^2 = 144$

$\Rightarrow \quad \dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$

The equation of any tangent to the given ellipse is

$y = mx + \sqrt{a^2m^2 + b^2}$

$\Rightarrow \quad y = mx + \sqrt{16m^2 + 9}$

which is passing through (2, 3), so

$3 = 2m + \sqrt{16m^2 + 9}$

$\Rightarrow \quad (3 - 2m)^2 = 16m^2 + 9$

$\Rightarrow \quad 9 - 6m + 4m^2 = 16m^2 + 9$

$\Rightarrow \quad 12m^2 + 6m = 0$

$\Rightarrow \quad 2m^2 + m = 0$

$\Rightarrow \quad m(2m + 1) = 0$

$\Rightarrow \quad m = 0, -\dfrac{1}{2}$

Hence, the equation of tangents are $y = 3$ and $y = -\dfrac{1}{2}x + \sqrt{13}$.

17. The given line is

$3x + 4y = 5$

$\Rightarrow \quad 4y = -3x + 5$

and the ellipse is

$9x^2 + 16y^2 = 144$

$\Rightarrow \quad y = -\dfrac{3}{4}x + \dfrac{5}{4}$ and $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$

As we know that the line $y = mx + c$ will be a tangent to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, the points of contacts are $\left(\pm\dfrac{a^2m}{c}, \pm\dfrac{b^2}{c}\right)$

$= \left(\dfrac{\pm 16\left(-\dfrac{3}{4}\right)}{\dfrac{5}{4}}, \dfrac{\pm 9}{\dfrac{5}{4}}\right)$

$= \left(\dfrac{\pm(-48)}{5}, \dfrac{\pm 36}{5}\right)$

18. The equation of the given ellipse is

$9x^2 + 16y^2 = 144$

$\Rightarrow \quad \dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$

The line $y = x + \lambda$ will be a tangent to the ellipse, if

$c^2 = a^2m^2 + b^2$

$\Rightarrow \quad \lambda^2 = 16.1 + 9 = 25$

$\Rightarrow \quad \lambda = \pm 5$

Hence, the values of λ are ± 5.

19. The given ellipse is $\dfrac{x^2}{3} + \dfrac{y^2}{4} = 1$

The equation of any ellipse is

$y = mx \pm \sqrt{a^2m^2 + b^2}$

$\Rightarrow \quad y = 2x \pm \sqrt{3.4 + 2} = 2x \pm \sqrt{14}$

20. The equation of any tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $P(\theta)$ is

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1 \quad \ldots(i)$$

The equation of perpendiculars from centre (0, 0) to tangent is

$$\frac{x}{b}\sin\theta-\frac{y}{a}\cos\theta=0 \quad \ldots(ii)$$

From Eq. (ii), we get

$$\frac{\sin\theta}{by}=\frac{\cos\theta}{ax}=\frac{1}{\sqrt{a^2x^2+b^2y^2}} \quad \ldots(iii)$$

The locus of the feet of perpendiculars is the point of intersection of (i) and (ii).

It is obtained by eliminating θ between Eqs (i) and (ii).

Squaring Eqs (i) and (ii) and adding, we get

$$\frac{x^2}{a^2}\cos^2\theta+\frac{y^2}{b^2}\sin^2\theta+\frac{x^2}{b^2}\sin^2\theta+\frac{y^2}{a^2}\cos^2\theta=1$$

$$\Rightarrow \left(\frac{x^2}{a^2}+\frac{y^2}{a^2}\right)\cos^2\theta+\left(\frac{y^2}{b^2}+\frac{x^2}{b^2}\right)\sin^2\theta=1$$

$$\Rightarrow (x^2+y^2)\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)=1$$

$$\Rightarrow (x^2+y^2)(b^2\cos^2\theta+a^2\sin^2\theta)=a^2b^2$$

$$\Rightarrow (x^2+y^2)\left(\frac{b^2a^2x^2}{a^2x^2+b^2y^2}+\frac{b^2a^2y^2}{a^2x^2+b^2y^2}\right)=a^2b^2$$

$$\Rightarrow (x^2+y^2)^2=(a^2x^2+b^2y^2)$$

Now, put, $x=r\cos\theta$ and $y=r\sin\theta$, we get

$$r^2=a^2\cos^2\theta+b^2\sin^2\theta.$$

Hence, the result.

21. The equation of circle is $x^2+y^2=r^2$.

So the equation of any tangent to the ellipse is

$$y=mx+\sqrt{a^2m^2+b^2}$$

If it is a tangent to a circle also, then the length of the perpendicular from the centre (0, 0) of a circle is equal to the radius of a circle.

Thus, $\left|\frac{\sqrt{a^2m^2+b^2}}{\sqrt{1+m^2}}\right|=r$

$$\Rightarrow a^2m^2+b^2=r^2(1+m^2)=r^2+r^2m^2$$

$$\Rightarrow (a^2-r^2)m^2=(r^2-b^2)$$

$$\Rightarrow m^2=\left(\frac{r^2-b^2}{a^2-r^2}\right)$$

$$\Rightarrow m=\sqrt{\left(\frac{r^2-b^2}{a^2-r^2}\right)}$$

$$\Rightarrow \tan\theta=\sqrt{\left(\frac{r^2-b^2}{a^2-r^2}\right)}$$

$$\Rightarrow \theta=\tan^{-1}\left(\sqrt{\left(\frac{r^2-b^2}{a^2-r^2}\right)}\right)$$

Hence, the result.

22. Let the end-points of a latus rectum are

$L(ae, 0)$ and $L(-ae, 0)$.

The tangent at L is $\frac{xe}{a}+\frac{y}{a}=1$ and the tangent at L' is $\frac{xe}{a}-\frac{y}{a}=1$.

On solving, we get,

$$x=\frac{a}{e} \text{ and } y=0.$$

$\Rightarrow x=\frac{a}{e}$ is the equation of directrix to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

23. The equation of any tangent to the ellipse at $P(\theta)$ is

$$AB: \frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1.$$

Thus $A=(a\sec\theta, 0)$ and $A=(0, b\operatorname{cosec}\theta)$

Let (h, k) be the mid-point of the tangent AB.

Therefore, $h=\frac{a\sec\theta}{2}$ and $k=\frac{b\operatorname{cosec}\theta}{2}$

$$\Rightarrow \cos\theta=\frac{a}{2h} \text{ and } \sin\theta=\frac{b}{2k}$$

Now squaring and adding, we get

$$\frac{a^2}{4h^2}+\frac{b^2}{4k^2}=1$$

$$\Rightarrow \frac{a^2}{h^2}+\frac{b^2}{k^2}=4$$

Hence the locus of (h, k) is

$$\frac{a^2}{x^2}+\frac{b^2}{y^2}=4$$

Now, put $x=r\cos\theta\, c\, y=r\sin\theta$, we get,

$$4r^2=a^2\sin^2+b^2\cos^2\theta,$$

Hence, the result.

24. Given ellipses are $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \ldots(i)$

and $\frac{x^2}{a}+\frac{y^2}{b}=a+b$

$$\Rightarrow \frac{x^2}{a(a+b)}+\frac{y^2}{b(a+b)}=1 \quad \ldots(ii)$$

Let $R(h, k)$ be the points of intersection of the tangents to the ellipse (ii) at P and Q. Then PQ will be the chord of contact.

Thus its equation will be

$$\frac{hx}{a(a+b)}+\frac{ky}{b(a+b)}=1$$

$$\Rightarrow \quad y = -\frac{bh}{ak}x + \frac{b(a+b)}{k} \qquad \ldots\text{(iii)}$$

Since the line (iii) is a tangent to the ellipse (i), so we have

$$\frac{b^2(a+b)^2}{k^2} = a^2 \times \frac{b^2}{a^2} \times \frac{h^2}{k^2} + b^2$$

$$\Rightarrow \quad b^2(a+b)^2 = b^2h^2 + b^2k^2$$

$$\Rightarrow \quad h^2 + k^2 = (a+b)^2$$

Thus the locus of (h, k) is

$$x^2 + y^2 = (a+b)^2$$

which is the director circle of the ellipse

$$\frac{x^2}{a} + \frac{y^2}{b} = a + b$$

Hence, the tangents at P and Q are at right angles.

25. Let two foci of an ellipse are

$S(ae, 0)$ and $S(-ae, 0)$.

The equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $P(\theta)$ is $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$.

Let p_1 and p_2 be the lengths of perpendiculars from the given foci to the given ellipse.

Thus, $p_1 = \left| \dfrac{1 - e\cos\theta}{\sqrt{\dfrac{\cos^2\theta}{a^2} + \dfrac{\sin^2\theta}{b^2}}} \right|$

and $\quad p_2 = \left| \dfrac{1 + e\cos\theta}{\sqrt{\dfrac{\cos^2\theta}{a^2} + \dfrac{\sin^2\theta}{b^2}}} \right|$

$$\Rightarrow \quad p_1 \times p_2 = \left| \frac{1 - e\cos\theta}{\sqrt{\dfrac{\cos^2\theta}{a^2} + \dfrac{\sin^2\theta}{b^2}}} \right| \left| \frac{1 + e\cos\theta}{\sqrt{\dfrac{\cos^2\theta}{a^2} + \dfrac{\sin^2\theta}{b^2}}} \right|$$

$$= \left(\frac{1 - e^2\cos^2\theta}{\dfrac{\cos^2\theta}{a^2} + \dfrac{\sin^2\theta}{b^2}} \right)$$

$$= \left(\frac{a^2b^2(1 - e^2\cos^2\theta)}{b^2\cos^2\theta + a^2\sin^2\theta} \right)$$

$$= \left(\frac{a^2b^2(1 - e^2\cos^2\theta)}{a^2(1-e^2)\cos^2\theta + a^2\sin^2\theta} \right)$$

$$= b^2$$

26. Consider P is the point of intersection of two perpendicular tangents. Thus the locus of P is the director circle.

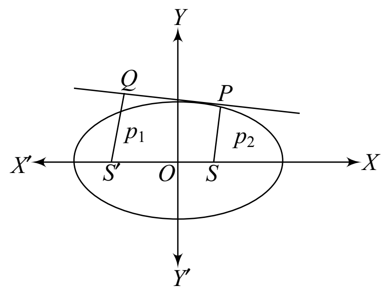
Therefore, the equation of a director circle is

$$x^2 + y^2 = a^2 + b^2$$

This means that the centre of the ellipse will always remain at a constant distance $\sqrt{a^2 + b^2}$ from P.

Hence, the locus of the centre is a circle.

27. The equation of any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $P(\theta)$ is



$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1 \qquad \ldots\text{(i)}$$

The equation of perpendiculars from foci $(\pm ae, 0)$ to the tangent is

$$\frac{x}{b}\sin\theta - \frac{y}{a}\cos\theta = \pm\frac{ae\sin\theta}{b} \qquad \ldots\text{(ii)}$$

Locus of the feet of perpendiculars is the point of intersection of (i) and (ii).

It is obtained by eliminating θ between Eqs (i) and (ii).

Squaring Eqs (i) and (ii) and adding, we get

$$\Rightarrow \quad x^2\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right) + y^2\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right)$$

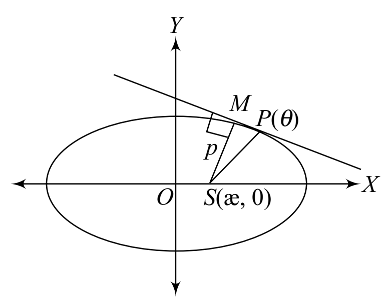
$$= 1 + \frac{a^2e^2\sin^2\theta}{b^2}$$

$$\Rightarrow \quad (x^2 + y^2)\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right) = a^2\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right)$$

$$\Rightarrow \quad (x^2 + y^2) = a^2$$

Hence, the result.

28. The tangent at $P(\theta)$ is $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$.



Also $SP = a - ex = a(1 - e\cos\theta)$

Since p is the length of perpendicular from the focus $S(ae, 0)$,

then, $p = \dfrac{e\cos\theta - 1}{\sqrt{\left(\dfrac{\cos^2\theta}{a^2} + \dfrac{\sin^2\theta}{b^2}\right)}}$

$$\Rightarrow \quad \frac{1}{p^2} = \frac{(b^2\cos^2\theta + a^2\sin^2\theta)}{a^2b^2(e\cos\theta - 1)^2}$$

$$\Rightarrow \quad \frac{b^2}{p^2} = \frac{(b^2\cos^2\theta + a^2\sin^2\theta)}{a^2(e\cos\theta - 1)^2}$$

$$= \left[\frac{a^2(1-e^2)\cos^2\theta + a^2\sin^2\theta}{a^2(e\cos\theta - 1)^2}\right]$$

$$= \left[\frac{(1-e^2)\cos^2\theta + \sin^2\theta}{(e\cos\theta - 1)^2}\right]$$

$$= \left[\frac{(1 - e^2\cos^2\theta)}{(e\cos\theta - 1)^2}\right]$$

$$= \left[\frac{1 + e\cos\theta}{1 - e\cos\theta}\right]$$

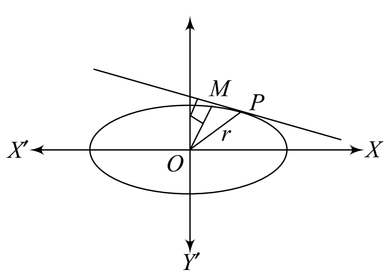
Also,

$$\frac{2a}{SP} - 1 = \frac{2a}{a(1 - e\cos\theta)} - 1 = \frac{1 + e\cos\theta}{1 - e\cos\theta}$$

Hence, $\dfrac{b^2}{p^2} = \dfrac{2a}{SP} - 1$

29. Let the point P be $(a\cos\theta\, c\, b\sin\theta)$.
The equation of the tangent at P be

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$



Now, $p = \left|\dfrac{0 + 0 - 1}{\sqrt{\dfrac{\cos^2\theta}{a^2} + \dfrac{\sin^2\theta}{b^2}}}\right|$

$$\Rightarrow \quad \frac{1}{p^2} = \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}$$

$$= \frac{a^2\sin^2\theta + b^2\cos^2\theta}{a^2b^2}$$

$$\Rightarrow \quad \frac{a^2b^2}{p^2} = a^2\sin^2\theta + b^2\cos^2\theta$$

$$= a^2(1 - \cos^2\theta) + b^2(1 - \sin^2\theta)$$
$$= a^2 + b^2 - (a^2\cos^2\theta + b^2\sin^2\theta)$$
$$= a^2 + b^2 - r^2$$

Hence, the result.

30. Any tangent to the ellipse

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

Let it meet the x-axis at A and y-axis at B. Then the co-ordinates of A and B are

$$A = \left(\frac{a}{\cos\theta}, 0\right) \text{ and } B = \left(0, \frac{b}{\sin\theta}\right)$$

Let $M(h, k)$ be the mid-point of AB.

Then $2h = \dfrac{a}{\cos\theta}$ and $2k = \dfrac{b}{\sin\theta}$

$$\Rightarrow \quad h = \frac{a}{2\cos\theta} \text{ and } k = \frac{b}{2\sin\theta}$$

$$\Rightarrow \quad \cos\theta = \frac{a}{2h} \text{ and } \sin\theta = \frac{b}{2k}$$

Squaring and adding, we get

$$\frac{a^2}{4h^2} + \frac{b^2}{4k^2} = 1$$

$$\Rightarrow \quad \frac{a^2}{h^2} + \frac{b^2}{k^2} = 4$$

Hence, the locus of $M(h, k)$ is

$$\frac{a^2}{x^2} + \frac{b^2}{y^2} = 4$$

31. The tangent at $P(a\cos\theta, b\sin\theta)$ is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

Let it meets the directrix $x = \dfrac{a}{e}$ at Q, where Q is $\left(\dfrac{a}{e}, \dfrac{b(e - \cos\theta)}{e\sin\theta}\right)$.

Slope of $SP = m_1 = \dfrac{b(e - \cos\theta)}{e\sin\theta}$

Slope of $SQ = m_2 = \dfrac{b(e - \cos\theta)}{e\sin\theta} \bigg/ \left(\dfrac{a}{e} - ae\right)$

$$= \frac{b(e - \cos\theta)}{a(1 - e^2)\sin\theta}$$

Thus, $m_1 \times m_2$

$$= \frac{b\sin\theta}{a(\cos\theta - e)} \times \frac{b(e - \cos\theta)}{a(1 - e^2)\sin\theta}$$

$$= -\frac{b^2}{a^2(1 - e^2)} = -\frac{b^2}{b^2} = -1$$

Hence, the result.

32. Any tangent to the ellipse is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1 \quad \ldots\text{(i)}$$

Its point of contact is $P(a\cos\theta\, b\sin\theta)$ and its slope is $-\frac{b}{a}\cot\theta$

Also the focus is $S(ae, 0)$.

Any line through the focus S and perpendicular to the tangent (i) is

$$y - 0 = \frac{a}{b}\tan\theta\,(x - ae) \quad \ldots\text{(ii)}$$

Also the equation of CP is

$$y - 0 = \frac{a}{b}\tan\theta(x - 0) \quad \ldots\text{(iii)}$$

Eliminating θ between Eqs (ii) and (iii), we get

$$\left(\frac{a^2}{b^2}\right)\left(\frac{x-ae}{x}\right) = 1$$

$$\Rightarrow \quad \left(\frac{x-ae}{x}\right) = \left(\frac{b^2}{a^2}\right)$$

$$\Rightarrow \quad \left(1 - \frac{ae}{x}\right) = \left(\frac{b^2}{a^2}\right)$$

$$\Rightarrow \quad \left(1 - \frac{b^2}{a^2}\right) = \left(\frac{ae}{x}\right)$$

$$\Rightarrow \quad \left(1 - \frac{a^2(1-e^2)}{a^2}\right) = \left(\frac{ae}{x}\right)$$

$$\Rightarrow \quad e^2 = \left(\frac{ae}{x}\right)$$

$$\Rightarrow \quad x = \frac{a}{e}$$

Hence, the result.

33. Consider a point P on the ellipse whose coordinates are

$(a\cos\theta\, b\sin\theta)$

The equation of the tangent at P is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

Since the tangent makes equal angles with the axes, so its slope is ± 1.

Thus, $-\frac{b\cos\theta}{a\sin\theta} = \pm 1$

$$\Rightarrow \quad \frac{b^2\cos^2\theta}{a^2\sin^2\theta} = 1$$

$$\Rightarrow \quad \frac{\sin^2\theta}{a^2} = \frac{\cos^2\theta}{b^2} = \frac{1}{a^2+b^2}$$

$$\Rightarrow \quad \sin^2\theta = \frac{a^2}{a^2+b^2} \text{ and } \cos^2\theta = \frac{b^2}{a^2+b^2}$$

Hence the point P is

$$\left(\pm\frac{a^2}{\sqrt{a^2+b^2}}, \pm\frac{b^2}{\sqrt{a^2+b^2}}\right)$$

34. As we know that the locus of the point of intersection of two perpendicular tangents is the director circle.

Hence, the equation of the director circle to the given ellipse is

$x^2 + y^2 = a^2 + b^2 = 16 + 9 = 25$

$\Rightarrow \quad x^2 + y^2 = 25$

35. As we know that the locus of the point of intersection of two perpendicular tangents to an ellipse is the director circle.

Hence, the equation of the director circle is

$x^2 + y^2 = a^2 + b^2 = 4 + 1 = 5$

$\Rightarrow \quad x^2 + y^2 = 5$

which is the required locus of P.

36. The equations of the pair of tangents to the ellipse

$2x^2 + 3y^2 = 1$ from the point $(1, 1)$ is

$(2x^2 + 3y^2 - 1)(2 + 3 - 1) = (2x + 3y - 1)^2$

$\Rightarrow \quad 4(2x^2 + 3y^2 - 1) = (2x + 3y - 1)^2$

$\Rightarrow \quad 4(2x^2 + 3y^2 - 1) = 4x^2 + 9y^2 + 1 + 12xy - 4x - 6y$

$\Rightarrow \quad 4x^2 - 3y^2 - 12xy - 4x + 6y - 3 = 0$

37. The equation of the given ellipse is

$3x^2 + 2y^2 = 5$

$$\Rightarrow \quad \frac{x^2}{5/3} + \frac{y^2}{5/2} = 1$$

The equation of any tangents to the ellipse are

$$y = mx \pm \sqrt{b^2m^2 + a^2}$$

which is passing through $(1, 2)$, so

$$2 = m \pm \sqrt{\frac{5}{2}m^2 + \frac{5}{3}}$$

$$\Rightarrow \quad (2-m)^2 = \frac{5}{2}m^2 + \frac{5}{3}$$

$$\Rightarrow \quad (2-m)^2 = \frac{15m^2+6}{5}$$

$\Rightarrow \quad 9m^2 + 24m - 14 = 0$

Let its roots are m_1 and m_2.

Then $m_1 + m_2 = -24/9$ and $m_1m_2 = -14/9$

Let θ be the angle between them.

Then

$$\tan(\theta) = \left|\frac{m_2 - m_1}{1 + m_1m_2}\right|$$

$$= \left|\frac{\sqrt{(m_1+m_2)^2 - 4m_1m_2}}{1 + m_1m_2}\right|$$

$$= \left| \frac{\sqrt{\frac{64}{9} + \frac{56}{9}}}{1 - \frac{14}{9}} \right| = \frac{\sqrt{120}}{3} \times \frac{9}{5} = 12 \times \sqrt{\frac{5}{6}}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{12\sqrt{6}}{\sqrt{5}}\right)$$

38. Do yourself.
39. Do yourself.
40. Do yourself.
41. Do yourself.
42. Let one end of a latus rectum is $L\left(ae, \frac{b^2}{a}\right)$ and minor axis be $B(0, -b)$.

The equation of the normal to the ellipse at L is

$$\frac{a}{e}x - ay = a^2 - b^2$$

which is passing through $B'(0, -b)$, so we have

$ab = a^2 - b^2$

$\Rightarrow \quad ab = a^2 - a^2(1 - 32) = a^2e^2$

$\Rightarrow \quad b = ae^2$

$\Rightarrow \quad b^2 = a^2c^4$

$\Rightarrow \quad a^2(1 - e^2) = a^2e^4$

$\Rightarrow \quad e^4 + e^2 - 1 = 0$

Hence, the result.

43. Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $P(a\cos\theta\, b\sin\theta)$ be any point on the ellipse.

Then,

$SP = a - ex = a - ae\cos\theta = a(1 - e\cos\theta)$

and

$SP = a + ex = a + ae\cos\theta = a(1 + e\cos\theta)$

The equation of the normal at P is

$ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2$

Now,

$$L = \left(\frac{a^2 - b^2}{a}\cos\theta, 0\right) = (ae^2\cos\theta, 0)$$

$\Rightarrow \quad OL = ae^2\cos\theta$

Thus, $S'L = ae + ae^2\cos\theta = a^3(1 + e\cos\theta)$,

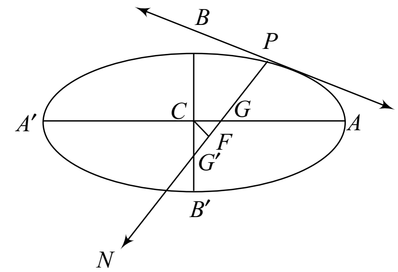
$SL = ae - ae^2\cos = ae(1 - e\cos\theta)$

Now, $\frac{S'P}{SP} = \frac{1 + e\cos\theta}{1 - e\cos\theta} = \frac{S'L}{SL}$

$\Rightarrow$ PL bisects the $\angle S'PS$ internally.

Since $PL \perp PT$, therefore, PT will bisect $\angle S'PS$ externally.

44. Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $P(a\cos\theta, b\sin\theta)$ be any point on the ellipse.



The equation of the normal at P is

$ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2$ ...(i)

Since the line (i) meets the major axis at G and minor axis at G' respectively, then

$$G = \left(\frac{a^2 - b^2}{a}\cos\theta, 0\right)$$

and $$E = \left(0, -\frac{a^2 - b^2}{a}\sin\theta\right)$$

$PF = CQ$ = length of perpendicular from $C(0, 0)$ on the tangent $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ at P.

$$= \frac{ab}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$

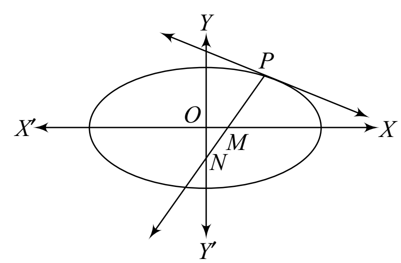
Also, $PG = \frac{b}{a}\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}$

and $PG' = \frac{a}{b}\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}$

Thus, $PF \cdot PG = b^2$ and $PF \cdot PG' = a^2$.

45. The equation of the normal to the ellipse at $P(\theta)$ is

$ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2)$



Thus the co-ordinates of M and N are

$$\left(0, \left(\frac{b^2 - a^2}{b}\right)\sin\theta\right)$$

and $$\left(\left(\frac{a^2 - b^2}{a}\right)\cos\theta, 0\right)$$

$$PM = \sqrt{a^2\cos^2\theta + \left(\frac{b^2 - a^2}{b} - b\right)^2\sin^2\theta}$$

$$= \sqrt{a^2\cos^2\theta + \frac{a^4}{b^2}\sin^2\theta}$$

$$= a^2\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}$$

Also, $PN = \sqrt{\left(a - \dfrac{a^2 - b^2}{a}\right)^2 \cos^2\theta + b^2\sin^2\theta}$

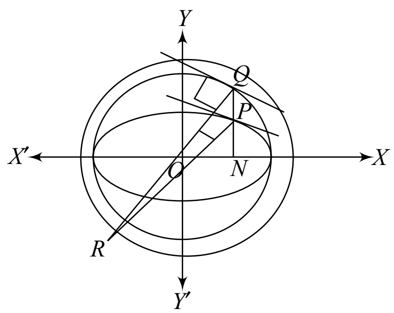
$$= \sqrt{\frac{b^4\cos^2\theta}{a^2} + b^2\sin^2\theta}$$

$$= b^2\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}$$

Thus, $PM : PN = a^2 : b^2$

Hence, the result.

46. Since the ordinate P meets the circle at Q, the co-ordinates of P and Q are (a cos θ, b sin θ) and (a cos θ, a sin θ), respectively.



The equation of the normal to the ellipse at $P(\theta)$ is

$$ax \sec\theta - by \operatorname{cosec}\theta = a^2 - b^2 \quad \ldots\text{(i)}$$

and the equation of normal to the auxiliary circle at Q(a cos θ, a sin θ) is

$$y = (\tan\theta)x \quad \ldots\text{(ii)}$$

Solving, we get

$$\cos\theta = \frac{x}{(a+b)} \text{ and } \sin\theta = \frac{y}{(a+b)}$$

Squaring and adding, we get

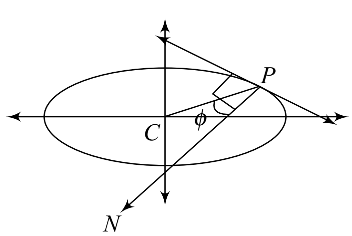
$$x^2 + y^2 = (a+b)^2$$

47. The equation of any normal at $P(\theta)$ is

$$x \sec\theta - by \operatorname{cosec}\theta = a^2 - b^2.$$

$\therefore$ Slope of normal = $m_1 = \dfrac{a}{b}\tan\theta$

and slope of CP = $m_2 = \dfrac{b}{a}\tan\theta$.



Let ϕ be the angle between CP and the normal at P.

$$\therefore \quad \tan\varphi = \left(\frac{\dfrac{a}{b}\tan\theta - \dfrac{b}{a}\tan\theta}{1 + \tan^2\theta}\right)$$

$$= \left(\frac{a^2 - b^2}{ab}\right)\frac{\tan\theta}{\sec^2\theta}$$

$$= \left(\frac{a^2 - b^2}{2ab}\right) \times \sin 2\theta$$

Thus, it is maximum when

$$2\theta = 90° \Rightarrow \theta = 45°.$$

Therefore, the maximum value is

$$\left(\frac{a^2 - b^2}{2ab}\right)$$

48. The equation of any normal to the ellipse at $P(\theta)$ is

$$ax \sec\theta - by \operatorname{cosec}\theta = a^2 - b^2 \quad \ldots\text{(i)}$$

Let C be the centre of the ellipse.

Then CM = Length of perpendicular from the centre C to the normal

$$= \frac{(a^2 - b^2)}{\sqrt{a^2\sec^2\theta + b^2\operatorname{cosec}^2\theta}}$$

$$= \frac{(a^2 - b^2)}{\sqrt{a^2 + b^2 + a^2\tan^2\theta + b^2\cot^2\theta}}$$

$$< \frac{(a^2 - b^2)}{\sqrt{a^2 + b^2 + 2ab}} = a - b$$

49. The equation of tangent at P is

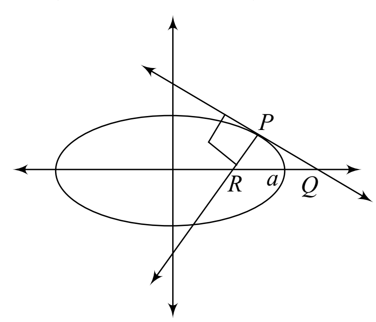
$$\frac{x}{a}\cos\varphi + \frac{y}{b}\sin\varphi = 1 \quad \ldots\text{(i)}$$

and the equation of normal at P is

$$ax \sec\varphi - by \operatorname{cosec}\varphi = a^2 - b^2 \quad \ldots\text{(ii)}$$

$\therefore \quad Q = (a \sec\phi, 0)$

and $\quad R = \left(\dfrac{(a^2 - b^2)\cos\varphi}{a}, 0\right)$



Therefore,

$$QR = a$$

$$\Rightarrow \quad a\sec\varphi - \frac{(a^2 - b^2)\cos\varphi}{a} = a$$

$$\Rightarrow \quad a^2\sin^2\varphi + b^2\cos^2\varphi = a^2\cos\varphi$$

$\Rightarrow \quad a^2 \sin^2 \varphi + a^2(1 - e^2) \cos^2 \varphi = a^2 \cos \varphi$

$\Rightarrow \quad a^2(\sin^2 + \cos^2) - a^2e^2\cos^2 \varphi = a^2\cos \varphi$

$\Rightarrow \quad a^2 - a^2e^2 \cos^2 = a^2 \cos \varphi$

$\Rightarrow \quad c^2 \cos^2 \varphi + \cos \varphi - 1 = 0$

50. The equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_2, y_3)$ are

$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2 \quad \ldots\text{(i)}$$

$$\frac{a^2x}{x_2} - \frac{b^2y}{y_2} = a^2 - b^2 \quad \ldots\text{(ii)}$$

and $$\frac{a^2x}{x_3} - \frac{b^2y}{y_3} = a^2 - b^2 \quad \ldots\text{(iii)}$$

Eliminating x and y from Eqs (i), (ii) and (iii), we get

$$\begin{vmatrix} \frac{a^2}{x_1} & -\frac{b^2}{y_1} & (a^2 - b^2) \\ \frac{a^2}{x_2} & -\frac{b^2}{y_2} & (a^2 - b^2) \\ \frac{a^2}{x_3} & -\frac{b^2}{y_3} & (a^2 - b^2) \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} \frac{1}{x_1} & \frac{1}{y_1} & 1 \\ \frac{1}{x_2} & \frac{1}{y_2} & 1 \\ \frac{1}{x_3} & \frac{1}{y_3} & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x_1 & y_1 & x_1y_1 \\ x_2 & y_2 & x_2y_2 \\ x_3 & y_3 & x_3y_3 \end{vmatrix} = 0$$

Also

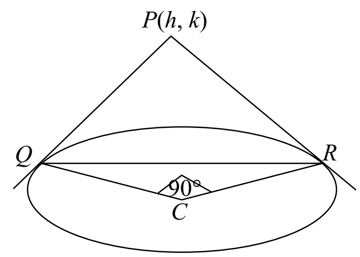
$$\begin{vmatrix} a\cos\alpha & b\sin\alpha & ab\cos\alpha\sin\alpha \\ a\cos\beta & b\sin\beta & ab\cos\beta\sin\beta \\ a\cos\gamma & b\sin\gamma & ab\cos\gamma\sin\gamma \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} \operatorname{cosec}\alpha & \sec\alpha & 1 \\ \operatorname{cosec}\beta & \sec\beta & 1 \\ \operatorname{cosec}\gamma & \sec\gamma & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} \sec\alpha & \operatorname{cosec}\alpha & 1 \\ \sec\beta & \operatorname{cosec}\beta & 1 \\ \sec\gamma & \operatorname{cosec}\gamma & 1 \end{vmatrix} = 0$$

Hence the result.

51. Let the point from which tangents are drawn be (h, k). The equation of the chord of contact from the point (h, k) to the given ellipse is $\frac{hx}{a^2} + \frac{ky}{b^2} = 1$.



It subtends a right angle at the centre (0, 0) of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \ldots\text{(ii)}$$

From Eqs (i) and (ii), we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \left(\frac{xh}{a^2} + \frac{yk}{b^2}\right)^2$$

$$\Rightarrow \quad \left(\frac{1}{a^2} - \frac{h^2}{a^4}\right)x^2 + \left(\frac{1}{b^2} - \frac{k^2}{b^4}\right)y^2 - \frac{2hxky}{a^2b^2} = 0$$

Since these lines are at right angles, therefore sum of the co-efficients of x^2 and y^2 is zero.

$$\Rightarrow \quad \left(\frac{1}{a^2} - \frac{h^2}{a^4}\right) + \left(\frac{1}{b^2} - \frac{k^2}{b^4}\right) = 0$$

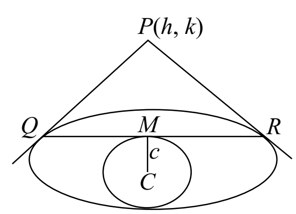
$$\Rightarrow \quad \left(\frac{h^2}{a^4} + \frac{k^2}{b^4}\right) = \frac{1}{a^2} + \frac{1}{b^2}$$

Hence, the locus of (h, k) is

$$\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right) = \frac{1}{a^2} + \frac{1}{b^2}$$

52. Let the point from which the tangents are drawn be (h, k).
So the equation of the chord of contact from the point (h, k) to the given ellipse is

$$\frac{hx}{a^2} + \frac{ky}{b^2} = 1 \quad \ldots\text{(i)}$$



It touches the circle $x^2 + y^2 = c^2$.
Therefore, the length of the perpendicular from the centre (0, 0) to the chord of contact (i) is equal to the radius of the circle.

Thus, $$\left|\frac{0 + 0 - 1}{\sqrt{\frac{h^2}{a^4} + \frac{k^2}{b^4}}}\right| = c$$

$$\Rightarrow \quad \frac{h^2}{a^4}+\frac{k^2}{b^4}=\frac{1}{c^2}$$

Hence, the locus of (h, k) is

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{c^2}$$

53. Let (x_1, y_1) be the point of intersection of the perpendicular tangents, so that (x_1, y_1) lies on the director circle

$$x_1^2+y_1^2=a^2+b^2 \qquad \ldots\text{(i)}$$

The equation of the chord of contact from (x_1, y_1) is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \qquad \ldots\text{(ii)}$$

Let its mid-point be (h, k).

$\therefore$ The equation of the chord bisected at (h, k) is

$$\frac{hx}{a^2}+\frac{ky}{b^2}=\frac{h^2}{a^2}+\frac{k^2}{b^2} \qquad \ldots\text{(iii)}$$

From Eqs (ii) and (iii), we get

$$\frac{x_1}{h}=\frac{y_1}{k}=\frac{1}{\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)}$$

Now from Eqs. (i), we get

$$\left(\frac{h}{\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)}\right)^2+\left(\frac{k}{\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)}\right)^2=(a^2+b^2)$$

$$\Rightarrow \quad (a^2+b^2)\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)^2=(h^2+k^2)$$

Hence, the locus of (h, k) is

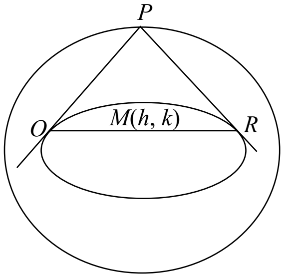
$$(a^2+b^2)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=(x^2+y^2)$$

54. Let the point be $(c\cos\theta, c\sin\theta)$.

The equation of the tangent to the ellipse at $(c\cos\theta, c\sin\theta)$ is

$$\frac{x\cdot c\cos\theta}{a^2}+\frac{y\cdot c\cos\theta}{b^2}=1 \qquad \ldots\text{(i)}$$

Let the co-ordinates of the mid-point be (h, k).



$\therefore$ The equation of the chord bisected at (h, k) is

$$\frac{hx}{a^2}+\frac{ky}{b^2}=\frac{h^2}{a^2}+\frac{k^2}{b^2} \qquad \ldots\text{(ii)}$$

From Eqs (i) and (ii), we get

$$\frac{h}{c\cos\theta}=\frac{k}{c\sin\theta}=\frac{1}{\frac{h^2}{a^2}+\frac{k^2}{b^2}}$$

$$\Rightarrow \quad \cos\theta=\frac{h}{c}\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)$$

and $$\sin\theta=\frac{k}{c}\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)$$

Squaring and adding, we get

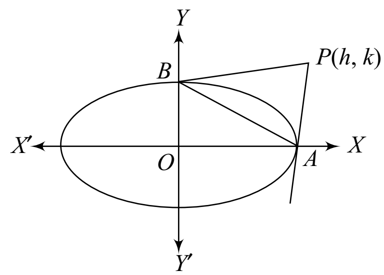
$$\left(\frac{h}{c}\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)\right)^2+\left(\frac{k}{c}\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)\right)^2=1$$

$$\Rightarrow \quad \left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)^2=\left(\frac{c^2}{h^2+k^2}\right)$$

Hence, the locus of (h, k) is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\left(\frac{c^2}{x^2+y^2}\right)$$

55. Let the co-ordinates of P be (h, k).



The equation of its chord of contact with respect to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ is}$$

$$\frac{hx}{a^2}+\frac{ky}{b^2}=1$$

It meets the axes in

$$A\left(\frac{a^2}{h}, 0\right) \text{ and } B\left(0, \frac{b^2}{k}\right)$$

Now, area of the triangle OAB $=\frac{1}{2}\cdot OA\cdot OB$

$$=\frac{1}{2}\cdot\frac{a^2}{h}\cdot\frac{b^2}{k}$$

$$=\frac{a^2b^2}{2\cdot h\cdot k}$$

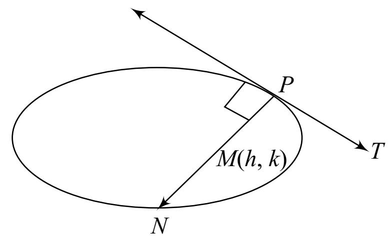
$$=\text{constant}=\frac{\lambda}{2}\text{ (say)}$$

Hence, the locus of (h, k) is $x\cdot y=\dfrac{a^2b^2}{\lambda}$ which represents a hyperbola.

56. Let the point P be $(a\cos\theta, b\sin\theta)$.

$\therefore$ The equation of normal at P to the ellipse is

$$a\cdot x\cdot \sec\theta - b\cdot y\cdot \operatorname{cosec}\theta = a^2 - b^2 \quad \ldots\text{(i)}$$



Let its mid-point be (h, k).

$\therefore$ The equation of the chord bisected at (h, k) is

$$\frac{hx}{a^2}+\frac{ky}{b^2}=\frac{h^2}{a^2}+\frac{k^2}{b^2} \quad \ldots\text{(ii)}$$

From Eqs (i) and (ii), we get

$$\frac{a\sec\theta}{(h/a^2)}=\frac{b\operatorname{cosec}\theta}{(-k/b^2)}=\frac{a^2-b^2}{\left(\dfrac{h^2}{a^2}+\dfrac{k^2}{b^2}\right)}$$

$$\Rightarrow \quad \cos\theta=\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)\Big/\left(\frac{h}{a^3(a^2-b^2)}\right)$$

and $\quad \sin\theta=\left(\dfrac{h^2}{a^2}+\dfrac{k^2}{b^2}\right)\Big/\left(\dfrac{-k}{b^3(a^2-b^2)}\right)$

Squaring and adding, we get

$$\left(\frac{a^6}{x^2}+\frac{b^6}{y^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=(a^2-b^2)^2$$

57. Let the point be (h, k).

The equation of the chord bisected at (h, k) to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is

$$\frac{hx}{a^2}+\frac{ky}{b^2}=\frac{h^2}{a^2}+\frac{k^2}{b^2} \quad \ldots\text{(i)}$$

and the equation of the tangent to the auxiliary circle $x^2+y^2=a^2$ at $(a\cos\theta c\, a\sin\theta)$ is

$$x\cos\theta + y\sin\theta = a \quad \ldots\text{(ii)}$$

From Eqs (i) and (ii), we get

$$\frac{\cos\theta}{h/a^2}=\frac{\sin\theta}{k/b^2}=\frac{a}{\left(\dfrac{h^2}{a^2}+\dfrac{k^2}{b^2}\right)}$$

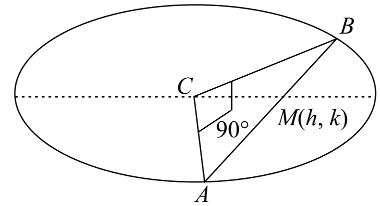
Squaring and adding, we get

$$a^2\left(\frac{h^2}{a^4}+\frac{k^2}{b^4}\right)=\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)^2$$

Hence the locus of (h, k) is

$$a^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2$$

58. Let the point be (h, k).



The equation of the chord bisected at (h, k) to the ellipse is $\dfrac{hx}{a^2}+\dfrac{ky}{b^2}=\dfrac{h^2}{a^2}+\dfrac{k^2}{b^2}$

$$\Rightarrow \quad \left(\frac{\left(\dfrac{hx}{a^2}+\dfrac{ky}{b^2}\right)}{\left(\dfrac{h^2}{a^2}+\dfrac{k^2}{b^2}\right)}\right)=1$$

Now we make it a homogenous equation of 2nd degree.

Thus, $\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\right)=\left(\dfrac{\left(\dfrac{hx}{a^2}+\dfrac{ky}{b^2}\right)}{\left(\dfrac{h^2}{a^2}+\dfrac{k^2}{b^2}\right)}\right)^2$

$$\Rightarrow \quad \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)^2=\left(\frac{hx}{a^2}+\frac{ky}{b^2}\right)^2$$

$$=\frac{h^2x^2}{a^4}+\frac{k^2y^2}{b^4}+\frac{2hkxy}{a^2b^2}$$

Since the chord subtends right angle at the centre, so co-efficient of x^2 + co-efficient of $y^2 = 0$

$$\frac{\alpha^2x^2}{a^4}+\frac{\beta^2y^2}{b^4}=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2$$

$$\Rightarrow \quad \left(\frac{h^2}{a^2}+\frac{\mathrm{k}^2}{\mathrm{b}^2}\right)^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)=\left(\frac{h^2}{a^4}+\frac{k^2}{b^4}\right)$$

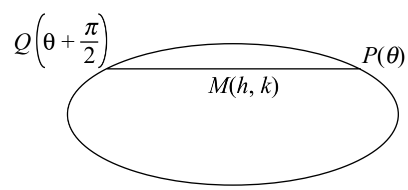
Hence, the locus of (h, k) is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)=\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)$$

59. Let the point P be $(a\cos\theta, b\sin\theta)$ and Q be

$$\left(a\cos\left(\frac{\pi}{2}+\theta\right), b\sin\left(\frac{\pi}{2}+\theta\right)\right),$$

i.e. $(-a\sin\theta s\, b\cos\theta)$.



Let $M(h, k)$ be the mid-point of PQ.

Then, $h = \frac{a}{2}(\cos\theta - \sin\theta)$

and $\quad k = \frac{b}{2}(\cos\theta + \sin\theta)$

$\Rightarrow \quad \frac{2h}{a} = (\cos\theta - \sin\theta)$ and $\frac{2k}{b} = (\cos\theta + \sin\theta)$

Squaring and adding, we get

$$\left(\frac{4h^2}{a^2} + \frac{4k^2}{b^2}\right) = 2$$

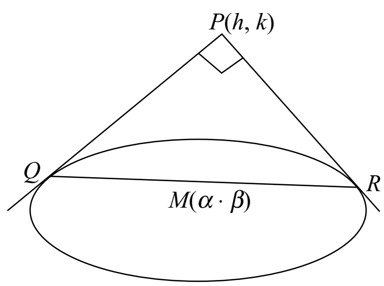
$$\Rightarrow \quad \left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right) = \frac{1}{2}$$

Hence, the locus of (h, k) is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = \frac{1}{2}$$

60. Equation of the chord of contact to the tangents at (h, k) is

$$\frac{hx}{a^2} + \frac{ky}{b^2} = 1 \qquad \ldots\text{(i)}$$



The equation of the chord of the ellipse whose mid-point (α, β) is

$$\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} = \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} \qquad \ldots\text{(ii)}$$

Since Eqs (i) and (ii) are the same, therefore

$$\frac{h}{\alpha} = \frac{k}{\beta} = \frac{1}{\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2}\right)}$$

$$\Rightarrow \quad h = \frac{\alpha}{\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2}\right)}$$

and $\quad k = \frac{\beta}{\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2}\right)}$

Also, (h, k) lies on the director circle of the given ellipse $x^2 + y^2 = a^2 + b^2$.

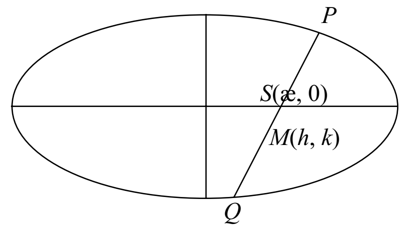
Thus, $h^2 + k^2 = a^2 + b^2$

$$\Rightarrow \quad \left(\frac{\alpha^2 + \beta^2}{a^2 + b^2}\right) = \left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2}\right)^2$$

Hence, the locus of (α, β) is

$$\left(\frac{x^2 + y^2}{a^2 + b^2}\right) = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2$$

61. Let the mid-point be (h, k).



The equation of the chord of the ellipse whose mid-point (h, k) is

$$\frac{hx}{a^2} + \frac{ky}{b^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2}$$

which passes through the focus $(ae, 0)$.

Thus, $\frac{hae}{a^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2}$

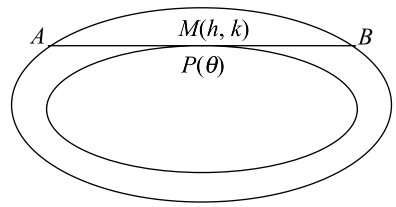
$$\Rightarrow \quad \frac{he}{a} = \frac{b^2h^2 + a^2k^2}{a^2b^2}$$

$$\Rightarrow \quad b^2h^2 + a^2k^2 = ab^2he$$

Hence, the locus of (h, k) is

$$b^2h^2 + a^2y^2 = ab^2xe$$

62. Let the point be (h, k).



The equation of the chord of the ellipse whose middle point (h, k) is

$$\frac{hx}{a^2} + \frac{ky}{b^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2} \qquad \ldots\text{(i)}$$

and the equation of the tangent to the ellipse $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$ at $(\alpha\cos\theta, \beta\sin\theta)$ is

$$\frac{x}{\alpha}\cos\theta + \frac{y}{\beta}\sin\theta = 1$$

Since both the equations are identical, so

$$\frac{\cos\theta/\alpha}{h/a^2} = \frac{\sin\theta/\beta}{k/b^2} = \frac{1}{\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)}$$

Squaring and adding, we get

$$\frac{\alpha^2h^2}{a^4} + \frac{\beta^2k^2}{b^4} = \left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)^2$$

Hence the locus of (h, k) is

$$\frac{\alpha^2 x^2}{a^4} + \frac{\beta^2 y^2}{b^4} = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2$$

63. Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and its focus is $(ae, 0)$.

The equation of the polar is

$$\frac{xx_1}{a^2} + \frac{y \cdot y_1}{b^2} = 1$$

$$\Rightarrow \quad \frac{x(ae)}{a^2} + \frac{y(0)}{b^2} = 1$$

$$\Rightarrow \quad x = \frac{a}{e}$$

which is the directrix.

64. Let the pole be (x_1, y_1).

The equation of the polar is $\frac{xx_1}{a^2} + \frac{y \cdot y_1}{b^2} = 1$ which is identical with $lx + my + n = 0$

So, $\frac{x_1/a^2}{l} = \frac{y_1/b^2}{m} = \frac{-1}{n}$

$$\Rightarrow \quad x_1 = -\frac{a^2 l}{n}, y_1 = -\frac{b^2 m}{n}$$

Hence, the pole is $\left(-\frac{a^2 l}{n}, -\frac{b^2 m}{n}\right)$

65. If (h, k) be the pole of a given line w.r.t. the ellipse, then its equation is

$$\frac{hx}{a^2} + \frac{ky}{b^2} = 1 \quad \ldots\text{(i)}$$

If tangents at its extremities meet at (α, β), then it is the chord of contact of (α, β) and hence its equation is

$$\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} = 1 \quad \ldots\text{(ii)}$$

Comparing Eqs (i) and (ii), we get

$$\frac{h/a^2}{\alpha/a^2} = \frac{k/b^2}{\beta/b^2} = 1$$

$$\Rightarrow \quad h = \alpha, k = \beta$$

Thus, the pole (h, k) is the same as (α, β), i.e. the intersection of tangents.

66. Let the pole be (h, k).
The equation of polar w.r.t. the ellipse $x^2 + 4y^2 = 4$ is

$$hx + 4ky = 4$$

which is identical with $x + 4y = 4$

So, $\frac{h}{1} = \frac{4k}{4} = \frac{4}{4}$

$$\Rightarrow \quad \frac{h}{1} = \frac{k}{1} = 1$$

Hence, the pole is (1, 1)

67. Let the pole be (h, k).
The equation of polar w.r.t. the ellipse $c^2x^2 + d^2y^2 = 1$ is

$$c^2hx + d^2ky = 1$$

$$\Rightarrow \quad d^2ky = -c^2 hx + 1$$

$$\Rightarrow \quad y = -\frac{c^2 h}{d^2 k} x + \frac{1}{d^2 k}$$

which is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

So, $c^2 = a^2m^2 + b^2$

$$\Rightarrow \quad \frac{1}{(d^2 k)^2} = a^2\left(-\frac{c^2 h}{d^2 k}\right)^2 + b^2$$

$$\Rightarrow \quad a^2c^4h^2 + b^2d^1k^2 = 1$$

Hence, the locus of (h, k) is

$$a^2c^4x^2 - b^2d^4y^2 = 1$$

68. Let the pole be (h, k).

The equation of the polar is $\frac{hx}{a^2} + \frac{ky}{b^2} = 1$

It is given that $\left|\frac{0 + 0 - 1}{\sqrt{\frac{h^2}{a^4} + \frac{k^2}{b^4}}}\right| = c$

$$\Rightarrow \quad \frac{h^2}{a^4} + \frac{k^2}{b^4} = \frac{1}{c^2}$$

Hence, the locus of (h, k) is

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{c^2}$$

69. Let the pole be (h, k).

The equation of the polar is $\frac{hx}{a^2} + \frac{ky}{b^2} = 1 \quad \ldots\text{(i)}$

and the equation of the normal to the ellipse is

$$ax \sec \varphi - by \operatorname{cosec} \varphi = a^2 - b^2 \quad \ldots\text{(ii)}$$

Comparing Eqs (i) and (ii), we get

$$\frac{h/a^2}{a \sec \varphi} = -\frac{k/b^2}{b \operatorname{cosec} \varphi} = \frac{1}{a^2 - b^2}$$

$$\Rightarrow \quad \frac{h \cos \varphi}{a^3} = -\frac{k \sin \varphi}{b^3 \operatorname{cosec} \varphi} = \frac{1}{a^2 - b^2}$$

$$\Rightarrow \quad \cos \varphi = \frac{a^3}{h(a^2 - b^2)}, \sin \varphi = -\frac{b^3}{k(a^2 - b^2)}$$

Eliminating ϕ, we get

$$\left(\frac{a^3}{h(a^2 - b^2)}\right)^2 + \left(-\frac{b^3}{k(a^2 - b^2)}\right)^2 = 1$$

$$\Rightarrow \quad \frac{a^6}{h^2} + \frac{b^6}{k^2} = (a^2 - b^2)^2$$

Hence, the locus of (h, k) is $\frac{a^6}{x^2} + \frac{b^6}{y^2} = (a^2 - b^2)^2$

70. Let the pole be (h, k)

The equation of the polar w.r.t. the parabola is

$$yk = 2a(x + h)$$

$$\Rightarrow \quad yk = 2ax + 2ah$$

$$\Rightarrow \quad y = \frac{2a}{k}x + \frac{2ah}{k}$$

which is a tangent to the ellipse $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$

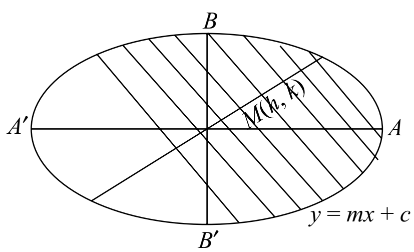
So, $\left(\frac{2ah}{k}\right)^2 = \alpha^2\left(\frac{2a}{k}\right)^2 + \beta^2$

$$\Rightarrow \quad 4a^2h^2 = 4a^2\alpha^2 + k^2\beta^2$$

Hence, the locus of pole (h, k) is

$$4a^2x^2 = y^2\beta^2 + 4a^2\alpha^2$$

71.



Let (h, k) be the mid-point of the chord $y = mx + c$ of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Then $T = S_1$

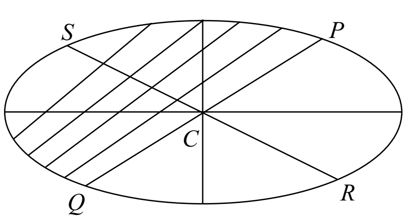
$$\Rightarrow \quad \frac{xh}{a^2} + \frac{yk}{b^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2}$$

$$\Rightarrow \quad k = -\frac{b^2h}{a^2m}$$

Hence, the locus of the mid-point is $y = -\frac{b^2x}{a^2m}$

72. Two diameters are said to be conjugate when each bisects all chords parallel to the other.

If $y = m_1x$ and $y = m_2x$ be two conjugate diameters of an ellipse, then $m_1m_2 = -\frac{b^2}{a^2}$.



Let PQ and RS be two conjugate diameters. Then the co-ordinates of the four extremities of two conjugate diameters are

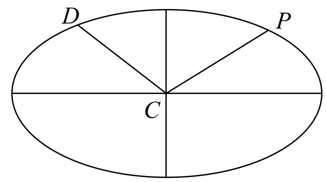
$P(a \cos \varphi, b \sin \varphi)$,

$Q(-a \cos \varphi, b \sin \varphi)$,

$S(-a \sin \varphi, b \cos \varphi)$

and $R(a \sin \varphi, -b \cos \varphi)$

73.



Let CP and CD be two conjugate semi-diameters of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and let the eccentric angle of P is ϕ. Thus the eccentric angle of D is $\frac{\pi}{2} + \varphi$.

Therefore the co-ordinates of P and D are $(a \cos \varphi, b \sin \varphi)$ and

$$\left(a \cos\left(\frac{\pi}{2} + \varphi\right), b \sin\left(\frac{\pi}{2} + \varphi\right)\right)$$

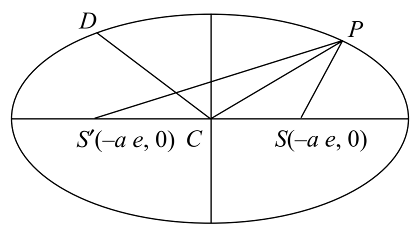
i.e. $(-a \sin \varphi, b \cos \varphi)$

Thus $CP^2 + CD^2$

$= (a^2 \cos^2 \varphi + b^2 \sin^2 \varphi) + (a^2 \sin^2 \varphi + b^2 \cos^2 \varphi)$

$= a^2 + b^2$

74.



Let CP and CD be the conjugate diameters of the ellipse.

Let $P = (a \cos \varphi, b \sin \varphi)$. then the co-ordinates of D is $(-a \sin \varphi, b \cos \varphi)$.

Thus,

$SP \cdot S'P = (a - ae \cos \varphi)(a + ae \cos \varphi)$

$= a^2 - a^2e^2 \cos^2 \varphi$

$= a^2 - (a^2 - b^2) \cos^2 \varphi$

$= a^2 \sin^2 + b^2 \cos^2 \varphi$

$= CD^2$

75 Let the eccentric angle of P is (ϕ) and the eccentric angle of M is $\left(\varphi + \frac{\pi}{2}\right)$.

Then the co-ordinates of P and M are $(a \cos \varphi, b \sin \varphi)$ and

$$\left(a \cos\left(\varphi + \frac{\pi}{2}\right), b \sin\left(\varphi + \frac{\pi}{2}\right)\right)$$

i.e. $(a \cos \varphi, b \sin \varphi)$ and $(-a \sin \varphi, b \cos \varphi)$

The equation of the tangent at P is

$$\frac{x}{a}\cos\varphi + \frac{y}{b}\sin\varphi = 1 \quad \text{...(i)}$$

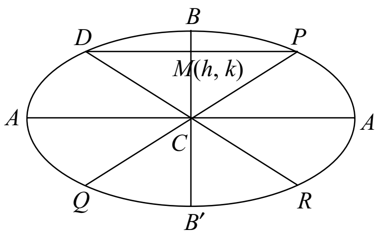
and the equation of the tangent at M is

$$\frac{x}{-a}\sin\varphi + \frac{y}{b}\cos\varphi = 1 \quad \text{...(ii)}$$

Squaring and adding Eqs (i) and (ii), we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$$

76. Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



Let the eccentric angle of P is (ϕ) and the eccentric angle of D is $\left(\varphi + \frac{\pi}{2}\right)$.

Then the co-ordinates of P and D are $(a\cos\varphi, b\sin\varphi)$ and

$$\left(a\cos\left(\varphi + \frac{\pi}{2}\right), b\sin\left(\varphi + \frac{\pi}{2}\right)\right)$$

i.e. $(a\cos\varphi, b\sin\varphi)$ and $(-a\sin\varphi, b\cos\varphi)$

Let $M(h, k)$ be the mid-point of PD.

Then

$$h = \frac{a\cos\varphi - a\sin\varphi}{2}$$

and $$k = \frac{a\cos\varphi + b\sin\varphi}{2} \Rightarrow \frac{(2h)^2}{a^2} + \frac{(2k)^2}{b^2}$$

$$= (\cos\varphi - \sin\varphi)^2 + (\cos\varphi + \sin\varphi)^2$$

$$\Rightarrow \quad \frac{h^2}{a^2} + \frac{k^2}{b^2} = \frac{1}{2}$$

Hence, the locus of (h, k) is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{2}$

77. The equation of the given diameter is

$$ax - by = 0$$

$$\Rightarrow \quad y = \frac{a}{b}x \quad \text{...(i)}$$

Thus, $m_1 = \frac{a}{b}$

Let the diameter conjugate to (i) be $y = m_2 x$

As we know that, two diameters $y = m_1 x$ and $y = m_2 x$ are conjugate, if $m_1 \cdot m_2 = -\frac{b^2}{a^2}$

$$\Rightarrow \quad \frac{a}{b} \times m_2 = -\frac{b^2}{a^2}$$

$$\Rightarrow \quad m_2 = -\frac{b^3}{a^3}$$

Hence, the required diameters is $y = -\frac{b^3}{a^3}x$

$$\Rightarrow \quad a^3 y + b^3 x = 0$$

78. The given ellipses are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{...(i)}$$

and $$\frac{x^2}{c^2} + \frac{y^2}{d^2} = 1 \quad \text{...(ii)}$$

The equation of lines passing through the point of intersection of the ellipses are

$$x^2\left(\frac{1}{a^2} - \frac{1}{c^2}\right) + y^2\left(\frac{1}{b^2} - \frac{1}{d^2}\right) = 0 \quad \text{...(iii)}$$

which represents a pair of lines through the origin.

If $y = mx$ be one of the lines, then $y = mx$ must satisfy (iii), then

$$\left(\frac{1}{a^2} - \frac{1}{c^2}\right) + m^2\left(\frac{1}{b^2} - \frac{1}{d^2}\right) = 0$$

$$\Rightarrow \quad \left(\frac{1}{b^2} - \frac{1}{d^2}\right)m^2 + \left(\frac{1}{a^2} - \frac{1}{c^2}\right) = 0$$

which is a quadratic in m. Let it has roots m_1 and m_2.

Then, $$m_1 m_2 = -\frac{\left(\frac{1}{a^2} - \frac{1}{c^2}\right)}{\left(\frac{1}{b^2} - \frac{1}{d^2}\right)}$$

$$\Rightarrow \quad -\frac{b^2}{a^2} = \frac{\left(\frac{1}{a^2} - \frac{1}{c^2}\right)}{\left(\frac{1}{b^2} - \frac{1}{d^2}\right)}$$

$$\Rightarrow \quad a^2\left(\frac{1}{a^2} - \frac{1}{c^2}\right) = -b^2\left(\frac{1}{b^2} - \frac{1}{d^2}\right)$$

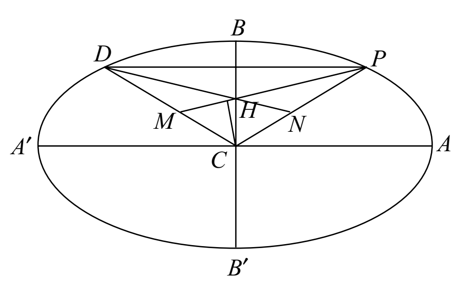
$$\Rightarrow \quad \left(1 - \frac{a^2}{c^2}\right) = \left(-1 + \frac{b^2}{d^2}\right)$$

$$\Rightarrow \quad \frac{a^2}{c^2} + \frac{b^2}{d^2} = 2$$

Hence, the result.

79. The co-ordinates of P and D are $(a\cos\varphi, b\sin\varphi)$ and $(-a\sin\varphi, b\cos\varphi)$

Let PM is the normal at P and DN is the normal at D.



The equations of the normals at P and D are

$$ax \sec \varphi - by \operatorname{cosec} \varphi = a^2 - b^2$$

and $-ax \operatorname{cosec} \varphi - by \sec \varphi = a^2 - b^2$

respectively.

Since, $H(\alpha, \beta)$ is the point of intersection of the normals, so

$$a\alpha \sec \varphi - b\beta \operatorname{cosec} \varphi - (a^2 - b^2) = 0 \quad \ldots\text{(i)}$$

and

$$a\alpha \operatorname{cosec} \varphi + b\beta \sec \varphi + (a^2 - b^2) = 0 \quad \ldots\text{(ii)}$$

Eliminating ϕ from Eqs (i) and (ii), we get

$$\frac{\sec \varphi}{(a\alpha - b\beta)} = \frac{\operatorname{cosec} \varphi}{-(a\alpha + b\beta)} = \frac{(a^2 - b^2)}{(a^2\alpha^2 + b^2\beta^2)}$$

$$\Rightarrow \quad \cos \varphi = \frac{(a^2\alpha^2 + b^2\beta^2)}{(a^2 - b^2)(a\alpha - b\beta)}$$

and $\sin \varphi = \dfrac{(a^2\alpha^2 + b^2\beta^2)}{-(a^2 - b^2)(a\alpha + b\beta)}$

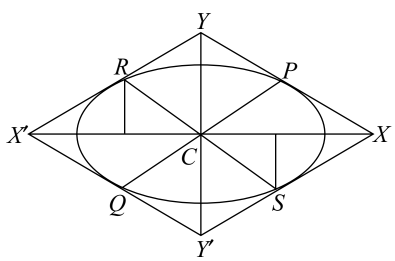
Squaring and adding, we get

$$2(a^2\alpha^2 + b^2\beta^2)^3 = (a^2 - b^2)(a^2\alpha^2 - b^2\beta^2)^2$$

Hence, the locus of $H(\alpha, \beta)$ is

$$2(a^2x^2 + b^2y^2)^3 = (a^2 - b^2)(a^2x^2 - b^2y^2)^2.$$

80.



Let PCQ and RCS be two conjugate diameters of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Then the co-ordinates of P, Q, R, and S are $P(a \cos \varphi, b \sin \varphi)$, $Q(-a \cos \varphi, -b \sin \varphi)$, $R(-a \sin \varphi, b \cos \varphi)$ and $S(a \sin \varphi, -b \cos \varphi)$ respectively.

The equations of tangents at P, R, Q and S are

$$\frac{x}{a}\cos \varphi + \frac{y}{b}\sin \varphi = 1,$$

$$-\frac{x}{a}\sin \varphi + \frac{y}{b}\cos \varphi = 1,$$

$$-\frac{x}{a}\cos \varphi - \frac{y}{b}\sin \varphi = 1,$$

and $\dfrac{x}{a}\sin \varphi - \dfrac{y}{b}\cos \varphi = 1$

Thus, the tangents at P and Q are parallel. Also the tangents at R and S are are parallel. Hence, the tangents at P, R, Q, S form a parallelogram.

Area of the parallelogram = $MNM'N'$

= 4 (the area of the parallelogram $CPMR$)

$$= 4 \times \sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi} \times \frac{ab}{\sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}}$$

$= 4ab$

= constant

Hence, the result.

81. Let the eccentric angles at P and Q be φ and $\left(\dfrac{\pi}{2} + \varphi\right)$ respectively

The equation of the tangents at P and Q

are $\dfrac{x}{a}\cos \varphi + \dfrac{y}{b}\sin \varphi = 1$

and $-\dfrac{x}{a}\sin \varphi + \dfrac{y}{b}\cos \varphi = 1$

respectively

Squaring and adding, we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$$

Hence, the result.

82. Thus, the equation of the ellipse is

$$y^2 + \frac{2}{3}x^2 = 1$$

$$\Rightarrow \quad \frac{x^2}{3} + \frac{y^2}{2} = 1$$

Hence, the eccentricity is

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{2}{3}} = \frac{1}{\sqrt{3}}$$

83. The given equation of an ellipse is

$$16x^2 + 25y^2 = 400$$

$$\Rightarrow \quad \frac{x^2}{25} + \frac{y^2}{16} = 1$$

From the reflection property of an ellipse, the reflection ray passes through the focus.

Thus, the co-ordinates of the other focus = (3, 0).

When $y = 4$, then $x = 0$.

So the point is (0, 4).

So the equation of the reflection ray is

$$\frac{x}{3} + \frac{y}{4} = 1$$

$$\Rightarrow \quad 4x + 3y = 12$$

84. The equation of the incident ray is

$$x - y + 2 = 0$$

$$\Rightarrow \quad \frac{x}{-2} + \frac{y}{2} = 1$$

The equation of the ellipse is

$$3x^2 + 4y^2 = 12$$

$$\Rightarrow \quad \frac{x^2}{4} + \frac{y^2}{3} = 1$$

Clearly, the co-ordinates of the foci are (–2, 0) and (2, 0).

Since the incident ray $x - y + 2 = 0$ intersects the ellipse at (0, 2), so, the equation of the reflection ray = the equation of the line joining (0, 2) and (2, 0)

$\Rightarrow \quad \frac{x}{2} + \frac{y}{2} = 1$

$\Rightarrow \quad x + y = 2$

LEVEL III

1. Given ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

So, $F_1 = (ac, 0)$ and $F_2 = (-ac, 0)$

Let P be $(a\cos\theta, b\sin\theta)$.

Then

$$\text{ar}(\Delta PF_1F_2) = \frac{1}{2}\begin{vmatrix} a\cos\theta & b\sin\theta & 1 \\ ae & 0 & 1 \\ -ae & 0 & 1 \end{vmatrix}$$

$$= \frac{1}{2} \times 2ae \times b\sin\theta$$

$$= abe \times \sin\theta$$

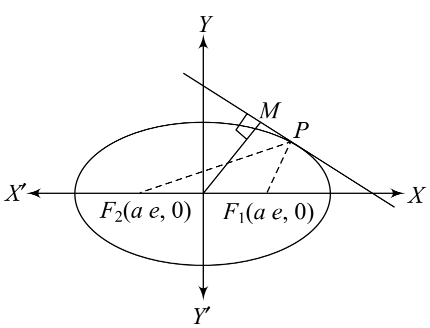
Maximum value of $A = abe$

$$= ab\sqrt{1 - \frac{b^2}{a^2}}$$

$$= b\sqrt{a^2 - b^2}$$

2. Given ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



The equation of the tangent to the given ellipse at $P(a\cos\theta, b\sin\theta)$ is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

Now, $OM = d$

$$\Rightarrow \left|\frac{0 + 0 - 1}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}}\right| = d$$

$$\Rightarrow \frac{1}{d^2} = \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}$$

Now,

$$4a^2\left(1 - \frac{b^2}{d^2}\right)$$

$$= 4a^2 - \frac{4a^2b^2}{d^2}$$

$$= 4a^2 - 4a^2b^2\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right)$$

$$= 4a^2 - 4b^2\cos^2\theta - 4a^2\sin^2\theta$$

$$= 4a^2(1 - \sin^2\theta) - 4b^2\cos^2\theta$$

$$= 4a^2\cos^2\theta - 4b^2\cos^2\theta$$

$$= 4\cos^2\theta(a^2 - b^2)$$

$$= 4\cos^2\theta(a^2e^2)$$

$$= (2ae\cos\theta)^2$$

$$= [(a + ae\cos\theta) - (a - ae\cos\theta)]^2$$

$$= (PF_1 - PF_2)^2$$

3. Given ellipse is

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

$$\therefore \quad \text{Foci} = (\pm ae, 0) = \left(\pm 4 \cdot \frac{\sqrt{7}}{4}, 0\right) = (\pm\sqrt{7}, 0)$$

and the radius of the circle $= \sqrt{7 + 9} = \sqrt{16} = 4$

4. The equation of the tangent to the parabola $y^2 = 4x$ at $(t^2, 2t)$ is

$yt = x + t^2$

$\Rightarrow \quad x - yt + t^2 = 0 \qquad \ldots(i)$

The equation of the normal

$4x^2 + 5y^2 = 20$

i.e. $\frac{x^2}{5} + \frac{y^2}{4} = 1$ at $(\sqrt{5}\cos\varphi, 2\sin\varphi)$

is $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$

$$\Rightarrow \frac{5x}{\sqrt{5}\cos\varphi} - \frac{4y}{2\sin\varphi} = 5 - 4 = 1$$

$$\Rightarrow (\sqrt{5}\sec\varphi)x - (2\,\text{cosec}\,\varphi)y = 1 \qquad \ldots(ii)$$

Since Eqs (i) and (ii) are the same line, so

$$\frac{\sqrt{5}\sec\varphi}{1} = \frac{2\,\text{cosec}\,\varphi}{t} = \frac{-1}{t^2}$$

$$\Rightarrow \sec\varphi = -\frac{1}{\sqrt{5}t^2}, \ \text{cosec}\,\varphi = -\frac{1}{2t}$$

$$\Rightarrow \cos\varphi = -t^2\sqrt{5}, \ \sin\varphi = -2t$$

Thus,

$$5t^2 + 4t^2 - 1 = 0$$

$\Rightarrow \quad 5t^4 + 5t^2 - t^2 - 1 = 0$

$\Rightarrow \quad 5t^2(t^2 + 1) - 1(t^2 + 1) = 0$

$\Rightarrow \quad (5t^2 - 1)(t^2 + 1) = 0$

$\Rightarrow \quad (5t^2 - 1) = 0$

$\Rightarrow \quad t = \pm\dfrac{1}{\sqrt{5}}$

Also, $\tan\varphi = \pm 2$

$\Rightarrow \quad \varphi = \tan^{-1}(\pm 2)$

5. We have $B = (0, b)$, $F = (ae, 0)$ and $F' = (-ae, 0)$

Now, $m(FB) = -\dfrac{b}{ae}$ and $m(BF') = \dfrac{b}{ae}$

It is given that,

$$m(FB) \times m(BF') = -1$$

$\Rightarrow \quad -\dfrac{b}{ae} \times \dfrac{b}{ae} = -1$

$\Rightarrow \quad b^2 = a^2e^2$

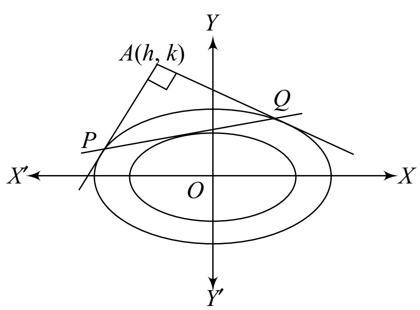
$\Rightarrow \quad a^2(1 - e^2) = a^2e^2$

$\Rightarrow \quad (1 - e^2) = e^2$

$\Rightarrow \quad 2e^2 = 1$

$\Rightarrow \quad e = \dfrac{1}{\sqrt{2}}$

6.



Given ellipse is

$$x^2 + 4y^2 = 4$$

$\Rightarrow \quad \dfrac{x^2}{4} + \dfrac{y^2}{1} = 1 \quad \ldots\text{(i)}$

The equation of the tangent to the ellipse (i) is

$$\frac{x}{2}\cos\theta + y\sin\theta = 1 \quad \ldots\text{(ii)}$$

and the equation of the 2nd ellipse can be written as

$$\frac{x^2}{6} + \frac{y^2}{3} = 1 \quad \text{(iii)}$$

Let the tangents at P and Q meet at $A(h, k)$.

So PQ is the chord of contact.

The equation of the chord of contact of the tangents through A is

$$\frac{hx}{6} + \frac{ky}{3} = 1 \quad \ldots\text{(iv)}$$

Since the eqs (ii) and (iv) are identical, so

$$\frac{\frac{h}{6}}{\frac{\cos\theta}{2}} = \frac{\frac{k}{3}}{\sin\theta} = 1$$

$\Rightarrow \quad h = 3\cos\theta, k = 3\sin\theta$

Now, squaring and adding, we get

$$h^2 = k^2 = 9$$

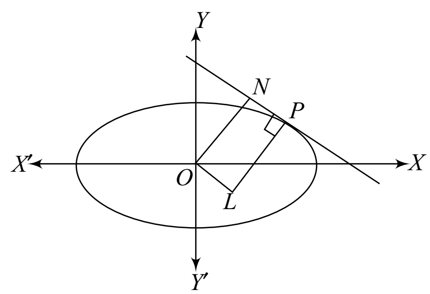
Therefore, the locus of A is

$$x^2 + y^2 = 9$$

which is the director circle of $\dfrac{x^2}{6} + \dfrac{y^2}{3} = 1$

Thus, the angle between the tangents at P and Q of the ellipse $x^2 + 2y^2 = 6$ is right angle.

7.



Given ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Let P lies in the first quadrant, so

$$P = (a\cos\theta, b\sin\theta)$$

The equation of the tangent at P is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

Now, $ON = \left|\dfrac{-1}{\sqrt{\dfrac{\cos^2\theta}{a^2} + \dfrac{\sin^2\theta}{b^2}}}\right|$

$$= \frac{ab}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$

The equation of ON is

$$\frac{x}{b}\sin\theta - \frac{y}{a}\cos\theta = 0$$

and the equation of the normal at P is

$$ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2$$

So, $OL = \dfrac{a^2 - b^2}{\sqrt{a^2\sec^2\theta + b^2\operatorname{cosec}^2\theta}}$

$= \dfrac{(a^2 - b^2)\sin\theta\cos\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$

Now, $NP = OL$

$\Rightarrow \quad NP = \dfrac{(a^2 - b^2)\sin\theta\cos\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$

Therefore,

ar ΔOPN

$= \dfrac{1}{2} \times ON \times NP$

$= \dfrac{1}{2} \times \dfrac{ab(a^2 - b^2)\sin\theta\cos\theta}{a^2\sin^2\theta + b^2\cos^2\theta}$

$= \dfrac{1}{2} \times \dfrac{ab(a^2 - b^2)}{a^2\tan\theta + b^2\cot\theta}$

$\leq \dfrac{1}{2} \times \dfrac{ab(a^2 - b^2)}{2ab} = \dfrac{(a^2 - b^2)}{4}$

at $\tan\theta = \dfrac{b}{a}$.

Thus, the point P is

$$\left(\frac{a^2}{\sqrt{a^2 + b^2}}, \frac{b^2}{\sqrt{a^2 + b^2}}\right)$$

By symmetry, we have four such points.

Thus, $\left(\pm\dfrac{a^2}{\sqrt{a^2 + b^2}}, \pm\dfrac{b^2}{\sqrt{a^2 + b^2}}\right)$

8. Given ellipse is $\dfrac{x^2}{(5/3)} + \dfrac{y^2}{(5/2)} = 1$

The equation of any tangent to the given ellipse is

$y = mx + \sqrt{b^2m^2 + a^2}$

$\Rightarrow \quad y = mx + \sqrt{\dfrac{5}{2}m^2 + \dfrac{5}{3}}$

which is passing through (1, 2).

So, $\quad 2 = m + \sqrt{\dfrac{5}{2}m^2 + \dfrac{5}{3}}$

$\Rightarrow \quad (2 - m)^2 = \left(\dfrac{5}{2}m^2 + \dfrac{5}{3}\right)$

$\Rightarrow \quad 6m^2 - 24m + 24 = 15m^2 + 10$

$\Rightarrow \quad 9m^2 + 24m - 14 = 0$

Let its roots are m_1 and m_2

So, $\quad m_1 + m_2 = -\dfrac{8}{3}, m_1m_2 = -\dfrac{14}{9}$

Let θ be the angle between the tangents

Then $\tan(\theta) = \dfrac{m_2 - m_1}{1 + m_1m_2}$

$= \dfrac{\sqrt{(m_2 + m_1)^2 - 4m_1m_2}}{1 + m_1m_2}$

$= \left|\dfrac{\sqrt{\dfrac{64}{9} + \dfrac{56}{9}}}{1 - \dfrac{14}{9}}\right| = \dfrac{2\sqrt{30}}{3} \times \dfrac{9}{5} = \dfrac{6\sqrt{6}}{\sqrt{5}}$

Thus, $\theta = \tan^{-1}\left(\dfrac{6\sqrt{6}}{\sqrt{5}}\right)$

9. Normal at $P(a\cos\theta, b\sin\theta)$ is

$ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2$

$\Rightarrow \quad ax\sec\theta - by\operatorname{cosec}\theta = 14 - 5 = 9$

which meets the ellipse again at Q. So

$a^2\cos 2\theta\sec\theta - b^2\sin 2\theta\operatorname{cosec}\theta = 9$

$\Rightarrow \quad 14\cos 2\theta\sec\theta - 5\sin 2\theta\operatorname{cosec}\theta = 9$

$\Rightarrow \quad 28\cos^2\theta - 14 - 10\cos^2\theta = 9\cos\theta$

$\Rightarrow \quad 18\cos^2\theta - 9\cos\theta - 14 = 0$

$\Rightarrow \quad (6\cos\theta - 7)(3\cos\theta + 2) = 0$

$\Rightarrow \quad \cos\theta = -\dfrac{2}{3}, \dfrac{7}{6}$

Thus, $\cos\theta = -\dfrac{2}{3}$

10. Given ellipse is $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$ …(i)

The equation of the chord bisected at (h, k) is

$T = S_1$

$\Rightarrow \quad \dfrac{hx}{a^2} + \dfrac{ky}{b^2} = \dfrac{h^2}{a^2} + \dfrac{k^2}{b^2}$

$\Rightarrow \quad \dfrac{hx}{25} + \dfrac{ky}{16} = \dfrac{h^2}{25} + \dfrac{k^2}{16}$

$\Rightarrow \quad \dfrac{(1/5)x}{25} + \dfrac{(2/5)y}{16} = \dfrac{(1/5)^2}{25} + \dfrac{(2/5)^2}{16}$

Solving, we get

$4x + 5y = 4$ …(ii)

Solving Eqs (i) and (ii), we get

$x = 4, -3$ and $y = -\dfrac{12}{5}, \dfrac{16}{5}$

Let the chord be AB, where

$A = \left(4, -\dfrac{12}{5}\right)$ and $B = \left(-3, \dfrac{16}{5}\right)$

Hence, the length the AB is

$= \sqrt{(7)^2 + \left(\dfrac{28}{5}\right)^2} = 7\sqrt{1 + \dfrac{16}{25}} = \dfrac{7\sqrt{41}}{5}$

11. Any tangent to the ellipse is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1 \quad \ldots\text{(i)}$$

Its point of contact is $P(a\cos\theta c - b\sin\theta)$ and its slope is $-\frac{b}{a}\cot\theta$. Also the focus is $S(ae, 0)$.

Any line through the focus S and perpendicular to tangent (i) is

$$y - 0 = \frac{a}{b}\tan\theta\,(x - ae) \quad \ldots\text{(ii)}$$

Also the equation of CP is

$$y - 0 = \frac{a}{b}\tan\theta(x - 0) \quad \ldots\text{(iii)}$$

Eliminating θ between Eqs (ii) and (iii), we get

$$\left(\frac{a^2}{b^2}\right)\left(\frac{x - ae}{x}\right) = 1$$

$$\Rightarrow \left(\frac{x - ae}{x}\right) = \left(\frac{b^2}{a^2}\right)$$

$$\Rightarrow \left(1 - \frac{ae}{x}\right) = \left(\frac{b^2}{a^2}\right)$$

$$\Rightarrow \left(1 - \frac{b^2}{a^2}\right) = \left(\frac{\text{ae}}{\text{x}}\right)$$

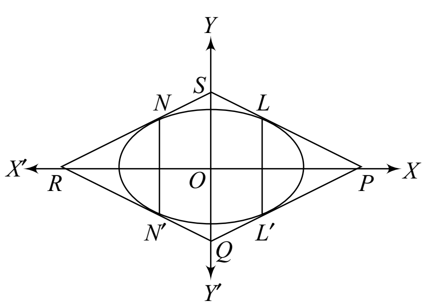
$$\Rightarrow \left(1 - \frac{a^2(1 - e^2)}{a^2}\right) = \left(\frac{ae}{x}\right)$$

$$\Rightarrow e^2 = \left(\frac{ae}{x}\right)$$

$$\Rightarrow x = \frac{a}{e}$$

Hence, the result.

12.



Given ellipse is $\frac{x^2}{9} + \frac{y^2}{5} = 1$

Now, $e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{5}{9}} = \frac{2}{3}$

$$L = \left(ae, \frac{b^2}{a}\right) = \left(2, \frac{5}{3}\right)$$

$$L' = \left(ae, -\frac{b^2}{a}\right) = \left(2, -\frac{5}{3}\right)$$

$$N = \left(-ae, \frac{b^2}{a}\right) = \left(-2, \frac{5}{3}\right) \text{ and}$$

$$N' = \left(-ae, -\frac{b^2}{a}\right) = \left(-2, -\frac{5}{3}\right)$$

Now, the tangent at L is

$$\frac{xx_1}{9} + \frac{yy_1}{5} = 1$$

$$\Rightarrow \frac{2x}{9} + \frac{\frac{5}{3}y}{5} = 1$$

$$\Rightarrow \frac{x}{9/2} + \frac{y}{3} = 1$$

Thus, $P = \left(\frac{9}{2}, 0\right)$ and $S = (0, 3)$

Therefore,

$$\begin{aligned} \text{ar(quad } PQRS) &= 4 \times \text{ar}(\Delta OPS) \\ &= 4 \times \frac{1}{2} \times \frac{9}{2} \times 3 \\ &= 27 \text{ s.u.} \end{aligned}$$

13. The equation of the tangent at $(3\sqrt{3}\cos\theta, \sin\theta)$ to the given ellipse is

$$\frac{x \cdot 3\sqrt{3}\cos\theta}{27} + y \cdot \sin\theta = 1$$

$$\Rightarrow \frac{x}{3\sqrt{3}\sec\theta} + \frac{y}{(\operatorname{cosec}\theta)} = 1$$

It is given that

$$S = 3\sqrt{3}\sec\theta + \operatorname{cosec}\theta$$

$$\Rightarrow \frac{dS}{d\theta} = 3\sqrt{3}\sec\theta\tan\theta - \operatorname{cosec}\theta\cot\theta$$

For maximum or minimum, $\frac{dS}{d\theta} = 0$ gives

$$\Rightarrow 3\sqrt{3}\sec\theta\tan\theta - \operatorname{cosec}\theta\cot\theta = 0$$

$$\Rightarrow 3\sqrt{3}\sec\theta\tan\theta = \operatorname{cosec}\theta\cot\theta$$

$$\Rightarrow \frac{\operatorname{cosec}\theta\cot\theta}{\sec\theta\tan\theta} = 3\sqrt{3}$$

$$\Rightarrow \cot^3\theta = 3\sqrt{3}$$

$$\Rightarrow \cot\theta = \sqrt{3}$$

$$\Rightarrow \quad \theta = \frac{\pi}{6}$$

Hence, the value of θ is $\frac{\pi}{6}$.

14. Let the mid-point be (h, k).
The equation of the ellipse is $x^2 + 2y^2 = 2$.

$$\frac{x^2}{2} + y^2 = 1$$

and the equation of the tangent to the ellipse at $(\sqrt{2}\cos\theta, \sin\theta)$ is

$$\frac{x \cdot \sqrt{2}\cos\theta}{2} + y \cdot \sin\theta = 1.$$

$$\Rightarrow \quad \frac{x}{\sqrt{2}\sec\theta} + \frac{y}{\operatorname{cosec}\theta} = 1.$$

Let $A = (\sqrt{2}\sec\theta, 0)$ and $B = (0, \operatorname{cosec}\theta)$

Thus, $h = \frac{\sqrt{2}\sec\theta}{2}, k = \frac{\operatorname{cosec}\theta}{2}$

$$\Rightarrow \quad \cos\theta = \frac{1}{\sqrt{2}h} \text{ and } \sin\theta = \frac{1}{2k}$$

Squaring and adding, we get

$$\frac{1}{2h^2} + \frac{1}{4k^2} = 1$$

Hence, the locus of (h, k) is

$$\frac{1}{2x^2} + \frac{1}{4y^2} = 1$$

15. The equation of any tangent to the given ellipse at $(a\cos\theta, b\sin\theta)$ is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

$$\Rightarrow \quad \frac{x}{a\sec\theta} + \frac{y}{b\operatorname{cosec}\theta} = 1$$

Let $A = (a\sec\theta, 0)$ and $B = (0, b\operatorname{cosec}\theta)$

$$\Delta = \text{ar}(\Delta OAB) = \frac{1}{2\,ab\sin\theta\operatorname{cosec}\theta}$$

$$= \frac{1}{ab\sin 2\theta}$$

Hence, the minimum area is $\frac{1}{ab}$ sq. unit.

16. The equation of any tangent to the ellipse is

$$y = mx + \sqrt{a^2m^2 + b^2}$$

$$\Rightarrow \quad y = mx + \sqrt{25m^2 + 4}$$

$$\Rightarrow \quad mx - y + \sqrt{25m^2 + 4} = 0$$

which is also a tangent to the given circle. So, the length of the perpendicular from the centre to the tangent is equal to the radius of a circle

$$\frac{\sqrt{25m^2 + 4}}{\sqrt{m^2 + 1}} = 4$$

$$\Rightarrow \quad 25m^2 + 4 = 16(m^2 + 1)$$

$$\Rightarrow \quad 9m^2 = 12$$

$$\Rightarrow \quad m^2 = \frac{4}{3}$$

$$\Rightarrow \quad m = \pm\frac{2}{\sqrt{3}}$$

Hence, the equation of the common tangent is

$$y = -\frac{2}{\sqrt{3}}x + \sqrt{\frac{100}{3} + 4}$$

$$y = -\frac{2}{\sqrt{3}}x + \sqrt{\frac{112}{3}}$$

Let $A = (2\sqrt{7}, 0)$ and $B = \left(0, \sqrt{\frac{112}{3}}\right)$

Thus, the length of the tangent

$$= \sqrt{28 + \frac{112}{3}} = \sqrt{\frac{196}{3}} = \frac{14}{\sqrt{3}}$$

17. Let (h, k) be the point of intersection of tangents at θ and φ. Then

$$\frac{h}{a} = \frac{\cos\left(\frac{\theta+\varphi}{2}\right)}{\cos\left(\frac{\theta-\varphi}{2}\right)} \text{ and } \frac{k}{b} = \frac{\sin\left(\frac{\theta+\varphi}{2}\right)}{\cos\left(\frac{\theta-\varphi}{2}\right)}$$

Squaring and adding, we get

$$\frac{h^2}{a^2} + \frac{k^2}{b^2} = \frac{1}{\cos^2\left(\frac{\theta-\varphi}{2}\right)} \quad \ldots\text{(i)}$$

It is given that,

$$b(\sin\theta + \sin\varphi) = 3$$

$$\Rightarrow \quad (\sin\theta + \sin\varphi) = 1 \quad (\because b = 3)$$

$$\Rightarrow \quad 2\sin\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right) = 1$$

Now, $$\frac{k}{b} = \frac{\sin\left(\frac{\theta+\varphi}{2}\right)}{\cos\left(\frac{\theta-\varphi}{2}\right)} = \frac{1}{2\cos^2\left(\frac{\theta-\varphi}{2}\right)} \quad \ldots\text{(ii)}$$

From Eqs (i) and (ii), we get

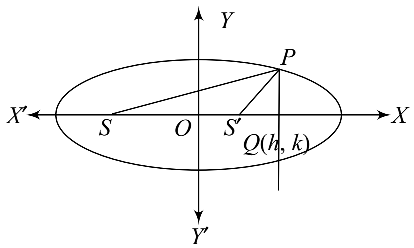
$$\frac{h^2}{a^2} + \frac{k^2}{b^2} = \frac{2k}{b}$$

$$\Rightarrow \quad \frac{h^2}{25} + \frac{k^2}{9} = \frac{2k}{3}$$

Hence, the locus of (h, k) is

$$\frac{x^2}{25} + \frac{y^2}{9} = \frac{2y}{3}$$

18. Given ellipse is $\frac{x^2}{25}+\frac{y^2}{16}=1$



Let the point Q be (h, k).
Clearly, k is negative.
It is given that,

$$SP = NQ$$

$\Rightarrow$ $-k = a + eh$

$\Rightarrow$ $-k = 5+\frac{3}{5}h$

Hence, the locus of Q is

$$-y = 5+\frac{3}{5}x$$

$\Rightarrow$ $3x + 5y + 25 = 0$

19. The Equation of the tangent to the ellipse is

$$y = mx+\sqrt{a^2m^2+b^2}$$
$$= 2x+\sqrt{4a^2+b^2}$$

which is a normal to the given circle
Clearly, it will pass through the centre of a circle, so

$$0 = -4+\sqrt{4a^2+b^2}$$

$\Rightarrow$ $\sqrt{4a^2+b^2}=4$

$\Rightarrow$ $4a^2 + b^2 = 16$

Let $P = ab = a\sqrt{16-4a^2}$

$\Rightarrow$ $P^2 = a^2(16 - 4a^2)$

$\Rightarrow$ $Q = 16a^2 - 4a^4$

$\Rightarrow$ $\frac{dQ}{da}=32a-16a^3$

$\Rightarrow$ $\frac{d^2Q}{da^2}=32-48a^2$

For maximum or minimum,

$$\frac{dQ}{da}=0$$

$\Rightarrow$ $32a - 16a^3 = 0$

$\Rightarrow$ $16a(a^2 - 2) = 0$

$\Rightarrow$ $a = 0, \pm\sqrt{2}$

when $a=\pm\sqrt{2}$, then $b=\pm 2\sqrt{2}$

Hence, the maximum value of ab is 4.

20. Any point on the ellipse be $(a\cos\theta + a\sin\theta)$.
Clearly, the centre is (0, 0)

Now, distance, $r=\sqrt{a^2\cos^2\theta+a^2\sin^2\theta}=a$

Also,

$$a^2\cos^2\theta + 2a^2\sin\theta\cos\theta + 2a^2\sin^2\theta = 1$$

$\Rightarrow$ $a^2=\frac{1}{1+\sin 2\theta+\sin^2\theta}$

$\Rightarrow$ $a=\frac{1}{\sqrt{1+\sin 2\theta+\sin^2\theta}}$

Therefore, $r=a=\frac{1}{\sqrt{1+\sin 2\theta+\sin^2\theta}}$

$\Rightarrow$ $r=\frac{1}{\sqrt{\frac{3}{2}+\sin 2\theta-\frac{1}{2}\cos 2\theta}}$

r will be maximum when Δr will provide us minimum value
Hence, the maximum value of r

$$=\frac{\sqrt{2}}{\sqrt{3-\sqrt{5}}}$$

21. Let $M(h, k)$ the mid-point of the chord PQ.
Since the length of PQ is $2c$, so P and Q can be considered as $(h + c\cos\theta, k + c\sin\theta)$ and $(h - c\cos\theta, k - c\sin\theta)$ respectively.

Thus, $\frac{(h+c\cos\theta)^2}{a^2}+\frac{(k+c\sin\theta)^2}{b^2}=1$

and $\frac{(h-c\cos\theta)^2}{a^2}+\frac{(k-c\sin\theta)^2}{b^2}=1$

Adding, we get

$$b^2h^2 + a^2k^2 - a^2b^2 + c^2(a^2\sin^2\theta + a^2\cos^2\theta) = 0 \quad \ldots\text{(i)}$$

and subtracting, we get

$$\frac{4ch}{a^2}\cos\theta-\frac{4ch}{b^2}\sin\theta=0$$

$$\frac{\sin\theta}{b^2h}=\frac{\cos\theta}{-a^2k}=\frac{1}{\sqrt{h^2b^4+k^2a^4}} \quad \ldots\text{(ii)}$$

From Eqs (i) and (ii), we get

$$b^2h^2 + a^2k^2 - a^2b^2$$
$$+c^2\left(\frac{b^4h^2a^2}{b^4h^2+k^2a^4}+\frac{a^4k^2b^2}{b^4h^2+k^2a^4}\right)=0$$

Hence, the locus of $M(h, k)$ is

$$\left(\frac{b^2x^2+a^2y^2}{a^4y^2+b^4x^2}\right)=\frac{1}{c^2}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)$$

22. Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and its pole be (h, k)

The equation of polar w.r.t. the given ellipse is

$$\frac{hx}{a^2}+\frac{ky}{b^2}=1$$

$$\Rightarrow \quad \frac{ky}{b^2}=-\frac{hx}{a^2}+1$$

$$\Rightarrow \quad y=-\frac{b^2h}{a^2k}x+\frac{b^2}{k} \qquad \ldots(\text{i})$$

which is a tangent to the parabola

$ay^2 = -2b^2x$

$$\Rightarrow \quad y^2=\left(-\frac{2b^2}{a}\right)x$$

$$\Rightarrow \quad y^2=4\left(-\frac{b^2}{2a}\right)x \qquad \ldots(\text{ii})$$

Since (i) is tangent to (ii), so

$$\frac{b^2}{k}=\frac{\left(-\frac{b^2}{2a}\right)}{\left(-\frac{b^2h}{a^2k}\right)}$$

$$\Rightarrow \quad \frac{b^2}{k}=\frac{ak}{2h}$$

$$\Rightarrow \quad k^2=\frac{2b^2}{a}h$$

Hence, the locus of (h, k) is

$$y^2=\left(\frac{2b^2}{a}\right)x$$

which represents a parabola.

23. Let the equation of the ellipse be $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=1$

The equation of a tangent to the given ellipse is

$$x\cos\alpha + y\sin\alpha = \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha} \qquad \ldots(\text{i})$$

After rotation, the equation of the tangent is

$$x\cos(\alpha+90°)+y\sin(\alpha+90°) = \sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha}$$

$$\Rightarrow \quad -x\sin\alpha + y\cos\alpha = \sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha} \qquad \ldots(\text{ii})$$

On subtraction, we get

$$(x+y)\sin\alpha + (x-y)\cos\alpha = 0$$

$$\Rightarrow \quad \frac{\sin\alpha}{(y-x)}=\frac{\cos\alpha}{(x+y)}=\frac{1}{\sqrt{2(x^2+y^2)}}$$

Putting the values of $\sin\alpha$ and $\cos\alpha$ in Eq. (i), we get

$$x(x+y)+y(y-x)=\sqrt{a^2(y+x)^2+b^2(y-x)^2}$$

$$\Rightarrow \quad (x^2+y^2)=\sqrt{a^2(y+x)^2+b^2(y-x)^2}$$

$$\Rightarrow \quad (x^2+y^2)=\sqrt{(a^2+b^2)(x^2+y^2)+2xy(a^2-b^2)}$$

$$\Rightarrow \quad (x^2+y^2)^2=(a^2+b^2)(x^2+y^2)+2xy(a^2-b^2)$$

$$\Rightarrow \quad (x^2+y^2)(x^2+y^2-a^2-b^2)=2xy(a^2-b^2)$$

which is the required locus.

24. Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the point P be (h, k).

The equation of any tangent to the ellipse be

$$y = mx+\sqrt{a^2m^2+b^2}$$

which is passing through P.

So, $\quad k = mh+\sqrt{a^2m^2+b^2}$

$$\Rightarrow \quad (k-mh)^2=(a^2m^2+b^2)$$

$$\Rightarrow \quad k^2-2hkm+m^2h^2=(a^2m^2+b^2)$$

$$\Rightarrow \quad (h^2-a^2)m^2-2hkm+(k^2-b^2)=0$$

It has two roots, say m_1 and m_2.

Thus, $m_1+m_2=\frac{2hk}{(h^2-a^2)}$

and $\quad m_1m_2=\frac{k^2-b^2}{h^2-a^2}$

It is given that, $\theta_1+\theta_2=2\alpha$

$$\Rightarrow \quad \tan(\theta_1+\theta_2)=\tan(2\alpha)$$

$$\Rightarrow \quad \frac{\tan\theta_1+\tan\theta_2}{1-\tan\theta_1\cdot\tan\theta_2}=\tan(2\alpha)$$

$$\Rightarrow \quad \frac{m_1+m_2}{1-m_1m_2}=\tan(2\alpha)$$

$$\Rightarrow \quad \frac{2hk}{h^2-a^2-k^2+b^2}=\tan(2\alpha)$$

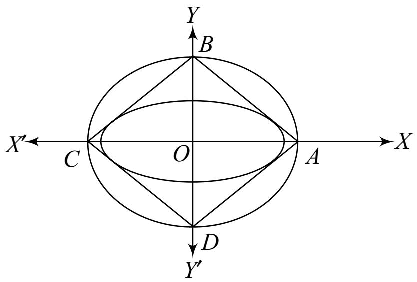
$$\Rightarrow \quad \frac{2hk}{(h^2-k^2)+(b^2-a^2)}=\tan(2\alpha)$$

Hence, the locus of $P(h, k)$ is

$$\frac{2xy}{(x^2-y^2)+(b^2-a^2)}=\tan(2\alpha)$$

$$\Rightarrow \{(x^2-y^2)+(b^2-a^2)\}\tan(2\alpha)=2xy$$

25. Given ellipse is $\frac{x^2}{16}+\frac{y^2}{9}=1$



Clearly, the vertices of the square lie on the director circle of the given ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$.

The equation of the director circle is

$x^2+y^2=16+9=25$

Thus, the length of AC is 10 which is diagonal of the square.

Thus, $a\sqrt{2}=10$

$\Rightarrow \quad a=5\sqrt{2}$

Hence, the length of the side of the square is $5\sqrt{2}$.

26. Let PQ be the double ordinate, where

$P=(3\cos\theta, 2\sin\theta)$ and $Q=(3\cos\theta, -2\sin\theta)$

Let the point $R(h, k)$ divides the double ordinate in the ratio 2 : 1

Thus $h=3\cos\theta$ and $k=\frac{2}{3}\sin\theta$

Squaring and adding, we get

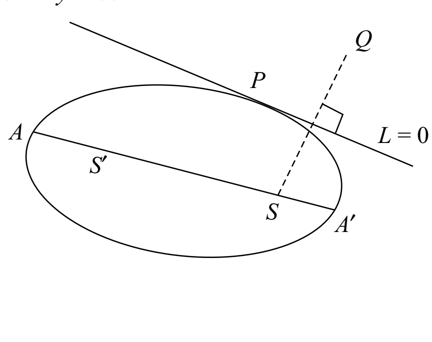
$$\left(\frac{h}{3}\right)^2+\left(\frac{3k}{2}\right)^2=1$$

$$\Rightarrow \quad \frac{h^2}{9}+\frac{9k^2}{4}=1$$

Hence, the locus of (h, k) is

$$\frac{x^2}{9}+\frac{9y^2}{4}=1$$

27. Let $S=(2,-1)$ and $S'=(1,-1)$ and Q is the image of S w.r.t. $x+y=5$.



So, $\frac{h-2}{1}=\frac{k-1}{1}=-\frac{2(-4)}{2}$

$\Rightarrow \quad \frac{h-2}{1}=\frac{k-1}{1}=4$

Thus, $Q=(6, 3)$

As we know that,

$SP+S'P=2a$

$\Rightarrow \quad SP+QP=2a$

Thus, S, P and Q are collinear.

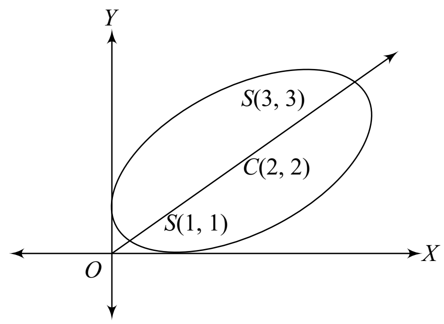
So, $\quad S'Q: 4x-5y=9$

Therefore P is a point of intersection of

$S'Q: 4x-5y=9$ and $L: x+y=5$

Hence, the point P is $\left(\frac{34}{9},\frac{11}{9}\right)$.

28. Clearly, the centre of the ellipse is (2, 2).



Since x-axis and y-axis are two perpendicular tangents to the ellipse, so (0, 0) lies on the director circle and (2, 2) is the centre of the director circle.

Thus, the radius $=\sqrt{2^2+2^2}=\sqrt{8}=2\sqrt{2}$

Hence, the area of the director circle $=8\pi$

29. The co-ordinates of any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be $(a\cos\theta, b\sin\theta)$.

If it lies on the line $bx=ay$, we have

$ba\cos\theta=ab\sin\theta$

$\Rightarrow \quad \tan\theta=1$

$\Rightarrow \quad \theta=\frac{\pi}{4}$ or $\frac{5\pi}{4}$

30. Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Its polar form is $\frac{1}{r^2}=\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}$

Let r_1 and r_2 be the lengths of the radius vectors CP and CQ which are inclined at angles θ and $\frac{\pi}{2}+\theta$.

So, $\quad \frac{1}{r_1^2}=\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}$

and $\quad \frac{1}{r_2^2}=\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}$

Now,

$$\frac{1}{r_1^2}+\frac{1}{r_2^2}=\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}+\frac{\sin^2\theta}{a^2}+\frac{\cos^2\theta}{b^2}$$

$$=\frac{1}{a^2}+\frac{1}{b^2}$$

31. The given ellipse is

$$4(x-2y+1)^2+9(2x+y+2)^2=25 \quad \ldots\text{(i)}$$

Let $X=x-2y+1$, $Y=2x+y+2$

Equation (i) reduces to

$$4X^2+9Y^2=25$$

$$\Rightarrow \frac{X^2}{25/4}+\frac{Y^2}{25/9}=1$$

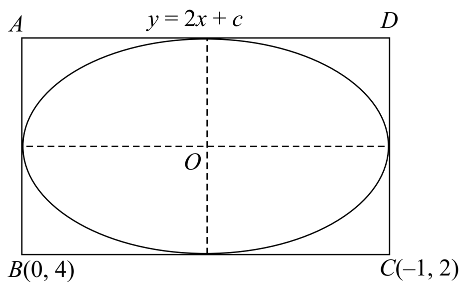
Thus, $e=\sqrt{1-\frac{b^2}{a^2}}$

$$=\sqrt{1-\frac{25/9}{25/4}}=\sqrt{1-\frac{4}{9}}=\frac{\sqrt{5}}{3}$$

32. Let AD: $y=2x+c$

So, BC: $y=2x+4$, AB: $x+2y=8$

and DC: $x+2y-3=0$



Let $BC=2a$ and $AB=2b$.

Clearly, $2a=\sqrt{5}\Rightarrow a=\frac{\sqrt{5}}{2}$

It is given that,

$$\pi ab=\frac{5}{2}\pi$$

$$\Rightarrow ab=\frac{5}{2}$$

$$\Rightarrow \frac{\sqrt{5}}{2}b=\frac{5}{2}$$

$$\Rightarrow b=\sqrt{5}$$

Also, $b=\left|\frac{c-4}{2\sqrt{5}}\right|$

$$\Rightarrow \left|\frac{c-4}{2\sqrt{5}}\right|=\sqrt{5}$$

$$\Rightarrow c-4=\pm 10$$

$$\Rightarrow c=4\pm 10=14,-6$$

when $c=14$

On solving the equations AB: $x+2y=8$ and AD: $y=2x+14$, and AB and DC, we get

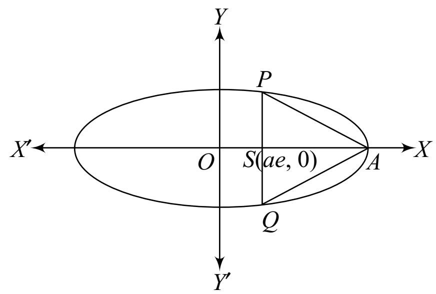
$A=(-4,6)$ and $D=(-5,4)$

When $c=-6$, we get,

$A=(3,2)$ and $D=(3,0)$.

Level IV

1. Given ellipse is



We have $A=\text{ar}(\Delta PQA)$

$$=\frac{1}{2}(2b\sin\theta)(a-a\cos\theta)$$

$$=ab\left(\sin\theta-\frac{1}{2}\sin 2\theta\right)$$

$$\frac{dA}{d\theta}=ab(\cos\theta-\cos 2\theta)$$

$$\Rightarrow \frac{d^2A}{d\theta^2}=ab(-\sin\theta+2\sin 2\theta)$$

For maximum or minimum

$$\frac{dA}{d\theta}=0$$

gives

$$ab(\cos\theta-\cos 2\theta)=0$$

$$\Rightarrow (\cos\theta-\cos 2\theta)=0$$

$$\Rightarrow \cos 2\theta=\cos\theta$$

$$\Rightarrow \cos 2\theta=\cos(2\pi-\theta)$$

$$\Rightarrow 2\theta=2\pi-\theta$$

$$\Rightarrow 3\theta=2\pi$$

$$\Rightarrow \theta=\frac{2\pi}{3}=120°$$

Clearly, $\frac{d^2A}{d^2\theta}$ is $-ve$

Thus, A is maximum.

Hence, the maximum value of A

$$=ab\left(\sin 120°-\frac{1}{2}\sin(240°)\right)$$

$$=\left(\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{4}\right)ab$$

$$=\frac{3\sqrt{3}}{4}ab \text{ s.u.}$$

2. The equation of the tangent to the parabola $y^2 = 4x$ at $(t^2, 2t)$ is

$$yt = x + t^2$$

$$\Rightarrow \quad x - yt + t^2 = 0 \qquad \text{...(i)}$$

and the equation of the normal

$$4x^2 + 5y^2 = 20$$

i.e. $$\frac{x^2}{5} + \frac{y^2}{4} = 1$$

at $(\sqrt{5}\cos\varphi, 2\sin\varphi)$ is

$$\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$$

$$\Rightarrow \quad \frac{5x}{\sqrt{5}\cos\varphi} - \frac{4y}{2\sin\varphi} = 5 - 4 = 1$$

$$\Rightarrow \quad (\sqrt{5}\sec\varphi)x - (2\operatorname{cosec}\varphi)\,y = 1 \qquad \text{...(ii)}$$

Since (i) and (ii) are the same line, so

$$\frac{\sqrt{5}\sec\varphi}{1} = \frac{2\operatorname{cosec}\varphi}{t} = \frac{-1}{t^2}$$

$$\Rightarrow \quad \sec\varphi = -\frac{1}{\sqrt{5}t^2}, \operatorname{cosec}\varphi = -\frac{1}{2t}$$

$$\Rightarrow \quad \cos\varphi = -t^2\sqrt{5}, \sin\varphi = -2t$$

Thus, $5t^4 + 4t^2 - 1 = 0$

$\Rightarrow \quad 5t^4 + 5t^2 - t^2 - 1 = 0$

$\Rightarrow \quad 5t^2(t^2 + 1) - 1(t^2 + 1) = 0$

$\Rightarrow \quad (5t^2 - 1)(t^2 + 1) = 0$

$\Rightarrow \quad (5t^2 - 1) = 0$

$$\Rightarrow \quad t = \pm\frac{1}{\sqrt{5}}$$

Also, $\tan\varphi = \pm 2$

$\Rightarrow \quad \varphi = \tan^{-1}(\pm 2)$

3. The equation of the given ellipse is

$$x^2 + 4y^2 = 16$$

$$\Rightarrow \quad \frac{x^2}{16} + y^2 = 1$$

Given centre of the circle is $C(1, 0)$.
Let the equation of the circle be

$$(x - 1)^2 + y^2 = r^2$$

Since the circle is the largest, so it will touch the ellipse at some point $P(a\cos\theta, b\sin\theta)$.
The equation of the tangent to the ellipse is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

whose slope is $m_1 = -\dfrac{b\cos\theta}{a\sin\theta} = -\dfrac{1}{2}\cot\theta$

and $m_2 = m(CP) = \dfrac{b\sin\theta}{(a\cos\theta - 1)} = \dfrac{2\sin\theta}{4\cos\theta - 1}$

But $m_1 m_2 = -1$

$$\Rightarrow \quad -\frac{1}{2}\frac{\cos\theta}{\sin\theta} \times \frac{2\sin\theta}{4\cos\theta - 1} = -1$$

$$\Rightarrow \quad \frac{\cos\theta}{4\cos\theta - 1} = 1$$

$\Rightarrow \quad 4\cos\theta - 1 = \cos\theta$

$\Rightarrow \quad 3\cos\theta = 1$

$$\Rightarrow \quad \cos\theta = \frac{1}{3}$$

Thus, the radius of the circle,

$$r = \sqrt{(a\cos\theta - 1)^2 + b^2\sin^2\theta}$$

$$= \sqrt{\left(\frac{4}{3} - 1\right)^2 + 4\left(1 - \frac{1}{9}\right)}$$

$$= \sqrt{\frac{1}{9} + \frac{32}{9}} = \frac{\sqrt{33}}{3}$$

Hence, the equation of the circle is

$$(x - 1)^2 + y^2 = \frac{33}{9} = \frac{11}{3}$$

4. Let the point P be $(a\cos\theta, b\sin\theta)$ and the point Q be

$$\left(a\cos\left(\theta + \frac{\pi}{4}\right), b\sin\left(\theta + \frac{\pi}{4}\right)\right)$$

The equation of the chord joining P and Q is

$$\frac{x}{a}\cos\left(\theta + \frac{\pi}{8}\right) + \frac{y}{b}\sin\left(\theta + \frac{\pi}{8}\right) = \cos\left(\frac{\pi}{8}\right)$$

which is identical with

$$px + qy = r$$

Comparing the co-efficients, we get

Thus, $$\frac{\cos\left(\theta + \frac{\pi}{8}\right)}{ap} = \frac{\sin\left(\theta + \frac{\pi}{8}\right)}{bq} = \frac{\cos\left(\frac{\pi}{8}\right)}{r}$$

$$\cos\left(\theta + \frac{\pi}{8}\right) = \frac{ap}{r}\cos\left(\frac{\pi}{8}\right)$$

and $$\sin\left(\theta + \frac{\pi}{8}\right) = \frac{bq}{r}$$

Squaring and adding, we get

$$\left(\frac{ap}{r}\right)^2 + \left(\frac{bq}{r}\right)^2 = 1$$

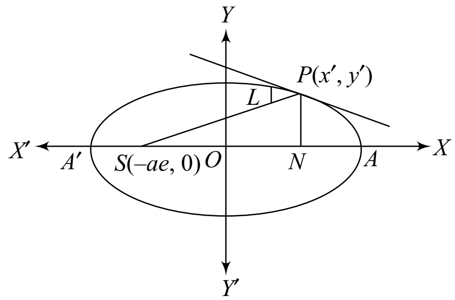
$$a^2p^2 + b^2q^2 = r^2$$

which is the required condition.

5. Let $R(h, k)$ be any point on the locus.
Let OP: $y = x \tan \alpha$ and $OQ : y = -x \tan \alpha$
It is given that,
$$(h \sin \alpha - k \cos \alpha)^2 + (h \sin \alpha + k \cos \alpha)^2 = 2\lambda^2$$
$$\Rightarrow \quad h^2 \sin^2\alpha + k^2 \cos^2\alpha = \lambda^2$$
$$\Rightarrow \quad \frac{h^2}{\lambda^2 \operatorname{cosec}^2\alpha} + \frac{k^2}{\lambda^2 \sec^2\alpha} = 1$$
which represents an ellipse.

6. Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
Let C be the centre and PQ be the chord whose equation is $x \cos \alpha + y \sin \alpha = p$.
Now, we make above two equations a homogeneous equation of 2nd degree.
Thus, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \left(\frac{x \cos \alpha + y \sin \alpha}{p}\right)^2$
$$\Rightarrow \quad \left(\frac{p^2}{a^2} - \cos^2\alpha\right)x^2 - 2xy \sin \alpha \cos \alpha + \left(\frac{p^2}{b^2} - \sin^2\alpha\right)y^2 = 0$$
Since the pair of diameters CP and CQ are at right angles, so
$$\left(\frac{p^2}{a^2} - \cos^2\alpha\right) + \left(\frac{p^2}{b^2} - \sin^2\alpha\right) = 0$$
$$\Rightarrow \quad \frac{p^2}{a^2} + \frac{p^2}{b^2} = 1$$
$$\Rightarrow \quad p^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = 1$$
$$\Rightarrow \quad p^2 = \frac{1}{\left(\frac{1}{a^2} + \frac{1}{b^2}\right)} = \frac{a^2b^2}{a^2 + b^2}$$
$$\Rightarrow \quad p = \frac{ab}{\sqrt{a^2 + b^2}}$$
Thus, $PQ : x \cos \alpha + y \sin \alpha = \frac{ab}{\sqrt{a^2 + b^2}}$
Now, the length of the perpendicular draw a from the centre to the line PQ
$$= \left|\frac{0 + 0 - \frac{ab}{\sqrt{a^2 + b^2}}}{\sqrt{\cos^2\alpha + \sin^2\alpha}}\right| = \frac{ab}{\sqrt{a^2 + b^2}}$$
= constant
Hence, the line PQ touches a fixed circle whose centre is (0, 0) and the radius is $\frac{ab}{\sqrt{a^2 + b^2}}$.

7. Given ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



The equation of tangent at P is
$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1$$
The slope of the tangent is $-\frac{b^2x'}{a^2y'}$
and the slope of SP is $\frac{y'}{x' + ae}$
Now, $\tan \theta = \frac{\frac{y'}{x' + ae} + \frac{b^2x'}{a^2y'}}{1 - \left(\frac{y'}{x' + ae} \cdot \frac{b^2x'}{a^2y'}\right)}$
$$= \frac{a^2b^2 + b^2x'ae}{x'y'a^2e^2 + a^2ey'}$$
$$= \frac{b^2a(a + ex')}{a^2ey'(a + ex')} = \frac{b^2}{aey'}$$
$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{b^2}{aey'}\right)$$

8. Given ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
and the circle is $x^2 + y^2 = ab$.
On solving, we get,
$$x^2 = \frac{a^2b}{a + b} \text{ and } y^2 = \frac{ab^2}{a + b}$$
The equation of tangent to the ellipse at (x_1, y_1) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ and the circle is $xx_1 + yy_1 = ab$
The slope of the tangent to the ellipse is
$$m_1 = -\frac{b^2x_1}{a^2y_1}$$
and the slope of tangent to the circle is
$$m_2 = -\sqrt{\frac{a}{b}}$$

Now, $\tan\theta = \dfrac{m_2 - m_1}{1 + m_1 m_2}$

$$= \frac{\sqrt{\frac{a}{b}}\left(1 - \frac{b^2}{a^2}\right)}{1 + \frac{ab^2}{ba^2}}$$

$$= \frac{a^2 - b^2}{a(a+b)}\sqrt{\frac{a}{b}}$$

$$= \frac{a-b}{\sqrt{ab}}$$

Thus, $\theta = \tan^{-1}\left(\dfrac{a-b}{\sqrt{ab}}\right)$

9. The normal at P is

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$$

and the normal at Q is

$$\Rightarrow \quad \frac{ax}{\cos\left(\theta + \frac{\pi}{2}\right)} - \frac{by}{\sin\left(\theta + \frac{\pi}{2}\right)} = a^2 - b^2$$

$$-\frac{ax}{\sin\theta} - \frac{by}{\cos\theta} = a^2 - b^2$$

$$\Rightarrow \quad \frac{ax}{\sin\theta} + \frac{by}{\cos\theta} = b^2 - a^2$$

The slope of the normal at P is

$$m_1 = \frac{a\sin\theta}{b\cos\theta} = \frac{a}{b}\tan\theta$$

and the slope of the normal at Q is

$$m_2 = -\frac{a}{b}\cot\theta$$

Now, $\tan\alpha = \dfrac{m_1 - m_2}{1 + m_1 m_2}$

$$= \frac{\frac{a}{b}(\tan\theta + \cot\theta)}{1 - \frac{a^2}{b^2}}$$

$$= \frac{ab}{(b^2 - a^2)}\frac{2}{\sin 2\theta}$$

$$= \frac{2ab}{a^2e^2} \times \frac{1}{\sin 2\theta}$$

$$= \frac{2a^2\sqrt{1-e^2}}{a^2e^2} \times \frac{1}{\sin 2\theta}$$

$$= \frac{2\sqrt{1-e^2}}{e^2(\sin 2\theta)}$$

10. The equation of the tangent at P is

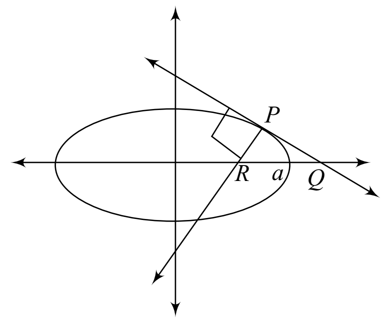
$$\frac{x}{a}\cos\varphi + \frac{y}{b}\sin\varphi = 1 \qquad \ldots\text{(i)}$$

and the equation of the normal at P is

$$ax\sec\varphi - by\operatorname{cosec}\varphi = a^2 - b^2 \qquad \ldots\text{(ii)}$$

Then $Q = (a\sec\varphi, 0)$

and $R = \left(\dfrac{(a^2 - b^2)\cos\varphi}{a}, 0\right)$



Therefore, $QR = a$

$$\Rightarrow \quad a\sec\varphi - \frac{(a^2 - b^2)\cos\varphi}{a} = a$$

$\Rightarrow \quad a^2\sin^2\varphi + b^2\cos^2\varphi = a^2\cos\varphi$

$\Rightarrow \quad a^2\sin^2\varphi + a^2(1 - e^2)\cos^2\varphi = a^2\cos\ \varphi)$

$\Rightarrow \quad a^2(\sin^2\varphi + \cos^2\varphi) - a^2e^2\cos^2\varphi = a^2\cos\varphi$

$\Rightarrow \quad a^2 - a^2e^2\cos^2\varphi = a^2\cos\varphi$

$\Rightarrow \quad e^2\cos^2\varphi + \cos\varphi - 1 = 0$

11. Let the point P be $(a\cos\varphi, b\sin\varphi)$

and $\quad S = (ae, 0), S' = (-ae, 0)$

$\therefore \quad SP = a - ex = a - e\cos\varphi$

and $\quad S'P = a + ae\cos\varphi$

Also, $SS' = 2ae$

Let (α, β) be the incentre of $\Delta PSS'$. So

$$\alpha = \frac{2ae \cdot a\cos\varphi + a(1 - e\cos\varphi)(-ae) + a(1 - e\cos\varphi)(ae)}{2ae + a(1 - e\cos\varphi) + a(1 + e\cos\varphi)}$$

$\Rightarrow \quad \alpha = ae\cos\varphi$

Similarly, $\beta = \dfrac{be\sin\varphi}{1+e}$

Eliminating ϕ, we get

$$\left(\frac{\alpha}{ae}\right)^2 + \left(\frac{b(1+e)}{be}\right)^2 = 1$$

$$\Rightarrow \quad \frac{\alpha^2}{a^2e^2} + \frac{\beta^2(1+e)^2}{b^2e^2} = 1$$

Hence, the locus of incentre is

$$\frac{x^2}{a^2e^2} + \frac{y^2}{\frac{b^2e^2}{(1+e)^2}} = 1$$

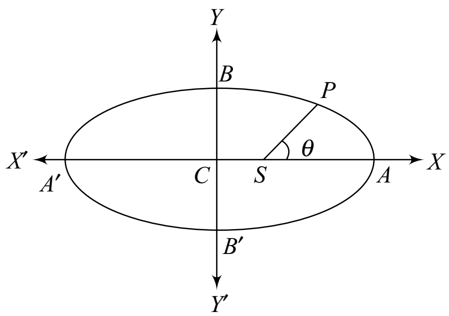
Let e_1 be its eccentricity, then

$$e_1 = \sqrt{1 - \frac{\frac{b^2e^2}{(1+e)^2}}{a^2e^2}}$$

$$= \sqrt{1 - \frac{a^2(1-e^2)}{a^2(1+e)^2}}$$

$$= \sqrt{1 - \frac{1-e}{1+e}} = \sqrt{\frac{2e}{1+e}}$$

12. Let the point P be $(a \cos \varphi, b \sin \varphi)$.



$$\tan \theta = \frac{b \sin \varphi}{a \cos \varphi - ae} = \frac{(b/a) \sin \varphi}{\cos \varphi - e}$$

$$= \frac{\sqrt{1-e^2} \sin \varphi}{\cos \varphi - e}$$

$$\Rightarrow \quad \tan \theta \cos \varphi - e \tan \theta = \sqrt{1-e^2} \sin \varphi$$

$$\Rightarrow \quad \frac{2 \tan (\theta/2)}{1 - \tan^2(\theta/2)} \times \left(\frac{1-\tan^2(\varphi/2)}{1+\tan^2(\varphi/2)} - e \right)$$

$$= \sqrt{1-e^2} \left(\frac{2 \tan (\theta/2)}{1+\tan^2(\theta/2)} \right) \left(\frac{1-\tan^2(\varphi/2)}{1+\tan^2(\varphi/2)} \right)$$

On simplification, we get

$$\Rightarrow \quad \sqrt{1+e} \tan\left(\frac{\varphi}{2}\right) - \sqrt{1-e} \tan\left(\frac{\theta}{2}\right) = 0$$

$$\sqrt{1+e} \tan\left(\frac{\varphi}{2}\right) = \sqrt{1-e} \tan\left(\frac{\theta}{2}\right)$$

$$\Rightarrow \quad \tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+e}{1-e}} \tan\left(\frac{\varphi}{2}\right)$$

13. Let the vertices of the equilateral triangle P, Q, R, whose eccentric angles are α, β, γ respectively.
Let (h, k) be the centroid of ΔPQR.

Then $\quad h = \frac{a}{3}(\cos \alpha + \cos \beta + \cos \gamma)$

and $\quad k = \frac{a}{3}(\sin \alpha + \sin \beta + \sin \gamma)$

Since ΔPQR is an equilateral triangle, so centroid = circumcentre.

$\therefore$ Circumcentre of ΔPQR be

$$h = \frac{(a^2-b^2)}{4a}(\cos \alpha + \cos \beta + \cos \gamma + \cos (\alpha + \beta + \gamma))$$

and $\quad k = \frac{(a^2-b^2)}{4a}(\sin \alpha + \sin \beta + \sin \gamma - \sin (\alpha + \beta + \gamma))$

On simplification, we get

$$\cos (\alpha + \beta + \gamma) = \frac{h(a^2+3b^2)}{a(a^2-b^2)}$$

and $\quad \sin (\alpha + \beta + \gamma) = \frac{(a^2+3b^2)k}{b(a^2-b^2)}$

Squaring and adding, we get

$$\left(\frac{(a^2+3b^2)h}{a(a^2-b^2)} \right)^2 + \left(\frac{(a^2+3b^2)k}{b(a^2-b^2)} \right)^2 = 1$$

Hence, the locus of the centroid (h, k) is

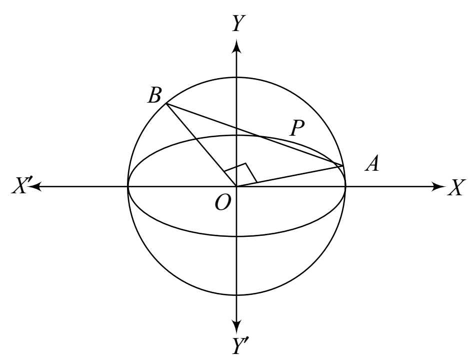
$$\left(\frac{(a^2+3b^2)x}{a(a^2-b^2)} \right)^2 + \left(\frac{(a^2+3b^2)y}{b(a^2-b^2)} \right)^2 = 1$$

14. The equation of the tangent to the ellipse at P is

$$\frac{x}{a} \cos \varphi + \frac{y}{b} \sin \varphi = 1 \qquad \ldots\text{(i)}$$

Let the equation of the auxilliary circle be

$$x^2 + y^2 = a^2 \qquad \ldots\text{(ii)}$$



The equation of the pair of lines OA and OB are obtained by making homogeneous of (i) and (ii). So

$$x^2 + y^2 = a^2 \left(\frac{x}{a} \cos \varphi + \frac{y}{b} \sin \varphi \right)^2$$

$$= a^2 \left(\frac{x^2}{a^2} \cos^2 \varphi + \frac{y^2}{b^2} \sin^2 \varphi + 2 \frac{xy}{ab} \sin \varphi \cos \varphi \right)$$

$$\Rightarrow (1 - \cos^2 \varphi) x^2 + \left(1 - \frac{a^2}{b^2} \sin^2 \varphi \right) y^2 + (....)\, xy = 0$$

It is given that, $\angle AOB = 90°$. So
co-efficient of x^2 + co-efficient of $y^2 = 0$

$\Rightarrow \quad 1-\cos^2\varphi+1-\dfrac{a^2}{b^2}\sin^2\varphi=0$

$\Rightarrow \quad \sin^2\varphi+1-\dfrac{a^2}{a^2(1-e^2)}\sin^2\varphi=0$

$\Rightarrow \quad \sin^2\varphi+1-\dfrac{1}{(1-e^2)}\sin^2\varphi=0$

$\Rightarrow \quad \left(1-\dfrac{1}{1-e^2}\right)\sin^2\varphi=-1$

$\Rightarrow \quad \left(\dfrac{-e^2}{1-e^2}\right)\sin^2\varphi=-1$

$\Rightarrow \quad e^2\sin^2\varphi=(1-e^2)$

$\Rightarrow \quad e^2(1+\sin^2\varphi)=1$

$\Rightarrow \quad e^2=\dfrac{1}{(1+\sin^2\varphi)}$

$\Rightarrow \quad e=\dfrac{1}{\sqrt{(1+\sin^2\varphi)}}$

15. Let two points on the ellipse be P and Q whose eccentric angles are φ_1 and φ_2 where $\varphi_1-\varphi_2=\alpha$.
The equation of the tangent at P and Q are

$$\frac{x}{a}\cos\varphi_1+\frac{y}{b}\sin\varphi_1=1$$

and $$\frac{x}{a}\cos\varphi_2+\frac{y}{b}\sin\varphi_2=1$$

Let $R(h, k)$ be the point of intersection of the tangents.

Thus, $$h=\frac{a\cos\left(\frac{\varphi_1+\varphi_2}{2}\right)}{\cos\left(\frac{\varphi_1-\varphi_2}{2}\right)} \text{ and } k=\frac{b\sin\left(\frac{\varphi_1+\varphi_2}{2}\right)}{\cos\left(\frac{\varphi_1-\varphi_2}{2}\right)}$$

$$\frac{h}{a}=\frac{\cos\left(\frac{\varphi_1+\varphi_2}{2}\right)}{\cos\left(\frac{a}{2}\right)} \text{ and } \frac{k}{b}=\frac{\sin\left(\frac{\varphi_1+\varphi_2}{2}\right)}{\cos\left(\frac{\alpha}{2}\right)}$$

Squaring and adding, we get

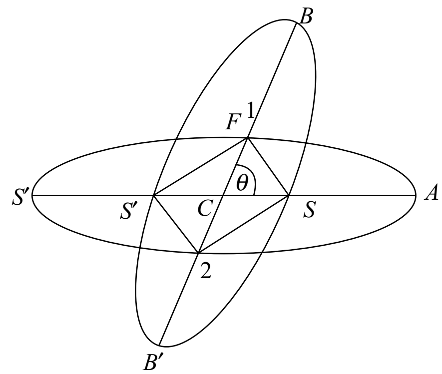
$$\left(\frac{h}{a}\right)^2+\left(\frac{k}{b}\right)^2=\frac{1}{\cos^2\left(\frac{\alpha}{2}\right)}$$

Hence, the locus of $R(h, k)$ is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\sec^2\left(\frac{\alpha}{2}\right)$$

16. Let S and S' be the foci of one ellipse and F_1 and F_2 of the other, where C being the common centre.
So, $SF_1S'F_2$ will form a parallelogram.

Since $SF_1+S'F_1=2a=F_1S'+F_2S'$



Clearly, $CS=ae$ and $CF_1=ae'$
Let θ be the angle between their axes.
Then, $SF_1^2=a^2e^2+a^2e'^2-2a^2ee'\cos\theta$
and $\quad S'F_1^2=a^2e^2+a^2e'^2+2a^2ee'\cos\theta$
Now, $2a=SF_1+S'F_1$

$\Rightarrow \quad 4a^2=(SF_1+S'F_1)^2$

$\Rightarrow \quad 4a^2=SF_1^2+S'F_1^2+2(SF_1)(S'F_1)$

$\Rightarrow \quad 4a^2=2a^2(e^2+e'^2)+2\sqrt{(a^2e^2+a^2e'^2)-4a^4e^2e'^2\cos^2\theta}$

$\Rightarrow \quad (2-e^2-e'^2)^2=(e^2+e'^2)^2-4e^2e'^2\cos^2\theta$

$\Rightarrow \quad 4-4(e^2+e'^2)=-4e^2e'^2\cos^2\theta$

$\Rightarrow \quad 1-(e^2+e'^2)=-e^2e'^2\cos^2\theta$

$\Rightarrow \quad \cos^2\theta=\left(\dfrac{e^2+e'^2-1}{e^2e'^2}\right)$

$\Rightarrow \quad \cos\theta=\left(\dfrac{\sqrt{e^2+e'^2-1}}{ee'}\right)$

17. Let the point of concurrency be (h, k).
The equation of the normal to the given ellipse at (x', y') is

$$\frac{a^2x}{x'}-\frac{b^2y}{y'}=a^2-b^2$$

which is passing through (h, k). So

$$\frac{a^2h}{x'}-\frac{b^2k}{y'}=a^2-b^2 \qquad \ldots\text{(i)}$$

Also, the point $(x'\ y')$ lies on $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. So

$$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1 \qquad \ldots\text{(ii)}$$

On simplification, we get

$$-(a^2-b^2)x'^4+2a^2(a^2-b^2)hx'^3+(\ldots)x'^2-2a^4(a^2-b^2)hx'+a^6h^2=0$$

which is a bi-quadratic equation. So, it has four roots, say x_1, x_2, x_3 and x_4.

Then $x_1+x_2+x_3+x_4=\dfrac{2ha^2}{(a^2-b^2)}$

and $x_1x_2x_3x_4=a^2$

Now, $\dfrac{1}{x_1}+\dfrac{1}{x_2}+\dfrac{1}{x_3}+\dfrac{1}{x_4}=\dfrac{\sum x_1x_2x_3}{x_1x_2x_3x_4}$

$=\dfrac{2(a^2-b^2)}{a^2h}$

Hence, the value of

$$(x_1+x_2+x_3+x_4)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)$$

$$=\frac{2ha^2}{(a^2-b^2)}\times\frac{(a^2-b^2)}{2ha^2}=1$$

18. Let the ellipse be $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

Let P and Q be two points lie on the ellipse whose eccentric angles are α and β such that $\theta=\alpha-\beta$.
Given that the tangents at P and Q are at right angles.

$$\left(-\frac{b}{a}\cot\alpha\right)\left(-\frac{b}{a}\cot\beta\right)=-1$$

$\Rightarrow\quad a^2\sin\alpha\sin\beta+b^2\cos\alpha\cos\beta=0$

But the diameter parallel to the tangent at P will be conjugate to the diameter CP, then its extremities will be $(-a\sin\alpha, b\cos\alpha)$.

Thus, $d_1^2=a^2\sin^2\alpha+b^2\cos^2\alpha$

Similarly, $d_2^2=a^2\sin^2\beta+b^2\cos^2\beta$

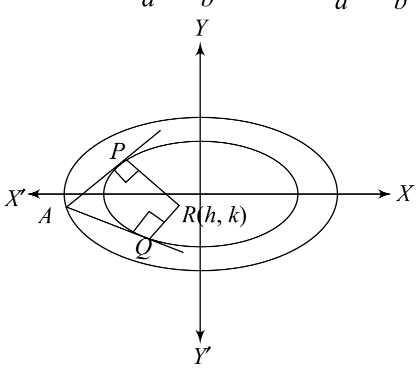
Now,

$$\begin{aligned}d_1^2d_2^2&=(a^2\sin^2\alpha+b^2\cos^2\alpha)(a^2\sin^2\beta+b^2\cos^2\beta)\\&=(a^2\sin\alpha\sin\beta+b^2\cos\alpha\cos\beta)^2\\&\qquad+a^2b^2(\sin\alpha\cos\beta-\cos\alpha\sin\beta)^2\\&=0+a^2b^2\sin^2(\alpha-\beta)\\&=a^2b^2\sin^2\theta\end{aligned}$$

$\Rightarrow\quad d_1d_2=ab\sin\theta$

Hence, the result.

19. Given conics are $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=4$ and $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.



The equations of tangents at P and Q are

$$\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha=1$$

and $\dfrac{x}{a}\cos\beta+\dfrac{y}{b}\sin\beta=1$

On solving, we get the point of intersection A, i.e.

$$A=\left(\frac{a\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)},\frac{a\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}\right)$$

Since the point A lies on $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=4$, so

$$\frac{\cos^2\left(\frac{\alpha+\beta}{2}\right)}{\cos^2\left(\frac{\alpha-\beta}{2}\right)}+\frac{\sin^2\left(\frac{\alpha+\beta}{2}\right)}{\cos^2\left(\frac{\alpha-\beta}{2}\right)}=4$$

$\Rightarrow\quad\cos^2\left(\dfrac{\alpha-\beta}{2}\right)=\dfrac{1}{4}$

$\Rightarrow\quad\cos\left(\dfrac{\alpha-\beta}{2}\right)=\pm\dfrac{1}{2}$

$\therefore$ The equations of the normals PR and QR are

$ax\sec\alpha-by\operatorname{cosec}\alpha=a^2-b^2$

and $ax\sec\beta-by\operatorname{cosec}\beta=a^2-b^2$

On simplification, we get

$$\frac{ax}{a^2-b^2}=\pm\cos\left(\frac{\alpha+\beta}{2}\right)\left\{\cos(\alpha+\beta)-\frac{1}{2}\right\}$$

$$\frac{by}{a^2-b^2}=\pm\sin\left(\frac{\alpha+\beta}{2}\right)\left\{\cos(\alpha+\beta)+\frac{1}{2}\right\}$$

Squaring and adding, we get

$$\left(\frac{ax}{a^2-b^2}\right)^2+\left(\frac{by}{a^2-b^2}\right)^2$$

$$=\cos^2(\alpha+\beta)+\frac{1}{4}-\cos^2(\alpha+\beta)$$

$\Rightarrow\quad a^2x^2+b^2y^2=\dfrac{1}{4}(a^2-b^2)^2$

Hence, the result.

20. Let the circle

$$x^2+y^2+2gx+2fy+c=0\qquad\ldots\text{(i)}$$

intersect the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$,

i.e. $b^2x^2+a^2y^2=a^2b^2$ …(ii)

in four points (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4), respectively.

It is given that, one point $(x_1, y_1)=(h, k)$ is fixed and other two points (x_2, y_2) and (x_3, y_3) are extremities of a diameter of the ellipse.

Since φ and $\pi+\varphi$ are the eccentric angles of the extremities of diameters of ellipse, So, $(x_2, y_2) = (a\cos\varphi, b\sin\varphi)$

and $(x_3, y_3) = (-a\cos\varphi, -b\sin\varphi)$

Thus, $x_1 + x_2 + x_3 = h$ and $y_1 + y_2 + y_3 = k$

Now, from Eq. (i), we get

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$$\Rightarrow a^2x^2 + a^2y^2 + 2ga^2x + 2fa^2y + a^2c = 0$$

$$\Rightarrow a^2x^2 + a^2y^2 + 2ga^2x + a^2c = -2fa^2y$$

$$\Rightarrow (a^2x^2 + a^2y^2 + 2ga^2x + a^2c)^2 = 4f^2a^4y^2$$

$$\Rightarrow (a^2x^2 + a^2b^2 - b^2x^2 + 2ga^2x + a^2c)^2 = 4f^2a^2(a^2b^2 - b^2x^2)$$

$$\Rightarrow (a^2 - b^2)^2x^4 + 4a^2g(a^2 - b^2)x^3 + \alpha x^2 + \beta x + \gamma = 0$$

where α, β and γ are constants which is a bi-quadratic equation.

Let x_1, x_2, x_3 and x_4 are its roots.

So, $$x_1 + x_2 + x_3 + x_4 = -\frac{4ga^2}{(a^2 - b^2)}$$

$$\Rightarrow h + x_4 = -\frac{4ga^2}{(a^2 - b^2)}$$

$$\Rightarrow x_4 = -\frac{4ga^2}{(a^2 - b^2)} - h$$

Similarly, $$y_4 = -\frac{4fb^2}{(a^2 - b^2)} - k$$

Since, (x_4, y_4) lies on the ellipse, so

$$\frac{x_4^2}{a^2} + \frac{y_4^2}{b^2} = 1$$

$$\Rightarrow \frac{\left(-\frac{4ga^2}{a^2 - b^2} - h\right)^2}{a^2} + \frac{\left(-\frac{4fb^2}{a^2 - b^2} - k\right)^2}{b^2} = 1$$

$$\Rightarrow \frac{16a^2g^2}{(a^2 - b^2)^2} + \frac{8gh}{a^2 - b^2} + \frac{16b^2f^2}{(a^2 - b^2)^2} - \frac{8fk}{(a^2 - b^2)} + \frac{h^2}{a^2} + \frac{k^2}{b^2} = 1$$

$$\Rightarrow \frac{2g^2a^2}{(a^2 - b^2)} + gh + \frac{2f^2b^2}{(a^2 - b^2)} - fk = 0$$

$$\Rightarrow \frac{2(g^2a^2 + f^2b^2)}{(a^2 - b^2)} + gh - fk = 0$$

$$\Rightarrow 2(g^2a^2 + f^2b^2) = (gh - fk)(a^2 - b^2)$$

Hence, the locus of the centre $(-g, -f)$ is

$$2(x^2a^2 + y^2b^2) = (hx - ky)(a^2 - b^2)$$

$$\Rightarrow 2(a^2x^2 + b^2y^2) = (a^2 - b^2)(hx - ky)$$

Hence, the result.

Integer Type Questions

1. The equation of the tangent to the ellipse is

$$\frac{x}{3\sec\theta} + \frac{y}{2\operatorname{cosec}\theta} = 1$$

Now,

$$\Delta = ar\,\Delta OAB$$

$$= \frac{1}{2} \times 3\sec\theta \times 2\operatorname{cosec}\theta = \frac{6}{\sin 2\theta}$$

Minimum area of the triangle = 6 s.u.

2. We have $F_1 = (2, 0)$ and $F_2 = (-2, 0)$

Now, $A = \text{ar}(\Delta PF_1F_2)$

$$= \frac{1}{2}\begin{vmatrix} \sqrt{5}\cos\theta & \sin\theta & 1 \\ 2 & 0 & 1 \\ -2 & 0 & 1 \end{vmatrix}$$

$$= \frac{1}{2}(4\sin\theta) = 2\sin\theta$$

Hence, the maximum value of A is 2 s.u.

3. Given ellipse is $16x^2 + 11y^2 = 256$.

$$\frac{x^2}{16} + \frac{y^2}{(16/\sqrt{11})^2} = 1$$

The equation of the tangent to the given ellipse at $\left(4\cos\varphi, \frac{16}{\sqrt{11}}\sin\varphi\right)$ is

$$\frac{x\cos\varphi}{4} + \frac{y\sin\varphi}{(16/\sqrt{11})} = 1 \quad \ldots\text{(i)}$$

which is also a tangent to the circle

$$x^2 + y^2 - 2x - 15 = 0$$

$$\Rightarrow (x - 1)^2 + y^2 = 16$$

So, the length of the perpendicular from the centre to the circle is equal to the radius of a circle.

$$\left|\frac{\left(\frac{\cos\varphi}{4}\right) - 1}{\sqrt{\frac{\cos^2\varphi}{16} + \frac{\sin^2\varphi}{256/11}}}\right| = 4$$

$$\Rightarrow \left(\left(\frac{\cos\varphi}{4}\right) - 1\right)^2 = 16\left(\frac{\cos^2\varphi}{16} + \frac{\sin^2\varphi}{256/11}\right)$$

$$\Rightarrow \left(\frac{\cos^2\varphi}{16} - 2\frac{\cos\varphi}{4} + 1\right) = \cos^2\varphi + \frac{11}{16}\sin^2\varphi$$

$$\Rightarrow 4\cos^2\varphi + 8\cos\varphi - 5 = 0$$

$$\Rightarrow (2\cos\varphi - 1)(2\cos\varphi + 5) = 0$$

$$\Rightarrow (2\cos\varphi - 1) = 0$$

$$\Rightarrow \cos\varphi = \frac{1}{2}$$

$\Rightarrow \quad \varphi = \frac{\pi}{6}, \frac{5\pi}{6}$

Hence, the number of values of ϕ is 2.

4. The required area of a parallelogram is made by the tangents at the extremities of a pair of conjugate diameter

$= 4ab$

$= 4 \times 3 \times \frac{1}{4}$

$= 3$ s.u.

5. Given curve is $\frac{x^2}{a-10} + \frac{y^2}{4-a} = 1$

$(4-a)x^2 + (a-10)y^2 - (a-10)(4-a) = 0$

which represents an ellipse, if $h^2 - ab < 0$

$\Rightarrow \quad -(4-a)(a-10) < 0$

$\Rightarrow \quad (a-4)(a-10) < 0$

$\Rightarrow \quad 4 < a < 10$

Hence, the number of integral values of a is 5.

6. Given ellipse is

$x^2 + 4y^2 = 4$

$\Rightarrow \quad \frac{x^2}{4} + \frac{y^2}{1} = 4$

Let A be the area of the rectangle.

So, $\quad A = (4\cos\theta)(2\sin\theta) = 4\sin(2\theta)$

Thus, the greatest area of the rectangle is 4.

7. Given ellipse is $9x^2 + 16y^2 = 144$

$\frac{x^2}{16} + \frac{y^2}{9} = 1$

Hence, the value of

$PF_1 + PF_2 - 2 = 2a - 2$
$= 8 - 2$
$= 6$

8. Clearly, $(S_1F_1) \cdot (S_2F_2) = b^2 = 3$

9. The minimum length of the intercept of the tangent to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ between the axes $= a + b = 7$.

10. The equation of the normal to the ellipse is

$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$

$\Rightarrow \quad \frac{169x}{13\cos\theta} - \frac{25y}{5\sin\theta} = 169 - 25$

$\Rightarrow \quad \frac{13x}{\cos\theta} - \frac{5y}{\sin\theta} = 144$

which is passing through (0, 6). So

$0 - \frac{30}{\sin\theta} = 144$

$\Rightarrow \quad -\frac{5}{\sin\theta} = 24$

$\Rightarrow \quad \sin\theta = -\frac{5}{24}$

$\Rightarrow \quad \theta = 2\pi - \sin^{-1}\left(\frac{5}{24}\right), \pi + \sin^{-1}\left(\frac{5}{24}\right)$

Also, y-axis is one of the normals.

Hence, it has three normals.

Previous Years' JEE-Advanced Examinations

1. Given line is

$x = -\frac{9}{2}$

$\Rightarrow \quad 2x + 9 = 0$

Let the point be $P(h, k)$.

Given $\sqrt{(x+2)^2 + y^2} = \frac{2}{3} \times \left(\frac{2x-9}{\sqrt{4}}\right)$

$\Rightarrow \quad (x+2)^2 + y^2 = \frac{4}{9} \times \left(\frac{2x-9}{\sqrt{4}}\right)^2$

$\Rightarrow \quad (x+2)^2 + y^2 = \frac{1}{9} \times (2x-9)^2$

$\Rightarrow \quad 9((x+2)^2 + y^2) = (2x-9)^2$

$\Rightarrow \quad 9(x^2 + y^2 + 4x + 4) = 4x^2 - 36x + 81$

$\Rightarrow \quad 5x^2 + 9y^2 + 72x - 55 = 0$

which represents an ellipse.

2. Clearly, $\frac{1}{9} + \frac{4}{4} - 1 > 0$

$\Rightarrow \quad P$ lies outside of E.

Also, $1 + 4 - 9 = -4 < 0$

$\Rightarrow \quad P$ lies inside C.

Thus, P lies inside C but outside E.

3. Given ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

So, $\quad F_1 = (ae, 0)$ and $F_2 = (-ae, 0)$

Let P be $(a\cos\theta, b\sin\theta)$

Then

$ar(\Delta PF_1F_2)$

$= \frac{1}{2}\begin{vmatrix} a\cos\theta & b\sin\theta & 1 \\ ae & 0 & 1 \\ -ae & 0 & 1 \end{vmatrix}$

$= \frac{1}{2} \times 2ae \times b\sin\theta$

$= abe \times \sin\theta$

Maximum value of $A = abe$

$= ab\sqrt{1 - \frac{b^2}{a^2}}$

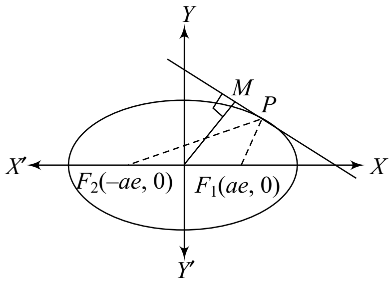
$= b\sqrt{a^2 - b^2}$

4. Given ellipse is $\dfrac{x^2}{16}+\dfrac{y^2}{9}=1$

$$\text{Foci} = (\pm ae, 0) = \left(\pm 4\cdot\frac{\sqrt{7}}{4}, 0\right) = (\pm\sqrt{7}, 0)$$

Radius of a circle $= \sqrt{7+9} = \sqrt{16} = 4$

5. Given ellipse is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$



The equation of the tangent to the given ellipse at $P(a\cos\theta, b\sin\theta)$ is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

Now,

$$OM = d$$

$$\Rightarrow \left|\frac{0+0-1}{\sqrt{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}}\right| = d$$

$$\Rightarrow \frac{1}{d^2} = \frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}$$

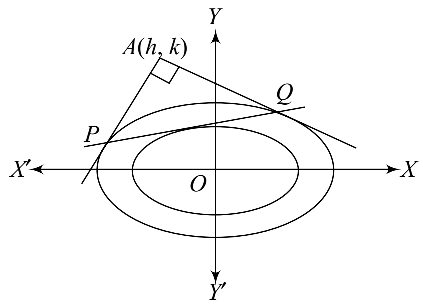
Now,

$$\begin{aligned}
4a^2\left(1-\frac{b^2}{d^2}\right) &= 4a^2 - \frac{4a^2b^2}{d^2}\\
&= 4a^2 - 4a^2b^2\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)\\
&= 4a^2 - 4b^2\cos^2\theta - 4a^2\sin^2\theta\\
&= 4a^2(1-\sin^2\theta) - 4b^2\cos^2\theta\\
&= 4a^2\cos^2\theta - 4b^2\cos^2\theta\\
&= 4\cos^2\theta(a^2-b^2)\\
&= 4\cos^2\theta(a^2e^2)\\
&= (2ae\cos\theta)^2\\
&= [(a+ae\cos\theta)-(a-ae\cos\theta)]^2\\
&= (PF_1 - PF_2)^2
\end{aligned}$$

6. Given ellipse is

$$x^2 + 4y^2 = 4$$

$$\Rightarrow \frac{x^2}{4}+\frac{y^2}{1} = 1 \qquad \text{...(i)}$$



The equation of the tangent to the ellipse (i) is

$$\frac{x}{2}\cos\theta + y\sin\theta = 1 \qquad \text{...(ii)}$$

The equation of the 2nd ellipse can be written as

$$\frac{x^2}{6}+\frac{y^2}{3} = 1 \qquad \text{...(iii)}$$

Let the tangents at P and Q meet at $A(h, k)$.
So PQ is the chord of contact.
The equation of the chord of contact of the tangents through A is

$$\frac{hx}{6}+\frac{ky}{3} = 1 \qquad \text{...(iv)}$$

Since the equations (ii) and (iv) are identical, so

$$\frac{\frac{h}{6}}{\frac{\cos\theta}{2}} = \frac{\frac{k}{3}}{\sin\theta} = 1$$

$\Rightarrow$ $h = 3\cos\theta$, $k = 3\sin\theta$

Now, squaring and adding, we get

$$h^2 + k^2 = 9$$

Therefore, the locus of A is

$$x^2 + y^2 = 9$$

which is the director circle of $\dfrac{x^2}{6}+\dfrac{y^2}{3}=1$.

Thus, the angle between the tangents at P and Q of the ellipse $x^2 + 2y^2 = 6$ right angle.

7. Let the ellipse be $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

$$B = (0, b),\ F = (ae, 0) \text{ and } F' = (-ae, 0)$$

It is given that, FBF' is a right angle. So

$$FB^2 + F'B^2 = (FF')^2$$

$$\Rightarrow a^2e^2 + b^2 + a^2e^2 + b^2 = 4a^2e^2$$

$$\Rightarrow 2a^2d^2 = 2b^2$$

$$\Rightarrow a^2e^2 = b^2 = a^2(1-e^2)$$

$$\Rightarrow a^2d^2 = a^2 - a^2e^2$$

$$\Rightarrow 2a^2e^2 = a^2$$

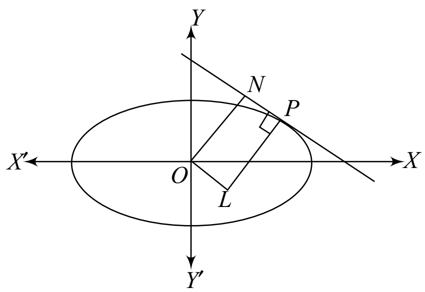
$$\Rightarrow 2e^2 = 1$$

$$\Rightarrow e^2 = \frac{1}{2}$$

$$\Rightarrow e = \frac{1}{\sqrt{2}}$$

8. Ans. (c)
Given ellipse is
$$16x^2 + 25y^2 = 400$$
$$\Rightarrow \quad \frac{x^2}{25}+\frac{y^2}{16}=1$$
Now,
$$PF_1 + PF_2 = 2a$$
$$= 10$$

9. Given ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



Let P lies in the first quadrant.
So $P = (a\cos\theta, b\sin\theta)$
The equation of the tangent at P is
$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$$
Now, $$ON=\left|\frac{-1}{\sqrt{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}}\right|$$
$$=\frac{ab}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}$$
The equation of ON is
$$\frac{x}{b}\sin\theta-\frac{y}{a}\cos\theta=0$$
and the equation of the normal at P is
$$ax\sec\theta - by\operatorname{cosec}\theta = a^2-b^2$$
So, $$OL=\frac{a^2-b^2}{\sqrt{a^2\sec^2\theta+b^2\operatorname{cosec}^2\theta}}$$
$$=\frac{(a^2-b^2)\sin\theta\cos\theta}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}$$
Now, $NP = OL$
$$\Rightarrow \quad NP=\frac{(a^2-b^2)\sin\theta\cos\theta}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}$$

Therefore, ar $\Delta OPN = \frac{1}{2}\times ON\times NP$
$$=\frac{1}{2}\times\frac{ab(a^2-b^2)\sin\theta\cos\theta}{a^2\sin^2\theta+b^2\cos^2\theta}$$
$$=\frac{1}{2}\times\frac{ab(a^2-b^2)}{a^2\tan\theta+b^2\cot\theta}$$
$$\le\frac{1}{2}\times\frac{ab(a^2-b^2)}{2ab}=\frac{(a^2-b^2)}{4}$$
at $\quad \tan\theta=\frac{b}{a}$

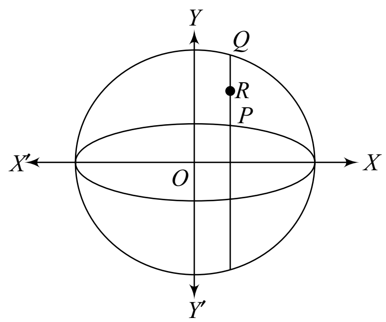
Thus, the point P is
$$\left(\frac{a^2}{\sqrt{a^2+b^2}},\frac{b^2}{\sqrt{a^2+b^2}}\right)$$
By symmetry, we have four such points.

Thus, $\left(\pm\frac{a^2}{\sqrt{a^2+b^2}},\pm\frac{b^2}{\sqrt{a^2+b^2}}\right)$

10. Given ellipse is $4x^2 + 9y^2 = 1$ …(i)
Differentiating w.r.t. x, we get,
$$8x+18y\cdot\frac{dy}{dx}=0$$
$$\Rightarrow \quad \frac{dy}{dx}=-\frac{8x}{18y}=-\frac{4x}{9y}$$
Since the tangent is parallel to $8x = 9y$, so
$$-\frac{4x}{9y}=\frac{8}{9}$$
$$\Rightarrow \quad -\frac{x}{y}=\frac{2}{1}$$
$\Rightarrow \quad x = -2y.$
Put $x = -2y$ in Eq. (i), we get
$\Rightarrow \quad 4(4y^2) + 9y^2 = 1$
$\Rightarrow \quad 25y^2 = 1$
$$\Rightarrow \quad y^2=\frac{1}{25}$$
$$\Rightarrow \quad y=\pm\frac{1}{5}$$
when $y=\pm\frac{1}{5}$, then $x=\mp\frac{2}{5}$

Hence, the points are $\left(-\frac{2}{5},\frac{1}{5}\right)$ and $\left(\frac{2}{5},-\frac{1}{5}\right)$.

11. Let the co-ordinates of P be $(a \cos \theta, b \sin \theta)$ and of Q be $(a \cos \theta, a \sin \theta)$ respectively.



Let $R(h, k)$ divides PQ in the ratio $r : s$.

Then $h = \dfrac{s(a \cos \theta) + r(a \cos \theta)}{r + s} = a \cos \theta$

and $k = \dfrac{s(b \sin \theta) + r(a \sin \theta)}{r + s} = \dfrac{(ar + bs) \sin \theta}{r + s}$

Thus, $\dfrac{h}{a} = \cos \theta, \dfrac{k(r + s)}{(ar + bs)} = \sin \theta$

Squaring and adding, we get

$$\frac{h^2}{a^2} + \frac{k^2(r + s)^2}{(ar + bs)^2} = 1$$

Hence, the locus of $R(h, k)$ is

$$\frac{x^2}{a^2} + \frac{(r + s)^2 y^2}{(ar + bs)^2} = 1$$

which represents an ellipse.

12. Any tangent to the ellipse is

$$\frac{x}{a}\cos \theta + \frac{y}{b}\sin \theta = 1 \quad \ldots(i)$$

Its point of contact is $P(a \cos \theta c\ b \sin \theta)$ and its slope is $-\dfrac{b}{a}\cot \theta$. Also the focus is $S(ae, 0)$.

Any line through the focus S and the perpendicular to tangent (i) is

$$y - 0 = \frac{a}{b}\tan \theta\,(x - ae) \quad \ldots(ii)$$

Also the equation of CP is

$$y - 0 = \frac{a}{b}\tan \theta (x - 0) \quad \ldots(iii)$$

Eliminating θ between Eqs (ii) and (iii), we get

$$\left(\frac{a^2}{b^2}\right)\left(\frac{x - ae}{x}\right) = 1$$

$$\Rightarrow \quad \left(\frac{x - ae}{x}\right) = \left(\frac{b^2}{a^2}\right)$$

$$\Rightarrow \quad \left(1 - \frac{ae}{x}\right) = \left(\frac{b^2}{a^2}\right)$$

$$\Rightarrow \quad \left(1 - \frac{b^2}{a^2}\right) = \left(\frac{ae}{x}\right)$$

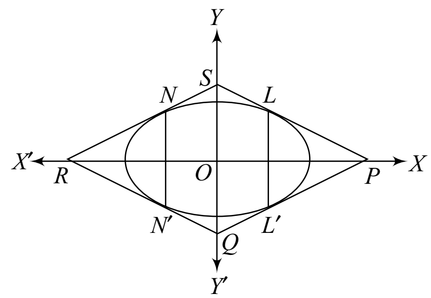
$$\Rightarrow \quad \left(1 - \frac{a^2(1 - e^2)}{a^2}\right) = \left(\frac{ae}{x}\right)$$

$$\Rightarrow \quad e^2 = \left(\frac{ae}{x}\right)$$

$$\Rightarrow \quad x = \frac{a}{e}$$

Hence, the result.

13. Given ellipse is $\dfrac{x^2}{9} + \dfrac{y^2}{5} = 1$



Now, $e = \sqrt{1 - \dfrac{b^2}{a^2}} = \sqrt{1 - \dfrac{5}{9}} = \dfrac{2}{3}$

$$L = \left(ae, \frac{b^2}{a}\right) = \left(2, \frac{5}{3}\right)$$

$$L' = \left(ae, -\frac{b^2}{a}\right) = \left(2, -\frac{5}{3}\right)$$

$$N = \left(-ae, \frac{b^2}{a}\right) = \left(-2, \frac{5}{3}\right)$$

and $N' = \left(-\text{ae}, -\dfrac{b^2}{a}\right) = \left(-2, -\dfrac{5}{3}\right)$

Now, the the tangent at L is

$$\frac{xx_1}{9} + \frac{yy_1}{5} = 1$$

$$\Rightarrow \quad \frac{2x}{9} + \frac{\frac{5}{3}y}{5} = 1$$

$$\Rightarrow \quad \frac{x}{9/2} + \frac{y}{3} = 1$$

Thus, $P = \left(\dfrac{9}{2}, 0\right)$ and $S = (0, 3)$

Therefore,

$$ar(\text{quad } PQRS) = 4 \times ar(\Delta OPS)$$

$$= 4 \times \frac{1}{2} \times \frac{9}{2} \times 3$$

$$= 27 \text{ s.u.}$$

14. Any tangent to the ellipse

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

Let it meet the x-axis at A and y-axis at B. Then the co-ordinates of A and B are

$$A = \left(\frac{a}{\cos\theta}, 0\right) \text{ and } B = \left(0, \frac{b}{\sin\theta}\right)$$

Let $M(h, k)$ be the mid-point of AB.

Then $2h = \dfrac{a}{\cos\theta}$ and $2k = \dfrac{b}{\sin\theta}$

$\Rightarrow \quad h = \dfrac{a}{2\cos\theta}$ and $k = \dfrac{b}{2\sin\theta}$

$\Rightarrow \quad \cos\theta = \dfrac{a}{2h}$ and $\sin\theta = \dfrac{b}{2k}$

Squaring and adding, we get

$$\frac{a^2}{4h^2} + \frac{b^2}{4k^2} = 1$$

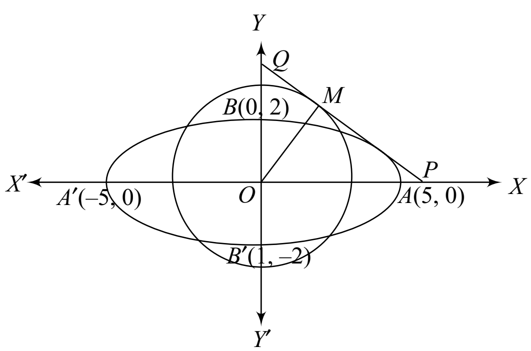
$$\Rightarrow \quad \frac{a^2}{h^2} + \frac{b^2}{k^2} = 4$$

Hence, the locus of $M(h, k)$ is

$$\frac{a^2}{x^2} + \frac{b^2}{y^2} = 4$$

15. The equation of any tangent to the given ellipse is

$$y = mx + \sqrt{25m^2 + 4} \quad \ldots(i)$$



Hence $OM = 4$

$$\Rightarrow \quad \left|\frac{m.0 - 0 + \sqrt{25m^2 + 4}}{\sqrt{m^2 + 1}}\right| = 4$$

$$\Rightarrow \quad \left|\frac{\sqrt{25m^2 + 4}}{\sqrt{m^2 + 1}}\right| = 4$$

$\Rightarrow \quad (25m^2 + 4) = 16(m^2 + 1)$

$\Rightarrow \quad 9m^2 = 12$

$\Rightarrow \quad 3m^2 = 4$

$$\Rightarrow \quad m = \pm\frac{2}{\sqrt{3}}$$

$$= -\frac{2}{\sqrt{3}} \quad (\because m < 0)$$

Thus, the equation of the common tangent is

$$y = -\frac{2}{\sqrt{3}}x + 4\sqrt{\frac{7}{3}}$$

This tangent meets the co-ordinate axes in P and Q respectively.

So, $\quad P = (2\sqrt{7}, 0)$ and $Q = \left(0, \dfrac{4\sqrt{7}}{\sqrt{3}}\right)$

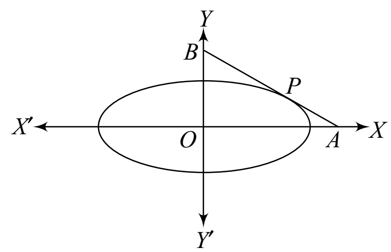
$$\text{Length of } PQ = \sqrt{28 + \frac{112}{3}} = \sqrt{\frac{84 + 112}{3}} = \sqrt{\frac{196}{3}} = \frac{14}{\sqrt{3}}$$

16. Given ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

The equation of any tangent to the given ellipse at $P(a\cos\theta, b\sin\theta)$ is

$$\Rightarrow \quad \frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

$$\Rightarrow \quad \frac{x}{a\sec\theta} + \frac{y}{b\operatorname{cosec}\theta} = 1$$



Here, $A = (a\sec\theta, 0)$ and $B = (0, \operatorname{cosec}\theta)$

Thus,

$$ar(\Delta OAB) = \left|\frac{1}{2} \times a\sec\theta \times b\operatorname{cosec}\theta\right|$$

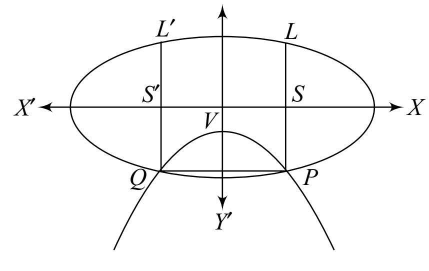
$$= \frac{ab}{|\sin 2\theta|}$$

$$\geq ab, \quad \left(\because |\sin 2\theta| \leq 1 \Rightarrow \frac{1}{|\sin 2\theta|} \geq 1\right)$$

17. Given ellipse is

$$x^2 + 4y^2 = 4$$

$$\Rightarrow \quad \frac{x^2}{4} + \frac{y^2}{1} = 1$$



Thus, $e=\sqrt{1-\dfrac{b^2}{a^2}}=\sqrt{1-\dfrac{1}{4}}=\dfrac{\sqrt{3}}{2}$

Foci:

$$S=(ae,0)=(\sqrt{3},0)$$

and $S'=(-ae,0)=(-\sqrt{3},0)$

End-points of locus recta:

$$L=\left(ae,\frac{b^2}{a}\right)=\left(\sqrt{3},\frac{1}{2}\right)$$

and $L'=\left(-ae,\dfrac{b^2}{a}\right)=\left(-\sqrt{3},\dfrac{1}{2}\right)$

Thus, $P=\left(\sqrt{3},-\dfrac{1}{2}\right)$ and $Q=\left(-\sqrt{3},-\dfrac{1}{2}\right)$

As we know that, the focus is the mid-point of P and Q.

Thus, the focus of a parabola is $\left(0,-\dfrac{1}{2}\right)$.

and the length of $PQ=2\sqrt{3}$

Now, $4a=2\sqrt{3}\Rightarrow a=\dfrac{\sqrt{3}}{2}$

Thus, the vertices of a desired parabola

$$=\left(0,-\frac{1}{2}\pm a\right)=\left(0,-\frac{1}{2}\pm\frac{\sqrt{3}}{2}\right)$$

Therefore, two desired parabolas are

$$\Rightarrow\quad x^2=\pm 4a\left(y-\left(-\frac{1}{2}\pm\frac{\sqrt{3}}{2}\right)\right)$$

$$\Rightarrow\quad x^2=2\sqrt{3}\left(y+\frac{1}{2}+\frac{\sqrt{3}}{2}\right)$$

or

$$x^2=-2\sqrt{3}\left(y+\frac{1}{2}-\frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow\quad x^2=2\sqrt{3}\,y+(3+\sqrt{3})$$

or

$$x^2=-2\sqrt{3}\,y+(3-\sqrt{3})$$

18. Given ellipse is

$$x^2+9y^2=9$$

$$\Rightarrow\quad \frac{x^2}{9}+\frac{y^2}{1}=1$$

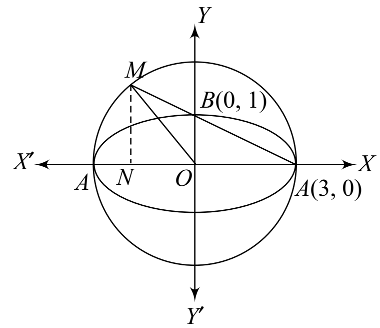
The equation of the auxiliary circle is

$$x^2+y^2=9 \qquad \ldots\text{(i)}$$

and the equation of the line AB is

$$\Rightarrow\quad \frac{x}{3}+\frac{y}{1}=1$$

$$x=3(1-y)$$



Put $x=3(1-y)$ in Eq. (i), we get

$$9(1-y)^2+y^2=9$$

$\Rightarrow\quad 9(y^2-2y+1)+y^2=9$

$\Rightarrow\quad 10y^2-18y=0$

$\Rightarrow\quad 5y^2-9y=0$

$\Rightarrow\quad y=0,\dfrac{9}{5}$

Thus, the y co-ordinate of M is $\dfrac{9}{5}$.

Now,

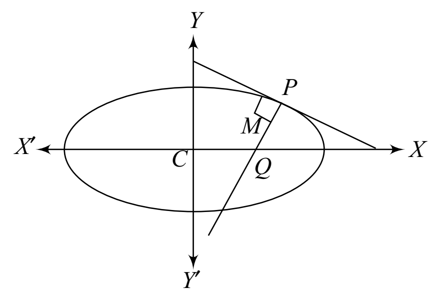
$$ar(\Delta OAM)=\frac{1}{2}\times OA\times MN$$

$$=\frac{1}{2}\times 3\times\frac{9}{5}$$

$$=\frac{27}{10}$$

19. Given ellipse is $x^2+4y^2=16$

$$\Rightarrow\quad \frac{x^2}{16}+\frac{y^2}{4}=1 \qquad \ldots\text{(i)}$$



Let the co-ordinates of P be $(4\cos\theta, 2\sin\theta)$.

The equation of the normal to the given ellipse at $P(4\cos\theta, 2\sin\theta)$ is

$$4x\sec\theta-2y\operatorname{cosec}\theta=4^2-2^2$$

$\Rightarrow\quad 4x\sec\theta-2y\operatorname{cosec}\theta=12$

$\Rightarrow\quad 2x\sec\theta-y\operatorname{cosec}\theta=6$

So, the point Q is $(3\cos\theta, 0)$.

Let $M(h,k)$ be the mid-point of PQ. So

$$h=\frac{7\cos\theta}{2}\ \text{and}\ k=\sin\theta$$

$\therefore \quad \dfrac{4h^2}{49} + k^2 = 1$

Hence, the locus of $M(h, k)$ is

$$\frac{4x^2}{49} + y^2 = 1 \qquad \ldots\text{(ii)}$$

The equation of the latus rectum of (i) is

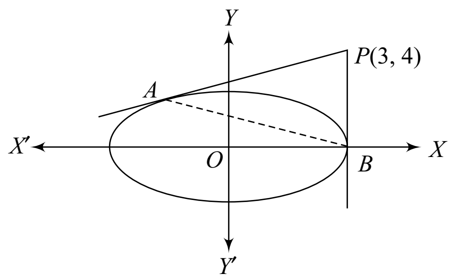
$$x = \pm ae = \pm 4 \times \frac{\sqrt{3}}{2} = \pm 2\sqrt{3}$$

Put $x = \pm 2\sqrt{3}$ in Eq. (ii), we get

$$y^2 = 1 - \frac{48}{49} = \frac{1}{49}$$

$\Rightarrow \quad y = \pm\dfrac{1}{7}$

Hence, the required points are $\left(\pm 2\sqrt{3}, \pm\dfrac{1}{7}\right)$.



20. Given ellipse is $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1 \qquad \ldots\text{(i)}$

Hence AB is the chord of contact. So

$$\frac{xx_1}{9} + \frac{yy_1}{4} = 1$$

$\Rightarrow \quad \dfrac{3x}{9} + \dfrac{4y}{4} = 1$

$\Rightarrow \quad \dfrac{x}{3} + y = 1 \qquad \ldots\text{(ii)}$

Solving Eqs (i) and (ii), we get

$$A = \left(-\frac{9}{5}, \frac{8}{5}\right), B = (3, 0)$$

21. The equation of the altitude through A is

$$y = \frac{8}{5} \qquad \ldots\text{(i)}$$

and the slope of AB is $-\dfrac{1}{3}$

The equation of the altitude through P is

$$y - 4 = 3(x - 3) \qquad \ldots\text{(ii)}$$

Solving Eqs (i) and (ii), we get

$$x = \frac{11}{5}, y = \frac{8}{5}$$

Hence, the orthocentre is $\left(\dfrac{11}{5}, \dfrac{8}{5}\right)$.

22. Let the point be $M(x, y)$.
It is given that PM = Length of perpendicular from Q to AB

$\Rightarrow \quad \sqrt{(x-3)^2 + (y-4)^2} = \dfrac{\left|\dfrac{x}{3} + y - 1\right|}{\sqrt{\left(\dfrac{1}{3}\right)^2 + 1}}$

$\Rightarrow \quad 10((x-3)^2 + (y-4)^2) = (x + 3y - 3)^2$

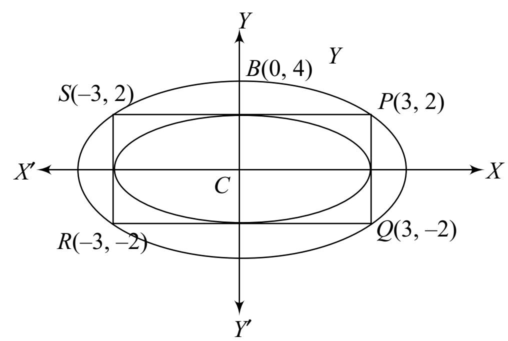
$\Rightarrow \quad 10(x^2 + y^2 - 6x - 8y + 25) = (x + 3y - 3)^2$

$\Rightarrow \quad 10(x^2 + y^2 - 6x - 8y + 25)$
$\qquad = x^2 + 9y^2 + 9 + 6xy - 6x - 18y$

$\Rightarrow \quad 9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$

23. Equation of ellipse is

$$(y+2)(y-2) + \lambda(x+3)(x-3) = 0$$



which is passing through $B(0, 4)$. So

$6 \times 2 + \lambda(-9) = 0$

$\Rightarrow \quad 9\lambda = 12$

$\Rightarrow \quad \lambda = \dfrac{4}{3}$

Thus, the required ellipse is

$$(y^2 - 4) + \frac{4}{3}(x^2 - 9) = 0$$

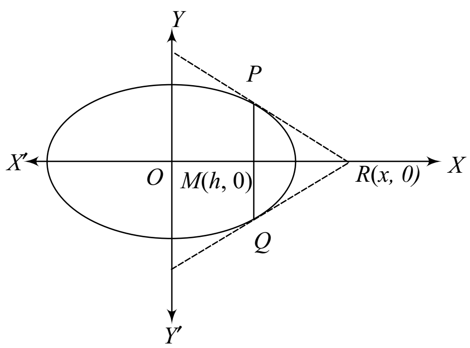
$\Rightarrow \quad 3(y^2 - 4) + 4(x^2 - 9) = 0$

$\Rightarrow \quad 4x^2 + 3y^2 = 48$

$\Rightarrow \quad \dfrac{x^2}{12} + \dfrac{y^2}{16} = 1$

Thus, $e = \sqrt{1 - \dfrac{12}{16}} = \dfrac{1}{2}$

24. Given ellipse is $\frac{x^2}{4}+\frac{y^2}{3}=1$



when $x = h$, then

$$\frac{y^2}{3}=1-\frac{h^2}{4}=\frac{4-h^2}{4}$$

$$\Rightarrow \quad y=\pm\frac{\sqrt{3}}{2}\sqrt{4-h^2}$$

Thus, the points P and Q are

$$\left(h, \frac{\sqrt{3}}{2}\sqrt{4-h^2}\right) \text{ and } \left(h, -\frac{\sqrt{3}}{2}\sqrt{4-h^2}\right)$$

Let the tangents at P and Q meet at $R(x_1, 0)$.
Therefore PQ is a chord of contact. So

$$\frac{xx_1}{4}=1$$

$$\Rightarrow \quad x=\frac{4}{x_1}$$

which is an equation of PQ at $x = h$. So

$$h=\frac{4}{x_1}$$

$$\Rightarrow \quad x_1=\frac{4}{h}$$

Now, $\Delta(h) = \text{area of } \Delta PQR$

$$=\frac{1}{2}\times PQ\times RT$$

$$=\frac{1}{2}\times\frac{2\sqrt{3}}{2}\times\sqrt{4-h^2}\times(x_1-h)$$

$$\Rightarrow \quad \Delta'(h)=\frac{\sqrt{3}(4+2h^2)}{2}\times\sqrt{4-h^2}\ h$$

which is always decreasing.

$$\Delta_1=\text{Maximum of } \Delta(h)=\frac{45\sqrt{5}}{8} \text{ at } h=\frac{1}{2}$$

$$\Delta_2=\text{Minimum of } \Delta(h)=\frac{9}{2} \text{ at } h=1$$

Therefore,

$$\frac{8}{\sqrt{5}}\Delta_1-8\Delta_2 =\frac{8}{\sqrt{5}}\times\frac{45\sqrt{5}}{8}-8\times\frac{9}{2}$$
$$=45-36$$
$$=9$$

25. Clearly, the point $P(h, 1)$ lies on the ellipse. So

$$\frac{h^2}{6}+\frac{1}{3}=1$$

$$\Rightarrow \quad \frac{h^2}{6}=1-\frac{1}{3}=\frac{2}{3}$$

$$\Rightarrow \quad \frac{h^2}{2}=2$$

$$\Rightarrow \quad h^2=4$$

$$\Rightarrow \quad h=\pm 2$$

Now, the tangent at (2, 1) is

$$\Rightarrow \quad \frac{2x}{6}+\frac{y}{3}=1$$

$$\frac{x}{3}+\frac{y}{3}=1$$

$$\Rightarrow \quad x+y=3$$

Hence, the value of h is 2.

26. The equation of P_1 is $y^2 - 8x = 0$ and P_2 is $y^2 + 16x = 0$
Tangent to $y^2 = 8x$ passes through (–4, 0)

$$0=m_1(-4)+\frac{2}{m_1}$$

$$\frac{1}{m_1^2}=2$$

Also, tangent to $y^2 + 16x = 0$ passes through (2, 0)

$$0=m_2\times 2-\frac{4}{m_2}$$

$$m_2^2=2$$

Hence, the value of $\left(\frac{1}{m_1^2}+m_2^2\right)$

$$=2+2$$
$$=4.$$

27. For the given line, point of contact for

$$E_1: \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ is } \left(\frac{a^2}{3}, \frac{b^2}{3}\right)$$

and for $E_2: \frac{x^2}{B^2}+\frac{y^2}{A^2}=1$ is $\left(\frac{B^2}{3}, \frac{A^2}{3}\right)$

Point of contact of $x + y = 3$ and the circle is (1, 2)
Also, the general point on $x + y = 3$ can be taken as
$\left(1\mp\frac{r}{\sqrt{2}}, 2\mp\frac{r}{\sqrt{2}}\right)$

where $r=\frac{2\sqrt{2}}{3}$

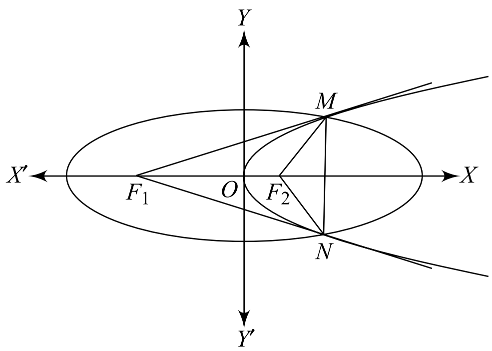
So, required points are $\left(\frac{1}{3}, \frac{8}{3}\right)$ and $\left(\frac{5}{3}, \frac{4}{3}\right)$

Comparing with points of contact of ellipse

$$a^2 = 5, B^2 = 8 \text{ and } b^2 = 4, A^2 = 1$$

$$e_1 e_2 = \frac{\sqrt{7}}{2\sqrt{10}} \text{ and } e_1^2 + e_2^2 = \frac{43}{40}$$

28. (i)



Here, $e = \frac{1}{3}, F_1 = (-1, 0), F_2 = (1, 0)$

The given parabola is $y^2 = 4x$

Thus, $M = \left(\frac{3}{2}, \sqrt{6}\right)$ and $N = \left(\frac{3}{2}, -\sqrt{6}\right)$

For orthocentre: One altitude is $y = 0$ (MN is perpendicular)

Other altitude is

$$(y - \sqrt{6}) = \frac{5}{2\sqrt{6}}\left(x - \frac{3}{2}\right)$$

Hence, the orthocentre is $\left(-\frac{9}{10}, 0\right)$

(ii) Equation of tangent at M and N are

$$\frac{x}{6} \pm \frac{y\sqrt{6}}{8} = 1$$

Thus, R is (6, 0)

Equation of normal at M is

$$(y - \sqrt{6}) = -\frac{\sqrt{6}}{2}\left(x - \frac{3}{2}\right)$$

Thus, Q is $\left(\frac{7}{2}, 0\right)$

$$ar(\Delta MQR) = \frac{1}{2} \times \sqrt{6} \times \frac{5}{2} = \frac{5\sqrt{6}}{4}$$

$$ar(MF_1NF_2) = \frac{\sqrt{6}}{2} + \frac{3\sqrt{6}}{2} = \frac{4\sqrt{6}}{2}$$

$$\text{Ratio} = \frac{5\sqrt{6}}{4} : \frac{4\sqrt{6}}{2} = \frac{5}{8}$$

CHAPTER 6

Hyperbola

Concept Booster

1. Introduction

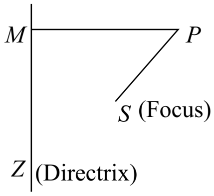
The word ‘hyperbola’ has ben derived from the Greek language meaning ‘over-thrown’ or ‘excessive’, from which the English term hyperbole is also derived. The term hyperbola is believed to have been coined by Apollonius of Perga (c.262–c.190 BC), who was a Greek geometer and astronomer noted for his writings on conic sections. His innovative methodology and terminology, especially in the field of the conic sections, the conics. According to him, hyperbola, the inclination of the plane to the base of the cone exceeds that of the side of the cone

2. Mathematical Definitions

Definition 1

It is the locus of a point which moves in a plane in such a way that its distance from a fixed point (focus) to a fixed straight line (directrix) is constant (> 1), i.e.

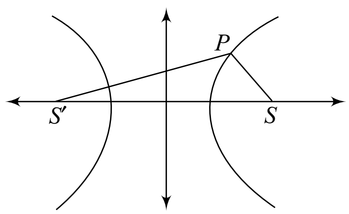
$$\frac{SP}{PM} = e \ (> 1)$$



Definition 2

A hyperbola is a conic section defined as the locus of all points P in the plane, the difference of whose distances from two fixed points (the foci S and S') is a constant,

i.e. $S'P - SP = 2a$



Definition 3

A conic section is said to be a hyperbola, if its eccentricity is more than 1, i.e. $e > 1$.

Definition 4

A conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents a hyperbola if

(i) $\Delta \neq 0$

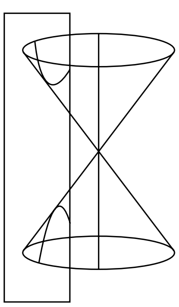
(ii) $h^2 - ab > 0$,

$$\text{where } \Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$$= abc + 2fgh - af^2 - bg^2 - ch^2$$

Definition 5

The section of a double right circular cone and a plane is said to be a hyperbola if the plane is parallel to the axis of a double right circular cone.



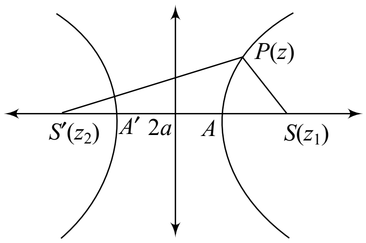
Definition 6

Let z, z_1 and z_2 be three complex numbers such that

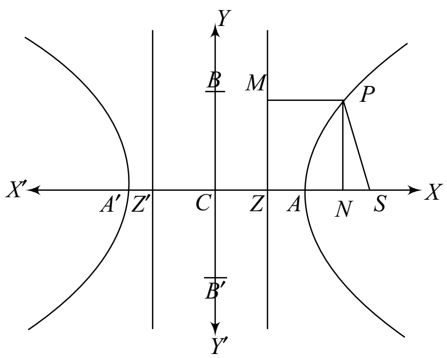
$$|z - z_1| - |z - z_2| = k,$$

where $k < |z_1 - z_2|$

Then the locus of z is known as a hyperbola.



3. Standard Equation of a Hyperbola



Let ZM be the directrix, S be the focus and SZ be the perpendicular to the directrix.

From the definition of hyperbola, we can write

$SA = e.AZ$ …(i)

and $SA' = eA'Z$ …(ii)

Let the length of $AA' = 2a$ and C be the mid-point of AA'.

Adding Eqs (i) and (ii), we get

$$SA + SA' = e(AZ + A'Z)$$

$\Rightarrow$ $(CS - CA) + (CS + CA) = e(AZ + AA' - AZ)$

$\Rightarrow$ $2CS = e(AA') = 2e$

$\Rightarrow$ $CS = ae$ …(iii)

Subtracting Eq. (i) from Eq. (ii), we get

$$(SA' - SA) = e(A'Z - AZ)$$

$\Rightarrow$ $AA' = e((CA' + CZ) - (CA - CZ))$

$\Rightarrow$ $2a = e \cdot 2CZ$

$\Rightarrow$ $CZ = \dfrac{a}{e}$ …(iv)

Let $P(x, y)$ be any point on the curve and PM be the perpendicular to the directrix.

Now from the definition of hyperbola, we get,

$$SP^2 = e^2 \cdot PM^2$$

$\Rightarrow$ $(x - ae)^2 + y^2 = e^2\left(\dfrac{ex - a}{\sqrt{e^2}}\right)^2 = (ex - a)^2$

$\Rightarrow$ $x^2 - 2aex + a^2e^2 + y^2 = (e^2x^2 - 2aex + a^2)$

$\Rightarrow$ $x^2(e^2 - 1) - y^2 = a^2(e^2 - 1)$

$\Rightarrow$ $\dfrac{x^2(e^2-1)}{a^2(e^2-1)} - \dfrac{y^2}{a^2(e^2-1)} = 1$

$\Rightarrow$ $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1,$

where, $b^2 = a^2(e^2 - 1)$

which is the standard equation of a hyperbola.

4. Definition and Basic Terminology of the Hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

(i) **Centre:** $C(0, 0)$. All chords passing through C and are bisected at C.

(ii) **Vertices:** $A(a, 0)$ and $A'(-a, 0)$

(iii) **Co-vertices:** $B(0, b)$ and $B'(0, -b)$

(iv) **Transverse axis:** $AA' = 2a$

(v) **Conjugate axis:** $BB' = 2b$

(vi) **Eccentricity:** $e = \sqrt{1 + \dfrac{b^2}{a^2}}$

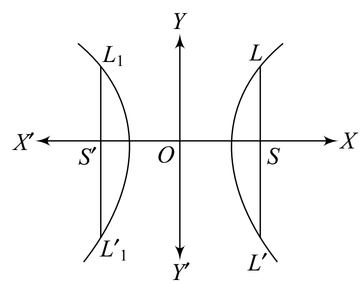
(vii) **Latus rectum:** Any chord passing through the focus and perpendicular to the axis is known as latus rectum.

The length of the latus rectum $= \dfrac{2b^2}{a}$.

(viii) **End-points of a latus recta:** Let LL' and L_1L_1' be two latus recta pass through the focus $S(ae, 0)$ and $S'(-ae, 0)$.

Then, $L\left(ae, \dfrac{b^2}{a}\right)$; $L'\left(ae, -\dfrac{b^2}{a}\right)$;

$$L_1\left(-ae, \frac{b^2}{a}\right); L_1'\left(-ae, -\frac{b^2}{a}\right)$$



(ix) **Equation of the latus recta:** $x = \pm ae$

(x) **Co-ordinates of Foci:** $S(ae, 0)$ and $S'(-ae, 0)$

(xi) **Distances between two foci (Focal length):** $2ae$

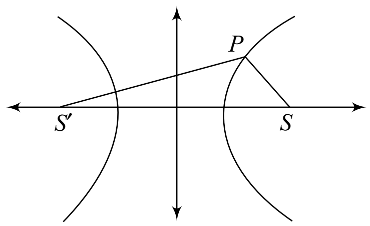
(xii) **Equation of directrices:** $x = \pm \dfrac{a}{e}$

(xiii) **Distance between the directrices:** $\dfrac{2a}{e}$

(xiv) **Focal distances**

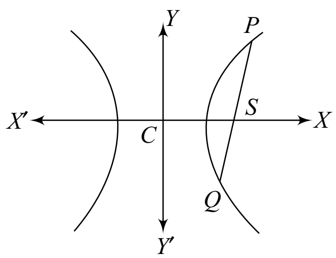
$$SP = ex - a \text{ and } S'P = ex + a$$

(xv) $|S'P - SP| = 2a$



(xvi) **Diameter:** Any chord passing through the centre of the hyperbola is known as the diameter.

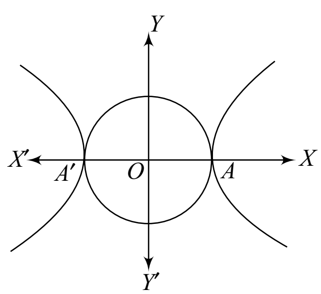
(xvii) **Focal chord:** Any chord passing through the focus is called the focal chord.



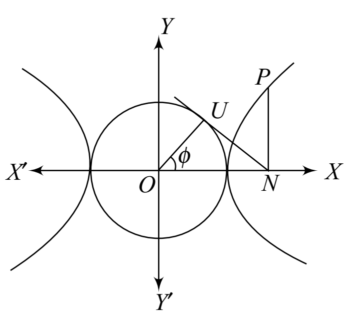
(xviii) **Auxiliary circle:** Any circle is drawn with centre C and the transverse axis as a diameter is called the auxiliary circle.

The equation of the auxiliary circle is given by

$$x^2 + y^2 = a^2$$



(xix) **Parametrics equations:** Let the auxiliary circle be $x^2 + y^2 = a^2$ and U $(a \cos \varphi, b \sin \varphi)$ be any point on the auxiliary circle.



Let $P(x, y)$ be any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Draw PN perpendicular to x-axis

Let NU ba tangent to the auxiliary circle.

Join NU.

Let $\angle UCN = \varphi$

Here P and U are the corresponding points of the hyperbola and the auxiliary circle and ϕ is the eccentric angle of P, where $0 \le \phi < 2\pi$.

Now, $U = (a \cos \varphi, a \sin \varphi)$

Also, $x = CN = \frac{CN}{CU} \cdot CU = a.\sec\varphi$

Thus the co-ordinates of P be $(a \sec \phi, y)$.

Since the point P lies on the hyperbola, so

$$\frac{a^2 \sec^2\varphi}{a^2} - \frac{y^2}{b^2} = 1$$

$$\Rightarrow \quad \sec^2\varphi - \frac{y^2}{b^2} = 1$$

$$\Rightarrow \quad \frac{y^2}{b^2} = \sec^2\varphi - 1 = \tan^2\varphi$$

$\Rightarrow \quad y^2 = b^2 \tan^2\varphi$

$\Rightarrow \quad y = \pm b \tan \varphi$

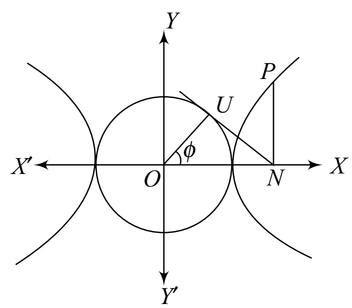
$\Rightarrow \quad y = b \tan \varphi$

(since P lies in the first quadrant)

Hence, the parametric equations of the hyperbola are $x = a \sec \varphi$ and $y = b \tan \varphi$.

(xx) Any point on the hyperbola can be considered as $(a \sec \varphi, b \tan \varphi)$.

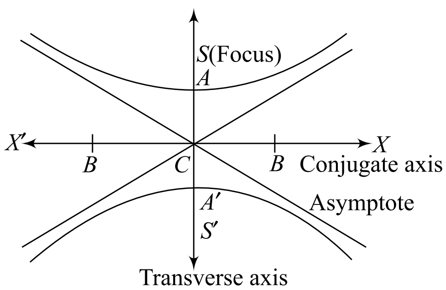
(xxi) **Eccentric angle:** If two points $P(a \sec \varphi, b \tan \varphi)$ and $U(a \cos \varphi, a \sin \varphi)$ are the corresponding points on the hyperbola and the auxiliary circle, then ϕ is called the eccentric angle of the point P on the hyperbola, where $0 \le \varphi < 2\pi$.



(xxii) **Conjugate hyperbola:** Corresponding to every hyperbola, there exists a hyperbola in which the conjugate and the transverse axes of one is equal to the transverse and the conjugate axes of the other. Such types of hyperbolas are known as the conjugate hyperbola.

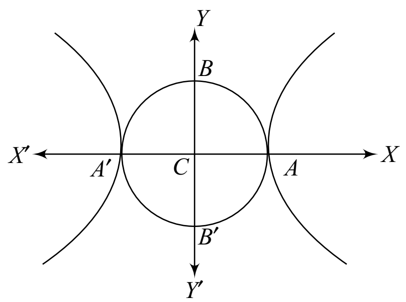
The equation of the conjugate hyperbola to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$



It is also known as vertical hyperbola.

(xxiii) **Rectangular or equilateral hyperbola:** If the semi-transverse axis is equal to the semi-conjugate axis of a hyperbola, i.e. $a = b$, then it is known as the rectangular hyperbola or the equilateral hyperbola.



The equation of a rectangular hyperbola is
$x^2 - y^2 = a^2$.
Clearly its eccentricity (e)

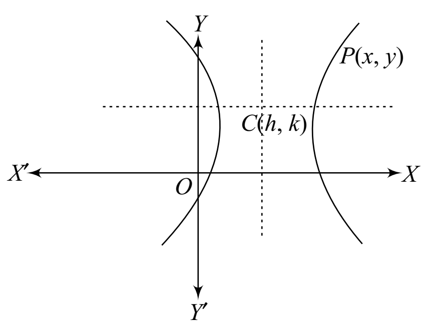
$$= \sqrt{1+\frac{b^2}{a^2}} = \sqrt{1+\frac{a^2}{a^2}} = \sqrt{1+1} = \sqrt{2}$$

(xxiv) Equation of a hyperbola whose axes are parallel to the co-ordinate axes and the centre (h, k) is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1,$$

where the foci are $(h \pm ae, k)$ and the directrix is

$$x = h \pm \frac{a}{e}$$



(xxvi) **Equation of the chord joining the points $P(\varphi_1)$ and $Q(\varphi_2)$:** The equation of the chord joining the points

$P(a \sec \varphi_1, b \tan \varphi_1)$ and $Q(a \sec \varphi_2, b \tan \varphi_2)$ is

$$\frac{x}{a}\cos\left(\frac{\varphi_1-\varphi_2}{2}\right) - \frac{y}{b}\sin\left(\frac{\varphi_1+\varphi_2}{2}\right) = \cos\left(\frac{\varphi_1+\varphi_2}{2}\right)$$

(xxvii) **Condition of a focal chord:** The equation of the chord joining the points $P(a \sec \varphi_1, b \tan \varphi_1)$ and $Q(a \sec \varphi_2, b \tan \varphi_2)$ is

$$\frac{x}{a}\cos\left(\frac{\varphi_1-\varphi_2}{2}\right) - \frac{y}{b}\sin\left(\frac{\varphi_1+\varphi_2}{2}\right) = \cos\left(\frac{\varphi_1+\varphi_2}{2}\right)$$

which is passing through the focus $S(ae, 0)$. So

$$e\cos\left(\frac{\varphi_1-\varphi_2}{2}\right) = \cos\left(\frac{\varphi_1+\varphi_2}{2}\right)$$

$$\Rightarrow \quad \frac{\cos\left(\frac{\varphi_1-\varphi_2}{2}\right)}{\cos\left(\frac{\varphi_1+\varphi_2}{2}\right)} = \frac{1}{e}$$

$$\Rightarrow \quad \frac{\cos\left(\frac{\varphi_1-\varphi_2}{2}\right) - \cos\left(\frac{\varphi_1+\varphi_2}{2}\right)}{\cos\left(\frac{\varphi_1-\varphi_2}{2}\right) + \cos\left(\frac{\varphi_1+\varphi_2}{2}\right)} = \frac{1-e}{1+e}$$

$$\Rightarrow \quad \tan\left(\frac{\varphi_1}{2}\right)\tan\left(\frac{\varphi_2}{2}\right) = \frac{1-e}{1+e}$$

(xxviii) **Condition of a focal chord with respect to eccentricity (e):** As we know that if the chord passing through the focus, then

$$e\cos\left(\frac{\varphi_1-\varphi_2}{2}\right) = \cos\left(\frac{\varphi_1+\varphi_2}{2}\right)$$

$$\Rightarrow \quad e \times 2\sin\left(\frac{\varphi_1+\varphi_2}{2}\right)\cos\left(\frac{\varphi_1-\varphi_2}{2}\right)$$

$$= 2\sin\left(\frac{\varphi_1+\varphi_2}{2}\right)\cos\left(\frac{\varphi_1+\varphi_2}{2}\right)$$

$$\Rightarrow \quad e \times (\sin\varphi_1 + \sin\varphi_2) = \sin(\varphi_1 + \varphi_2)$$

$$\Rightarrow \quad e = \frac{\sin(\varphi_1+\varphi_2)}{(\sin\varphi_1 + \sin\varphi_2)}$$

which is the required condition.

(xxix) **Rule to find out the centre of the hyperbola**

If $f(x, y) = 0$ is the equation of a hyperbola, the centre of the hyperbola is obtained by the relation $\frac{\delta f}{dx} = 0$ and $\frac{\delta f}{dy} = 0$.

Let

$f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

be the equation of a hyperbola.

Now, $\frac{\delta f}{dx} = 0$

$$\Rightarrow \quad 2ax + 2hy + 2g = 0$$

$$\Rightarrow \quad ax + y + g = 0 \qquad \ldots\text{(i)}$$

and $\frac{\delta f}{\delta y} = 0$

$$\Rightarrow \quad 2hx + 2by + 2f = 0$$

$$\Rightarrow \quad hx + by + f = 0 \qquad \ldots\text{(ii)}$$

Solving Eqs (i) and (ii), we get the required centre.

(xxx) **Polar form of a hyperbola**

The equation of hyperbola is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Putting $x = r\cos\theta$ and $y = r\sin\theta$, we get

$$\frac{1}{r^2}=\frac{\cos^2\theta}{a^2}-\frac{\sin^2\theta}{b^2}$$

$$\Rightarrow \quad r^2=\frac{a^2b^2}{b^2\cos^2\theta-a^2\sin^2\theta}=\frac{a^2(e^2-1)}{(e^2\cos^2\theta-1)}$$

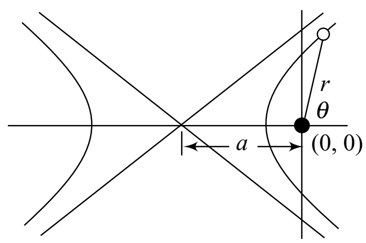
(xxxi) **Polar form of a hyperbola if centred at focus**

The equation of a hyperbola is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

If centred $C(0, 0)$ at the focus $(ae, 0)$, then the polar form of the hyperbola becomes

$$r=\frac{a(e^2-1)}{1-e\cos\theta}$$



(xxxii) **Polar form of a rectangular hyperbola**

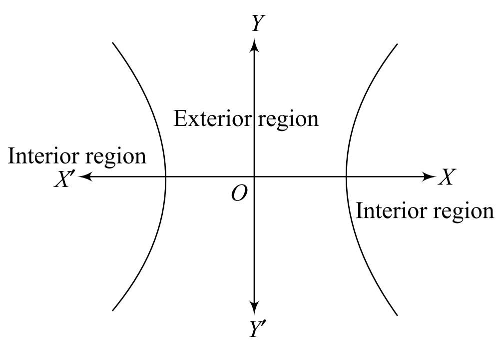
The equation of rectangular hyperbola is

$x^2 - y^2 = a^2$

Putting $x = r\cos\theta$ and $y = r\sin\theta$, we get,

$r^2\cos 2\theta = a^2$

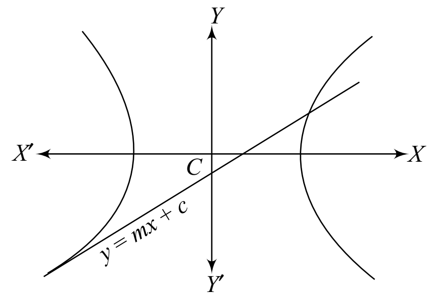
5. Position of a Point with Respect to a Hyperbola



The point (x_1, y_1) lies outside, on or inside the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ according as

$$\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1<0,=0,>0$$

6. Intersection of a Line and a Hyperbola



Let the hyperbola be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ …(i)

and the line be $y = mx + c$ …(ii)

From Eqs (i) and (ii), we get

$$\frac{x^2}{a^2}-\frac{(mx+c)^2}{b^2}=1$$

$\Rightarrow \quad b^2x^2 - a^2(mx + c)^2 = a^2b^2$

$\Rightarrow \quad (a^2m^2 - b^2)x^2 + 2mca^2x + a^2(b^2 + c^2) = 0$

Now,

$$\begin{aligned}\Delta &= 4m^2c^2a^4 - 4a^2(a^2m^2 - b^2)(b^2 + c^2)\\ &= 4(m^2c^2a^4 - a^4m^2b^2 - a^4m^2c^2 + a^2b^4 + a^2b^2c^2)\\ &= 4(-a^4m^2b^2 + a^2b^4 + a^2b^2c^2)\\ &= 4a^2b^2(b^2 + c^2 - a^2m^2)\end{aligned}$$

(i) The line $y = mx + c$ will never intersect the hyperbola, if

$D < 0$

$\Rightarrow \quad c^2 < a^2m^2 - b^2$

(ii) The line $y = mx + c$ will be a tangent to the hyperbola if $D = 0$

$\Rightarrow \quad c^2 = a^2m^2 - b^2$

This is known as the condition of tangency.

(iii) The line $y = mx + c$ will intersect the hyperbola in two real and distinct points, if

$D > 0$

$\Rightarrow \quad c^2 > a^2m^2 - b^2$

(iv) Any tangent to the hyperbola can be considered as $y=mx+\sqrt{a^2m^2-b^2}$.

(v) **Co-ordinates of the point of contact.**

If the line $y = mx + c$ be a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, the co-ordinates of the point of contact is $\left(\pm\frac{a^2m}{c},\pm\frac{b^2}{c}\right)$.

which is also known as ***m*-point** on the hyperbola.

(vi) **Number of tangents:** If a point lies outside, on and inside of a hyperbola, the number of tangents are 2, 1 and 0 respectively.

(vii) If the line $\lambda x + my + n = 0$ be a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then

$a^2l^2 - b^2m^2 = n^2$

7. Different Forms of Tangents

(i) **Point form:** The equation of the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point (x_1, y_1) is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$$

(ii) **Parametric form:** Equation of tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at (a sec φs b tan φ) is

$$\frac{x}{a}\sec\varphi-\frac{y}{b}\operatorname{cosec}\varphi=1$$

(iii) The point of intersection of the tangents at $P(\theta)$ and $Q(\varphi)$ on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$\left(\frac{a\cos\left(\frac{\theta-\varphi}{2}\right)}{\cos\left(\frac{\theta+\varphi}{2}\right)}, \frac{b\sin\left(\frac{\theta+\varphi}{2}\right)}{\cos\left(\frac{\theta+\varphi}{2}\right)}\right)$$

(iv) **Slope form:** The equation of the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ in terms of the slope m is

$$y=mx+\sqrt{a^2m^2-b^2}$$

The co-ordinates of the point of contact are $\left(\pm\frac{a^2m}{\sqrt{a^2m^2-b^2}}, \pm\frac{b^2}{\sqrt{a^2m^2-b^2}}\right)$.

(v) The equation of the tangent to the conjugate hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$ is

$$y=mx+\sqrt{b^2-a^2m^2}$$

The co-ordinates of the point of contact are

$$\left(\pm\frac{a^2m}{\sqrt{b^2-a^2m^2}}, \pm\frac{b^2}{\sqrt{b^2-a^2m^2}}\right)$$

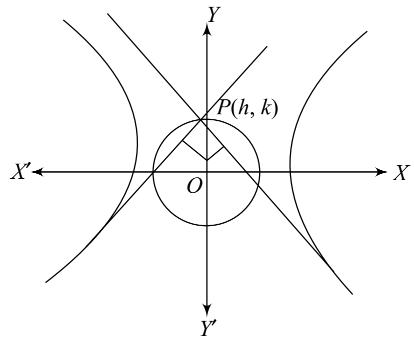
(vi) The line $y = mx + c$ be a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, if

$$c^2 = a^2m^2 - b^2$$

(vii) **Director circle:** The locus of the point of intersection of two perpendicular tangents to a hyperbola is known as the director circle of the hyperbola.
The equation of the director circle to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is
$$x^2 + y^2 = a^2 - b^2$$

The equation of any tangent to a hyperbola is

$$y=mx+\sqrt{a^2m^2-b^2}.$$



Let it passes through the point (h, k).
Then,

$$k=mh+\sqrt{a^2m^2-b^2}$$

$\Rightarrow\quad (k-mh)^2=(\sqrt{a^2m^2-b^2})^2$

$\Rightarrow\quad k^2 + m^2h^2 - 2kmh = a^2m^2 - b^2$

$\Rightarrow\quad (h^2 - a^2)m^2 - 2khm + (k^2 + b^2) = 0$

It has two roots, say m_1 and m_2. Then

$$m_1\cdot m_2=\frac{k^2+b^2}{h^2-a^2}$$

$\Rightarrow\quad \frac{k^2+b^2}{h^2-a^2}=-1$

$\Rightarrow\quad k^2 + b^2 = -h^2 + a^2$

$\Rightarrow\quad h^2 + k^2 = a^2 - b^2$

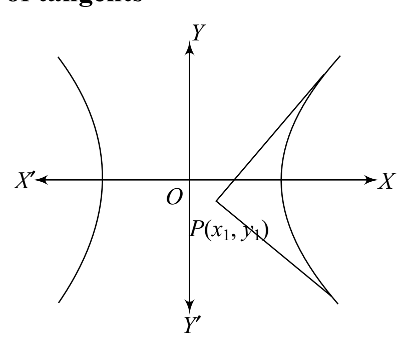
Hence, the locus of (h, k) is

$$x^2 + y^2 = a^2 - b^2$$

Notes

1. The director circle of a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ exists only when $a > b$.
2. If $a < b$, the equation of the director circle $x^2 + y^2 = a^2 - b^2$ does not exist.
3. The equation of the director circle to the conjugate hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$ is
   $$x^2 + y^2 = b^2 - a^2$$
   It exists only when $b > a$.
4. If $b < a$, the equation of the director circle does not exist.

(viii) **Pair of tangents**

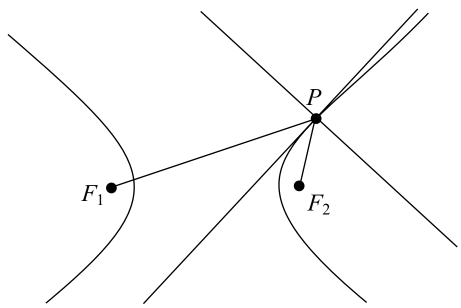


The combined equation of the pair of tangents drawn from a point $P(x_1, y_1)$ lies outside of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$SS_1 = T^2$$

$$\Rightarrow \quad \left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1\right)^2$$

8. Different Forms of Normals



(i) **Point form:** The equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at (x_1, y_1) is

$$\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$$

(ii) **Parametric form:** The equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at $(a \sec \varphi, b \tan \varphi)$ is

$$ax \cos \varphi + by \cot \varphi = a^2 + b^2$$

(iii) **Slope form:** The equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ in terms of the slope m is

$$y = mx \mp \frac{m(a^2+b^2)}{\sqrt{a^2-m^2b^2}}$$

The co-ordinates of the point of contact are

$$\left(\pm\frac{a^2}{\sqrt{a^2-b^2m^2}}, \mp\frac{mb^2}{\sqrt{a^2-b^2m^2}}\right)$$

(iv) The line $y = mx + c$ will be a normal to the hyperbola if

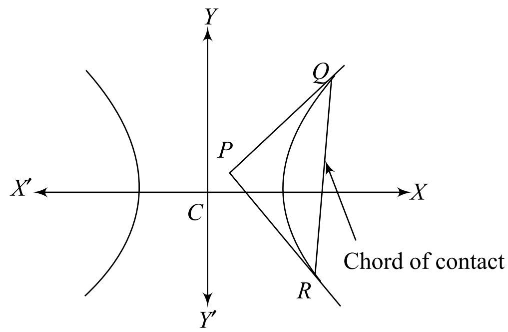
$$c^2=\left(\frac{m^2(a^2+b^2)^2}{(a^2-m^2b^2)}\right)$$

which is also known as the **condition of the normalcy to a hyperbola**.

9. Chord of Contact

If a tangent is drawn from a point $P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ touching the hyperbola at Q and R, the equation of the chord of contact QR is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$$

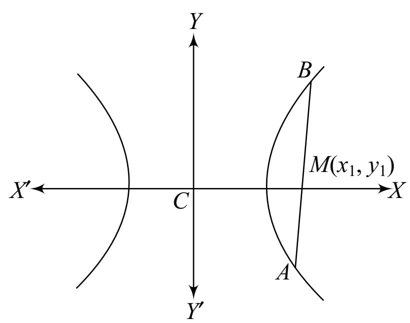


10. Equation of the Chord Bisected at a Point

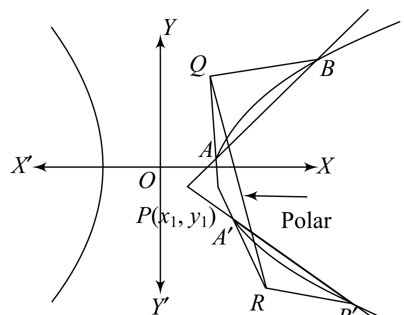
The equation of the chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ bisected at the point (x_1, y_1) is

$$SS_1 = T^2$$

i.e. $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1\right)\left(\frac{x_1{}^2}{a^2}-\frac{y_1{}^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1\right)^2$



11. Pole and Polar



Let P be any point inside or outside of the hyperbola. Suppose any straight line through P intersects the hyperbola at

A and B. Then the locus of the point of intersection of the tangents to the hyperbola at A and B is called the polar of the given point P with respect to the hyperbola and the point P is called the pole of the polar.

The equation of the polar from a point (x_1, y_1) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

Properties related to pole and polar

(i) The polar of the focus is the directrix.

(ii) Any tangent is the polar of the point of contact.

(iii) The pole of a line $\lambda x + my + n = 0$ with respect to the ellipse $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$\left(-\frac{a^2 l}{n}, -\frac{b^2 m}{n}\right)$$

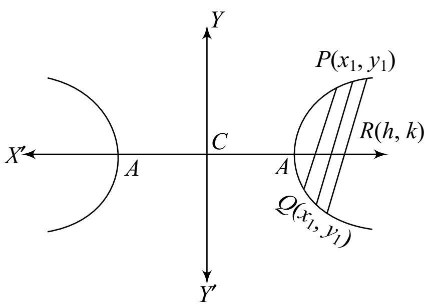
(iv) The pole of a given line is same as point of intersection of tangents at its extremities.

(v) If the polar of $P(x_1, y_1)$ passes through $Q(x_2, y_2)$, the polar of $Q(x_2, y_2)$ goes through $P(x_1, y_1)$ and such points are said to be **conjugate points**.

(vi) If the pole of a line $lx + my + n = 0$ lies on the another line $l'x + m'y + n' = 0$, the pole of the second line will lie on the first and such lines are said to be **conjugate lines**.

12. Diameter

The locus of the mid-points of a system of parallel chords of a hyperbola is called a diameter of the hyperbola.



The equation of a diameter to a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$y = \frac{b^2}{a^2 m} x$$

Let (h, k) be the mid-point of the chord $y = mx + c$ of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Then, $T = S_1$

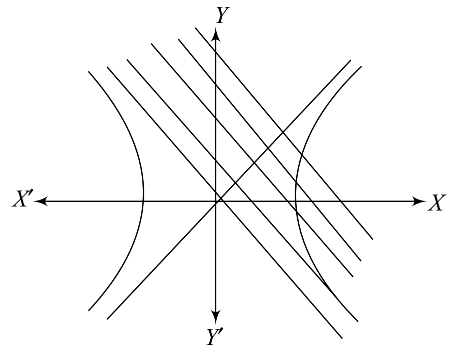
$$\Rightarrow \quad \frac{xh}{a^2} - \frac{yk}{b^2} - 1 = \frac{h^2}{a^2} - \frac{k^2}{b^2} - 1$$

Slope $= \frac{b^2 h}{a^2 k}$

Hence, the locus of the mid-point is $y = \frac{b^2 x}{a^2 m}$.

13. Conjugate Diameters

Two diameters are said to be conjugate when each bisects all the chords parallel to the others.



If $y = m_1 x$ and $y = m_2 x$ be two conjugate diameters, then,

$$m_1 \cdot m_2 = \frac{b^2}{a^2}$$

Properties of diameters

Property-I

If a pair of the diameters be conjugate with respect to a hyperbola, they are conjugate with respect to its conjugate hyperbola also.

Property-II

The parallelogram formed by the tangents at the extremities of the conjugate diameters of a hyperbola has its vertices lying on the asymptotes and is of conjugate area.

Property-III

If the normal at P meets the transverse axis in G, then $SG = e.SP$

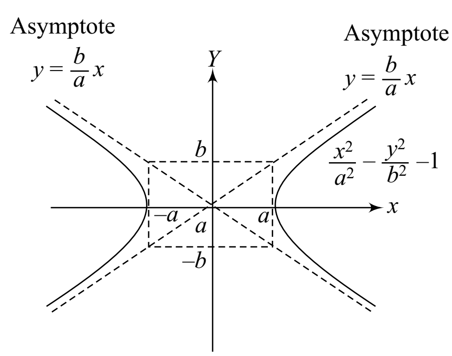
Also, the tangent and the normal bisects the angle between the focal distances of P.

Property-IV

If a pair of conjugate diameters meet the hyperbola in P and P' and its conjugate in D and D, then the asymptotes bisect

PD, $P'D \cdot PD'$ and $P'D'$.

14. Asymptotes



An asymptotes of any hyperbola or a curve is a straight line which touches it in two points at infinity.

The equation of the asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are

$$y=\pm\frac{b}{a}x$$

As we know that the difference between the 2nd degree curve and pair of asymptotes is constant.

Given hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

The equation of the pair of asymptotes are

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}+\lambda=0 \quad \ldots\text{(i)}$$

Equation (i) represents a pair of straight lines, then

$$\Delta=0$$

$$\Rightarrow \quad \frac{1}{a^2}\left(-\frac{1}{b^2}\right)\cdot\lambda+0-0-0-\lambda\cdot 0=0$$

$$\Rightarrow \quad \lambda=0$$

From Eq. (i) we get the equation of asymptotes as $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0.$

$$\Rightarrow \quad y=\pm\frac{b}{a}x$$

14.1 Important Points Related to Asymptotes

(i) The asymptotes pass through the centre of the hyperbola.
(ii) A hyperbola and its conjugate have the same asymptotes.
(iii) The equation of the hyperbola and its asymptotes differ by a constant only.
(iv) The equation of the asymptotes of a rectangular hyperbola $x^2-y^2=a^2$ are $y=\pm x$
(v) The angle between the asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$2\tan^{-1}\left(\frac{b}{a}\right)$$

(vi) The bisectors of the angles between the asymptotes are the co-ordinate axes.
(vii) No tangent to the hyperbola can be drawn from its centre.
(viii) Only one tangent to the hyperbola can be drawn from a point lying on its asymptotes other than its centre.
(ix) Two tangents can be drawn to the hyperbola from any of its external points which does not lie at its asymptotes.
(x) Let H: $\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0$, A: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$ and C: $\frac{x^2}{a^2}-\frac{y^2}{b^2}+1=0$ be the equations of the hyperbola, its asymptote and its conjugate, respectively, we can write from the above equations,

$$C+H=2A$$

15. Rectangular Hyperbola

A hyperbola is said to be rectangular, if the angle between its asymptote is 90°.

Thus, $\quad 2\tan^{-1}\left(\frac{b}{a}\right)=90°$

$$\Rightarrow \quad \tan^{-1}\left(\frac{b}{a}\right)=45°$$

$$\Rightarrow \quad \frac{b}{a}=\tan 45°$$

$$\Rightarrow \quad b=a$$

Hence, the equation of the rectangular hyperbola is

$$x^2-y^2=a^2$$

The eccentricity (e) of the rectangular hyperbola is

$$e=\sqrt{1+\frac{a^2}{a^2}}=\sqrt{1+1}=\sqrt{2}$$

16. Rectangular Hyperbola $xy = c^2$

The equation of a rectangular hyperbola is $x^2-y^2=a^2$ and its asymptote are

$$x-y=0 \text{ and } x+y=0,$$

where the asymptotes are inclined at 45° and 135°, respectively

If we rotate the axes through an angle of 45° in clockwise direction without changing the origin, then we replace x by $[x\cos(-45°)-y\sin(-45°)]$ and y by $[x\sin(-45°)+y\cos(-45°)]$,

i.e. x by $\left(\frac{x+y}{\sqrt{2}}\right)$ and y by $\left(\frac{-x+y}{\sqrt{2}}\right)$

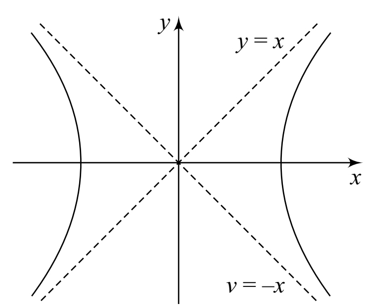
Then the equation, $x^2-y^2=a^2$ reduces to

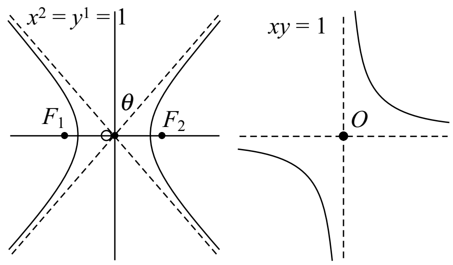
$$\left(\frac{x+y}{\sqrt{2}}\right)^2-\left(\frac{-x+y}{\sqrt{2}}\right)^2=a^2$$

$$\Rightarrow \quad \frac{1}{2}(2xy+2xy)=a^2$$

$$\Rightarrow \quad xy=\frac{a^2}{2}=c^2 \text{ (say)}$$

$$\Rightarrow \quad xy=c^2$$





Rectangular hyperbola

Properties of rectangular hyperbola

(i) The asymptotes of the rectangular hyperbola $xy = c^2$ are $x = 0$ and $y = 0$

(ii) The parametric equation of the rectangular hyperbola $xy = c^2$ are $x = ct$ and $y = \frac{c}{t}$.

(iii) Any point on the rectangular hyperbola $xy = c^2$ can be considered as $\left(ct, \frac{c}{t}\right)$.

(iv) The equation of the chord joining the points t_1 and t_2 is $x + yt_1t_2 - c(t_1 + t_2) = 0$

(v) The equation of the tangent to the rectangular hyperbola $xy = c^2$ at (x_1, y_1) is
$$xy_1 + x_1y = c^2$$

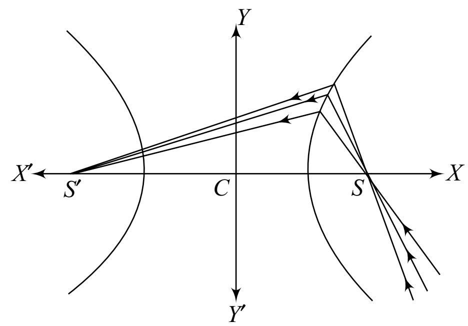
(vi) The equation of the tangent at $\left(ct, \frac{c}{t}\right)$ to the hyperbola $xy = c^2$ is $\frac{x}{t} + yt = 2c$.

(vii) The equation of the normal at (x_1, y_1) to the hyperbola $xy = c^2$ is
$$xx_1 - yy_1 = x_1^2 - y_1^2$$

(viii) The equation of the normal at t to the hyperbola $xy = c^2$ is
$$xt^3 - yt - ct^4 + c = 0$$

17. Reflection Property of a Hyperbola

If an incoming light ray passing through one focus, after striking the convex side of the hyperbola, it will get reflected towards other focus.



Exercises

Level I

(Problems Based on Fundamentals)

ABC OF HYPERBOLA

1. Find the centre, the vertices, the co-vertices, the length of transverse axis, the conjugate axis and the latus rectum, the eccentricity, the foci and the equation of directrices of each of the following hyperbolas.

   (i) $\frac{x^2}{9} - \frac{y^2}{4} = 1$

   (ii) $\frac{x^2}{16} - \frac{y^2}{9} = -1$

   (iii) $9x^2 - 16y^2 - 36x + 96y - 252 = 0$

2. Find the equation of the hyperbola, whose centre is (1, 0), one focus is (6, 0) and the length of transverse axis is 6.
3. Find the equation of the hyperbola, whose centre is (3, 2), one focus is (5, 2) and one vertex is (4, 2).
4. Find the equation of the hyperbola, whose centre is (–3, 2), one vertex is (–3, 4) and eccentricity is 5/2.
5. Find the equation of the hyperbola, whose one focus is (2, 1), the directrix is $x + 2y = 1$ and the eccentricity is 2.
6. Find the equation of the hyperbola, whose distance between foci is 16 and the eccentricity is $\sqrt{2}$.
7. Find the equation of the hyperbola,whose foci are (6, 4) and (–6, 4) and the eccentricity is 2.
8. Find the eccentricity of the hyperbola whose latus rectum is half of its transverse axis.
9. If e_1 and e_2 be the eccentricities of a hyperbola and its conjugate, prove that $\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1$.
10. An ellipse and a hyperbola are confocal (have the same focus) and the conjugate axis of the hyperbola is equal to the minor axis of the ellipse. If e_1 and e_2 are the eccentricities of the ellipse and the hyperbola, prove that $e_1^2 + e_2^2 = 2$.
11. Find the centre of the hyperbola
$$4(2y - x - 3)^2 - 9(2x + y - 1)^2 = 80$$
12. Find the centre of the hyperbola
$$3x^2 - 5y^2 - 6x + 20y - 32 = 0$$

13. Prove that the straight lines
$$\frac{x}{a}-\frac{y}{b}=2013 \text{ and } \frac{x}{a}+\frac{y}{b}=\frac{1}{2013}$$
always meet on a hyperbola.
14. Prove that the locus represented by $x=3\left(\frac{1+t^2}{1-t^2}\right)$ and $y=\frac{4t}{t^2-1}$ is a hyperbola.
15. Prove that the locus represented by $x=\frac{1}{2}(e^t+e^{-t})$ and $y=\frac{1}{2}(e^t-e^{-t})$ is a hyperbola.
16. If the equation $\frac{x^2}{2014-\lambda}+\frac{y^2}{2013-\lambda}=1$ represents a hyperbola, find λ.
17. If the foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{b^2}=1$ and the hyperbola $\frac{x^2}{144}-\frac{y^2}{81}=\frac{1}{25}$ coincide, find the value of b^2.
18. If the latus rectum subtends right angle at the centre of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, find its eccentricity.
19. Find the location of point (1, 4) w.r.t the hyperbola $2x^2-3y^2=6$.
20. If $(\lambda, -1)$ is an exterior point of the curve $4x^2-3y^2=1$ such that the length of the interval where λ lies is m, find the value of $m+10$.

INTERSECTIONS OF A LINE AND A HYPERBOLA

21. Find the points common to the hyperbola $25x^2-9y^2=225$ and the straight line $25x+12y-45=0$.
22. For what value of λ, does the line $y=3x+\lambda$ touch the hyperbola $9x^2-5y^2=45$?
23. For all real values of m, the straight line $y=mx+\sqrt{9m^2-4}$ is a tangent to a hyperbola, find the equation of the hyperbola.
24. Find the equations of tangents to the curve $4x^2-9y^2=36$, which is parallel to $5x-4y+7=0$.
25. Find the equations of tangents to the curve $9x^2-16y^2=144$, which is perpendicular to the straight line $3x+4y+10=0$.
26. If the line $5x+12y-9=0$ touches the hyperbola $x^2-9y^2=9$, then find its point of contact.
27. Find the equation of tangents to the curve $4x^2-9y^2=36$ from the point (3, 2).
28. Find the number of tangents from the point (1, –2) to the curve $2x^2-3y^2=12$.
29. Find the equation of the tangent to the curve $3x^2-4y^2=12$ having slope 4.

TANGENT AND TANGENCY

30. Find the equation of the tangent to the curve $x^2-y^2-8x+2y+11=0$ at (2, 1).
31. Find the equation of the tangent to the curve $4x^2-3y^2=24$ at $y=2$.
32. Find the angle between the tangents to the curve $9x^2-16y^2=144$ drawn from the point (4, 3).
33. Find the equations of the common tangent to the curves $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $\frac{x^2}{b^2}-\frac{y^2}{a^2}=-1$.
34. Find the equations of the common tangents to the curves $\frac{x^2}{9}-\frac{y^2}{4}=1$ and $\frac{x^2}{9}+\frac{y^2}{4}=1$.
35. Find the equation of the common tangents to the curves $\frac{x^2}{16}-\frac{y^2}{9}=1$ and $x^2+y^2=9$.
36. Find the equation of the common tangents to the curves $y^2=8x$ and $\frac{x^2}{9}-\frac{y^2}{5}=1$.
37. Find the locus of the point of intersection of the perpendicular tangents to the curve $\frac{x^2}{16}-\frac{y^2}{9}=1$.
38. Find the product of the perpendiculars from foci upon any tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.
39. If the tangent to the parabola $y^2=4ax$ intersects the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at A and B respectively, find the locus of the points of intersection of tangents at A and B.

NORMAL AND NORMALCY

40. Find the equation of the normal to the curve $\frac{x^2}{16}-\frac{y^2}{9}=1$ at $(8, 3\sqrt{3})$.
41. A normal is drawn at one end of the latus rectum of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ which meets the axes at points A and B respectively. Find the area of the ΔOAB.
42. Prove that the locus of the foot of the perpendicular from the centre upon any normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is
$$(x^2+y^2)(a^2y^2-b^2x^2)=(a^2+b^2)^2x^2y^2$$
43. A normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meets the axes in M and N and the lines MP and NP are drawn perpendiculars to the axes meeting at P. Prove that the locus of P is the hyperbola $a^2x^2-b^2y^2=(a^2+b^2)^2$.

44. If the normal at φ on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meets the transverse axis at G such that
$$AG \cdot A'G = a^m(e^n \sec^p \theta - 1),$$
where A, A' are the vertices of the hyperbola and m, n and p are positive integers, find the value of $(m+n+p)^2+36$.
45. If the normals at (x_i, y_i), $i = 1, 2, 3, 4$ on the rectangular hyperbola $xy = c^2$ meet at the point (α, β), prove that
(i) $x_1 + x_2 + x_3 + x_4 = \alpha$
(ii) $y_1 + y_2 + y_3 + y_4 = \beta$
(iii) $x_1^2 + x_2^2 + x_3^2 + x_4^2 = \alpha^2$
(iv) $y_1^2 + y_2^2 + y_3^2 + y_4^2 = \beta^2$
(v) $x^1 \cdot x^2 \cdot x^3 \cdot x^4 = -c^4$
(vi) $y_1 \cdot y_2 \cdot y_3 \cdot y_4 = -c^4$.
46. If the normals at (x_i, y_i), $i = 1, 2, 3, 4$ on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are concurrent, prove that
(i) $(x_1+x_2+x_3+x_4)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)=4$
(ii) $(y_1+y_2+y_3+y_4)\left(\frac{1}{y_1}+\frac{1}{y_2}+\frac{1}{y_3}+\frac{1}{y_4}\right)=4$

CHORD OF CONTACT/CHORD BISECTED AT A POINT

47. Find the equation of the chord of contact of tangents from the point (2, 3) to the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$.
48. Find the locus of the mid-points of the portions of the tangents to the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$ included between the axes.
49. From the points on the circle $x^2 + y^2 = a^2$, tangents are drawn to the hyperbola $x^2 - y^2 = a^2$. Prove that the locus of the mid-points of the chord of contact is
$$(x^2 - y^2)^2 = a^2(x^2 + y^2)$$
50. Prove that the locus of the mid-points of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ which subtend right angle at the centre is
$$\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right)=\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)$$
51. Tangents are drawn from a point P to the parabola $y^2 = 4ax$. If the chord of contact of the parabola be a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, find the locus of the point P.
52. A tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ cuts the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in points P and Q. Find the locus of the mid-point PQ.
53. Chords of the hyperbola $x^2 - y^2 = a^2$ touch the parabola $y^2 = 4ax$. Prove that the locus of their mid-points is the curve $y^2(x - a) = x^3$.
54. A variable chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is a tangent to the circle $x^2 + y^2 = c^2$. Prove that the locus of its mid-points is
$$\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^2=c^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)$$
55. A variable chord of the circle $x^2 + y^2 = a^2$ touches the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Prove that the locus of its mid-points is $(x^2 + y^2)^2 = a^2x^2 - b^2y^2$.
56. A tangent to the parabola $y^2 = 4ax$ meets the hyperbola $xy = c^2$ in two points P and Q. Prove that the locus of the mid-point of PQ lies on a parabola.
57. From a point P, tangents are drawn to the circle $x^2 + y^2 = a^2$. If the chord of contact of the circle is a normal chord of a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, prove that the locus of the point P is $\left(\frac{a^2}{x^2}-\frac{b^2}{y^2}\right)=\left(\frac{a^2+b^2}{a^2}\right)^2$.
58. Prove that the locus of the mid-points of the focal chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{ex}{a}$.
59. If the chords of contact of tangents from two points (x_1, y_1) and (x_2, y_2) to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are at right angles such that $\frac{x_1x_2}{y_1y_2}=-\frac{a^m}{b^n}$, where m, n are positive integers, find the value of $\left(\frac{m+n}{4}\right)^{10}$.

POLE AND POLAR

60. Find the polar of the focus $(-ae, 0)$ with respect to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.
61. If the polars of (x_1, y_1) and (x_2, y_2) with respect to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are at right angles, prove that $\frac{x_1x_2}{y_1y_2}+\frac{a^4}{b^4}=0$.

62. Find the pole of the line x – y = 3 w.r.t. the hyperbola $x^2 - 3y^2 = 3$.
63. Prove that the locus of the poles of the normal chords with respect to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is the curve $y^2a^6 - x^2b^6 = (a^2 + b^2)2x^2y^2$.
64. Prove that the locus of the poles with respect to the parabola $y^2 = 4ax$ of the tangent to the hyperbola $x^2 - y^2 = a^2$ is the ellipse $4x^2 + y^2 = 4a^2$.
65. Prove that the locus of the pole with respect to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ of any tangent to the circle, whose diameter is the line joining the foci, is the ellipse $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2 + b^2}$.

DIAMETER

66. Prove that the equation of the diameter which bisects the chord $7x + y - 2 = 0$ of the hyperbola $\frac{x^2}{3} - \frac{y^2}{7} = 1$ is $x + 3y = 0$.
67. Find the equation of the diameter of the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$, which corresponds the line $3x + 4y + 10 = 0$.
68. Find the equation of the diameter to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ parallel to the chord $2x + 3y + 5 = 0$.
69. In the hyperbola $16x^2 - 9y^2 = 144$, find the equation of the diameter which is conjugate to the diameter whose equation is $x = 2y$.

ASYMPTOTES

70. Find the asymptotes of the curve $xy - 3y - 2x = 0$.
71. Find the equations of the asymptotes of the curve $(a \sec \varphi, a \tan \varphi)$.
72. Find the eccentricity of the hyperbola whose asymptotes are $3x + 4y = 10$ and $4x - 3y = 5$.
73. Find the equation of a hyperbola whose asymptotes are $2x - y = 3$ and $3x + y = 7$ and which pass through the point (1, 1).
74. The asymptotes of a hyperbola having centre at the point (1, 2) are parallel to the lines $2x + 3y = 0$ and $3x + 2y = 0$. If the hyperbola passes through the point (5, 3), prove that its equation is $(2x + 3y - 8)(3x + 2y - 7) - 154 = 0$.
75. Find the product of the lengths of the perpendiculars from any point on the hyperbola $x^2 - 2y^2 = 2$ to its asymptotes.
76. Find the area of the triangle formed by any tangent to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ and its asymptotes.
77. Let P be a variable point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ such that its distance from the transverse axis is equal to its distance from an asymptote to the given hyperbola. Prove that the locus of P is $(x^2 - y^2)^2 = 4x^2(x^2 - a^2)$.
78. Show that the tangent at any point of a hyperbola cuts off a triangle of constant area from the asymptotes and that the portion of it intercepted between the asymptotes is bisected at the point of contact.
79. If p_1 and p_2 are the perpendiculars from any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ on its asymptotes, prove that $\frac{1}{p_1 p_2} = \frac{1}{a^2} + \frac{1}{b^2}$.
80. If the normal at t_1 to the hyperbola $xy = c^2$ meets it again at t_2, prove that $t_1^3 t_2 = -1$.
81. A triangle has its vertices on a rectangular hyperbola. Prove that the orthocentre of the triangle also lies on the same hyperbola.
82. Find the locus of the poles of the normal chords of the rectangular hyperbola $xy = c^2$
83. If the angle between the asymptote is 2α, prove that the eccentricity of the hyperbola is sec α.
84. A circle cuts the rectangular hyperbola $xy = 1$ in points (x_r, y_r), $r = 1, 2, 3, 4$, prove that $x_1x_2x_3x_4 = 1$ and $y_1y_2y_3y_4 = 1$.
85. If the tangent and the normal to a rectangular hyperbola $xy = c^2$ at a point cuts off intercepts a_1 and a_2 on one axis and b_1, b_2 on the other axis, prove that $a_1a_2 + b_1b_2 = 0$.
86. If e_1 and e_2 be the eccentricities of the hyperbola $xy = c^2$ and $x^2 - y^2 = a^2$, find the value of $(e_1 + e_2)^2$.
87. Find the product of the lengths of the perpendiculars drawn from any point on the hyperbola $\frac{x^2}{2} - y^2 = 1$ to its asymptote.
88. If A, B and C be three points on the rectangular hyperbola $xy = c^2$, find
    (i) the area of the ΔABC
    (ii) the area of the triangle formed by the tangents at A, B and C.
89. Find the length of the transverse axis of the rectangular hyperbola $xy = 18$.
90. Prove that the locus of a point whose chord of contact with respect to the circle $x^2 + y^2 = 4$ is a tangent to the hyperbola $xy = 1$ is a hyperbola.
91. Find the asymptotes of the hyperbola $xy = hx + ky$.
92. If e be the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and θ is the angle between the asymptotes, find cos $(\theta/2)$.
93. A ray is emanating from the point (5, 0) is incident on the hyperbola $9x^2 - 16y^2 = 144$ at a point P with abscissa 8. Find the equation of the reflected ray after first reflection and point P lies in first quadrant.

94. A ray is coming along the line $2x - y + 3 = 0$ (not through the focus) to the hyperbola $\frac{x^2}{4} - \frac{y^2}{3} = 1$.

After striking the hyperbolic mirror, it is reflected (not through the other focus).

Find the equation of the line containing the reflected ray.

Level II

(Mixed Problems)

1. The magnitude of the gradient of the tangent at extremity latus rectum of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is equal to

(a) be (b) e (c) ab (d) ae

2. The eccentricity of the hyperbola conjugate to the hyperbola $\frac{x^2}{4} - \frac{y^2}{12} = 1$ is

(a) $\frac{2}{\sqrt{3}}$ (b) 2 (c) $\sqrt{3}$ (d) $\frac{4}{3}$

3. The asymptote of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ form with any tangent to the hyperbola a triangle whose area is $a^2 \tan \lambda$ in magnitude, its eccentricity is

(a) $\sec \lambda$ (b) $\operatorname{cosec} \lambda$
(c) $\sec^2 \lambda$ (d) $\operatorname{cosec}^2 \lambda$

4. The equation $\frac{x^2}{29 - p} + \frac{y^2}{4 - p} = 1 \ (p \neq 4, 9)$ represents

(a) an ellipse if p is any constant greater than 4
(b) a hyperbola if p is any constant between 4 and 29
(c) a rectangular hyperbola if p is any constant greater than 29
(d) no real curve if p is less than 29.

5. Locus of the feet of the perpendiculars drawn from either foci on a variable tangent to the hyperbola $16y^2 - 9x^2 = 1$

(a) $x^2 + y^2 = 9$ (b) $x^2 + y^2 = \frac{1}{9}$
(c) $x^2 + y^2 = \frac{7}{144}$ (d) $x^2 + y^2 = \frac{1}{16}$

6. The locus of the point of intersection of the lines

$$\sqrt{3}x - y = 4\sqrt{3}t = 0$$

and $$\sqrt{3}tx + ty - 4\sqrt{3} = 0$$

(where t is a parameter) is a hyperbola, whose eccentricity is

(a) $\sqrt{3}$ (b) 2 (c) $2/\sqrt{3}$ (d) $4/3$

7. If the eccentricity of the hyperbola $x^2 - y^2 \sec^2\alpha = 5$ is $\sqrt{3}$ times the eccentricity of the ellipse $x^2 \sec^2\alpha + y^2 = 5$, the value of α is

(a) $\pi/6$ (b) $\pi/4$ (c) $\pi/3$ (d) $\pi/2$

8. For all real values of m, the straight line $y = mx + \sqrt{9m^2 - 4}$ is a tangent to the curve

(a) $9x^2 + 4y^2 = 36$ (b) $4x^2 + 9y^2 = 36$
(c) $9x^2 - 4y^2 = 36$ (d) $4x^2 - 9y^2 = 36$

9. The foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide. The value of b^2 is

(a) 5 (b) 7 (c) 9 (d) 4

10. The locus of the mid-points of the parallel chords with gradient m of the rectangular hyperbola $xy = c^2$ is

(a) $mx + y = 0$ (b) $y - mx = 0$
(c) $my - x = 0$ (d) $my + x = 0$

11. The locus of the foot of the perpendicular from the centre of the hyperbola $xy = c^2$ on a variable tangent is

(a) $(x^2 - y^2)^2 = 4c^2xy$ (b) $(x^2 + y^2)^2 = 2c^2xy$
(c) $(x^2 + y^2) = 4c^2xy$ (d) $(x^2 + y^2)^2 = 4c^2xy$

12. P is a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, N is the foot of the perpendicular from P on the transverse axis. The tangent to the hyperbola at P meets transverse axis at T. If O be the centre of the hyperbola, $OT.ON$ is

(a) e^2 (b) a^2 (c) b^2 (d) b^2/a^2

13. If PN be the perpendicular from a point on the rectangular hyperbola $(x^2 - y^2) = a^2$ on any on its asymptotes, the locus of the mid-point of PN is a/an

(a) circle (b) parabola
(c) ellipse (d) hyperbola

14. The equation to the chord joining two points (x_1, y_1) and (x_2, y_2) on the rectangular hyperbola $xy = c^2$ is

(a) $\frac{x}{x_1 + x_2} + \frac{y}{y_1 + y_2} = 1$ (b) $\frac{x}{x_1 - x_2} + \frac{y}{y_1 - y_2} = 1$
(c) $\frac{x}{y_1 + y_2} + \frac{y}{x_1 + x_2} = 1$ (d) $\frac{x}{y_1 - y_2} + \frac{y}{x_1 - x_2} = 1$

15. If $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ and $S(x_4, y_4)$ are 4 concyclic points on the rectangular hyperbola $xy = c^2$, the co-ordinates of the orthocentre of ΔPQR is

(a) $(x_4 - y_4)$ (b) (x_4, y_4)
(c) $(-x_4, -y_4)$ (d) $(-x_4, y_4)$

16. The chord PQ of the rectangular hyperbola $xy = a^2$ meets the axis of x at A, C is the mid-point of PQ and O the origin. The ΔACO is a/an

(a) equilateral (b) isosceles triangle
(c) right angled Δ (d) right isosceles triangle.

17. A conic passes through the point (2, 4) and is such that the segment of any of its tangents at any point con-

tained between the co-ordinates is bisected at the point of tangency. The foci of the conic are

(a) $(2\sqrt{2}, 0)$ and $(-2\sqrt{2}, 0)$
(b) $(2\sqrt{2}, 2\sqrt{2})$ and $(-2\sqrt{2}, -2\sqrt{2})$
(c) (4, 4) and (–4, –4)
(d) $(4\sqrt{2}, 4\sqrt{2})$ and $(-4\sqrt{2}, -4\sqrt{2})$

18. The latus rectum of the conic satisfying the differential equation, $xdy + ydx = 0$ and passing through the point (2, 8) is
(a) $4\sqrt{2}$ (b) 8 (c) $8\sqrt{2}$ (d) 16

19. If the normal to the rectangular hyperbola $xy = c^2$ at the point t meets the curve again at t_1, the value of t^3t_1 is
(a) 1 (b) –1 (c) 0 (d) None

20. With one focus of the hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=1$ as the centre, a circle is drawn which is the tangent to the hyperbola with no part of the circle being outside the hyperbola. The radius of the circle is
(a) < 2 (b) 2 (c) 11/3 (d) None

21. AB is a double ordinate of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ such that ΔAOB (where O is the origin) is an equilateral triangle, the eccentricity of the hyperbola satisfies
(a) $e>\sqrt{3}$ (b) $1<e<\frac{2}{\sqrt{3}}$
(c) $e=\frac{2}{\sqrt{3}}$ (d) $e>\frac{2}{\sqrt{3}}$

22. If the product of the perpendicular distances from any point on the hyperbola of the eccentricity from its asymptotes is equal to 6, the length of the transverse axis of the hyperbola is
(a) 3 (b) 6 (c) 8 (d) 12

23. If $x+iy=\sqrt{\varphi+i\psi}$ where $i=\sqrt{-1}$ and φ and ψ are non-zero real parameters, then φ = constant and ψ = constant, represents two systems of rectangular hyperbola which intersect at an angle of
(a) $\pi/6$ (b) $\pi/3$ (c) $\pi/4$ (d) $\pi/2$

24. The tangent to the hyperbola $xy = c^2$ at the point P intersects the x-axis at T and the y-axis at T'. The normal to the hyperbola at P intersects the x-axis at N and the y-axis at N'. The area of ΔPNT and $PN'T'$ are Δ and Δ' respectively, then $\frac{1}{\Delta}+\frac{1}{\Delta'}$ is
(a) equal to 1 (b) depends on t
(c) depends on c (d) equal to 2

25. The ellipse $4x^2 + y = 5$ and the hyperbola $4x^2 - y^2 = 4$ have the same foci and they intersect at right angles, the equation of the circle through the points of intersection of two conics is
(a) $x^2 + y^2 = 5$
(b) $\sqrt{5}(x^2+y^2)-3x-4y=0$
(c) $\sqrt{5}(x^2+y^2)+3x+4y=0$
(d) $x^2 + y^2 = 25$

26. At the point of intersection of the rectangular hyperbola $xy = c^2$ and the parabola $y^2 = 4ax$, tangents to the rectangular hyperbola and the parabola make angles θ and φ, respectively with the axis of x, then
(a) $\theta = \tan^{-1}(-2\tan\varphi)$ (b) $\varphi = \tan^{-1}(-2\tan\varphi)$
(c) $\theta=\frac{1}{2}\tan^{-1}(-\tan\varphi)$ (d) $\varphi=\frac{1}{2}\tan^{-1}(-\tan\theta)$

27. The area of the quadrilateral formed with the foci of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$ is
(a) $4a^2 + b^2)$ (b) $2(a^2 + b^3)$
(c) $(a^2 + b^2)$ (d) $\frac{1}{2}(a^2+b^2)$

28 The eccentricity of the hyperbola whose latus rectum is 8 and the conjugate axis is equal to half the distance between the foci is
(a) $\frac{4}{3}$ (b) $\frac{4}{\sqrt{3}}$ (c) $\frac{2}{\sqrt{3}}$ (d) None.

29. If $P(\sqrt{2}\sec\theta, \sqrt{2}\tan\theta)$ is a point on the hyperbola whose distance from the origin is $\sqrt{6}$, where P is in the first quadrant, then θ is equal to
(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) None

30. An ellipse and a hyperbola have same centre origin, the same foci and the minor axis of the one is the same as the conjugate axis of the other. If e_1 and e_2 be their eccentricities, respectively, then $\frac{1}{e_1^2}+\frac{1}{e_2^2}$ is equal to
(a) 1 (b) 2 (c) 3 (d) None

31. The number of possible tangents can be drawn to the curve $4x^2\ 9y^2 = 36$, which are perpendicular to the straight line $5x + 2y - 10 = 0$ is
(a) 0 (b) 1 (c) 2 (d) 4

32. The equation of a tangent passing through (2, 8) to the hyperbola $5x^2 - y^2 = 5$ is
(a) $3x - y + 2 = 0$ (b) $3x + y - 14 = 0$
(c) $23x - 3y - 2\ 2 = 0$ (d) $3x - 23y + 178 = 0$.

33. If $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3$ and (Sx_4, y_4) are four concyclic points on the rectangular hyperbola $xy = c^2$, the co-ordinates of the orthocentre of ΔPQR are
(a) (x_4, y_4) (b) $(x_4, -y_4)$
(c) $(-x_4, -y_4)$ (d) $(-x_4, y_4)$.

34. If the hyperbolas $x^2 + 3xy + 2y^2 + 2x + 3y + 2 = 0$ and $x^2 + 3xy + 2y^2 + 2x + 3y + c = 0$ are conjugate of each other, the value of c is
(a) -2 (b) 0 (c) 4 (d) 1
35. A rectangular hyperbola circumscribes a ΔABC, it will always pass through its
(a) orthocentre (b) circumcentre
(c) centroid (d) incentre
36. The co-ordinates of a point on the hyperbola $\frac{x^2}{24} - \frac{y^2}{18} = 1$, which is nearest to the line $3x + 2y + 1 = 0$ are
(a) (6, 3) (b) (–6, –3)
(c) (6, –3) (d) (–6, 3)
37. The latus rectum of the hyperbola $9x^2 - 16y^2 - 18x - 32y - 151 = 0$ is
(a) 9/4 (b) 9 (c) 3/2 (d) 9/2
38. If the eccentricity of the hyperbola $x^2 - y^2 \sec^2 \alpha = 5$ is $\sqrt{3}$ times the eccentricity of the ellipse $x^2 \sec^2 \alpha + y^2 = 25$, the value of α can be
(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$
39. If values of m for which the line $y = mx + 2\sqrt{5}$ touches the hyperbola $16x^2 - 9y^2 = 144$ are the roots of $x^2 - (a + b)x - 4 = 0$, then the value of $(a + b)$ is
(a) 2 (b) 4 (c) 0 (d) none
40. The locus of the feet of the perpendiculars drawn from either focus on a variable tangent to the hyperbola $16x^2 - 9y = 1$ is
(a) $x^2 + y^2 = 9$ (b) $x^2 + y^2 = 1/9$
(c) $x^2 + y^2 = 7/144$ (d) $x^2 + y^2 = 1/16$
41. The locus of the foot of the perpendicular from the centre of the hyperbola $xy = 1$ on a variable tangent is
(a) $(x^2 - y^2)^2 = 4xy$ (b) $(x^2 - y^2)^2 = 2xy$
(c) $(x^2 + y^2)^2 = 2xy$ (d) $(x^2 + y^2) = 4xy$
42. The tangent at a point P on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets one of the directrix in F. If PF subtends an angle θ at the corresponding focus, the value of θ is
(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{4}$ (d) None
43. The number of points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 3$, from which mutually perpendicular tangents can be drawn to the circle $x^2 + y^2 = a^2$, is
(a) 0 (b) 2 (c) 3 (d) 4
44. If the sum of the slopes of the normal from a point P to the hyperbola $xy = c^2$ is equal to λ, where $\lambda \in R^+$, the locus of the point P is
(a) $x^2 = \lambda c^2$ (b) $y^2 = \lambda c^2$
(c) $xy = \lambda c^2$ (d) $x/y = \lambda c^2$.
45. If two distinct tangents can be drawn from the point $(\alpha, 2)$ on different branches of the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$, then
(a) $|\alpha| < \frac{3}{2}$ (b) $|\alpha| > \frac{2}{3}$
(c) $|\alpha| > 3$ (d) $|\alpha| > 5$
46. From any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, tangents are drawn to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2$. The area cut off by the chord of contact on the asymptotes is
(a) $a/2$ (b) ab (c) $2ab$ (d) $4ab$
47. A hyperbola passes through (2, 3) and has asymptotes $3x - 4y + 5 = 0$ and $12x + 5y = 40$, the equation of its transverse axis is
(a) $77x - 21y - 265 = 0$ (b) $21x - 77y + 265 = 0$
(c) $21x + 77y - 326 = 0$ (d) $21x + 77y - 273 = 0$
48. The centre of a rectangular hyperbola lies on the line $y = 2x$. If one of the asymptotes is $x + y + c = 0$, the other asymptote is
(a) $x - y - 3c = 0$ (b) $2x - y + c = 0$
(c) $x - y - c = 0$ (d) None
49. The equation of a rectangular hyperbola, whose asymptotes are $x = 3$ and $y = 5$ and passing through (7, 8) is
(a) $xy - 3y + 5x + 3 = 0$ (b) $xy + 3y + 4x + 3 = 0$
(c) $xy - 3y + 5x - 3 = 0$ (d) $xy - 3y + 5x + 5 = 0$
50. The equation of the conjugate axis of the hyperbola $xy - 3y - 4x + 7 = 0$ is
(a) $x + y = 3$ (b) $x + y = 7$
(c) $y - x = 3$ (d) none
51. The curve $xy = c$ $(c > 0)$ and the circle $x^2 + y^2 = 1$ touch at two points, the distance between the points of contact is
(a) 2 (b) 3 (c) 4 (d) $2\sqrt{2}$
52. Let the curves $(x - 1)(y - 2) = 5$ and $(x - 1)^2 + (y - 2)^2 = r^2$ intersect at four points P, Q, R, S. If the centroid of ΔPQR lies on the line $y = 3x - 4$, the locus of S is
(a) $y = 3x$ (b) $x^2 + y^2 + 3x + 1 = 0$
(c) $3y = x + 1$ (d) $y = 3x + 1$
53. The ellipse $4x^2 + 9y^2 = 36$ and the hyperbola $a^2x^2 - y^2 = 4$ intersect at right angles, the equation of the circle through the point of intersection of the two conics is
(a) $x^2 + y^2 = 5$
(b) $\sqrt{5}(x^2 + y^2) = 3x + 4y$
(c) $\sqrt{5}(x^2 + y^2) + 3x + 4y = 0$
(d) $x^2 + y^2 = 25$
54. The angle between the lines joining the origin to the points of intersection of the line $\sqrt{3}x + y = 2$ and the curve $\sqrt{3}x + y = 2$ is

(a) $\tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$ (b) $\frac{\pi}{6}$

(c) $\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (d) $\frac{\pi}{2}$

55. If $S = 0$ be the equation of the hyperbola $x^2 + 4xy + 3y^2 - 4x + 2y + 1 = 0$, the value of k for which $S + k = 0$ represents its asymptotes is
(a) 20 (b) –16 (c) –22 (d) 18

LEVEL III

(Problems for JEE Advanced)

1. For all real values of m, the straight line $y = mx + \sqrt{9m^2 - 4}$ is a tangent to a hyperbola, find the equation of the hyperbola.
2. Find the equations of the common tangent to the curves $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{x^2}{b^2} - \frac{y^2}{a^2} = -1$ and also find its length.
3. Find the equation of the common tangents to the curves $\frac{x^2}{16} - \frac{y^2}{9} = 1$ and $x^2 + y^2 = 9$.
4. If the normal at ϕ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the transverse axis at G such that
$AG \cdot A'G = a^m(e^n \sec^p \theta - 1)$,
where A, A' are the vertices of the hyperbola and m, n and p are positive integers, find the value of
$(m + n + p)^2 + 36$
5. If the normals at (x_i, y_i), $i = 1, 2, 3, 4$ on the rectangular hyperbola $xy = c^2$ meet at the point (α, β), prove that
(i) $x_1 + x_2 + x_3 + x_4 = \alpha$
(ii) $y_1 + y_2 + y_3 + y_4 = \alpha$
(iii) $x_1^2 + x_2^2 + x_3^2 + x_4^2 = \beta^2$
(iv) $y_1^2 + y_2^2 + y_3^2 + y_4^2 = \beta^2$
(v) $x_1 \cdot x_2 \cdot x_3 \cdot x_4 = -c^4$
(vi) $y_1 \cdot y_2 \cdot y_3 \cdot y_4 = -c^4$
6. If the normals at (x_i, y_i), i = 1, 2, 3, 4 on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are concurrent, prove that
(i) $(x_1 + x_2 + x_3 + x_4)\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}\right) = 4$.
(ii) $(y_1 + y_2 + y_3 + y_4)\left(\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} + \frac{1}{y_4}\right) = 4$.
7. The perpendicular from the centre upon the normal at any point of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets at Q. Prove that the locus of Q is
$(x^2 + y^2)(a^2y^2 - b^2x^2) = (a^2 + b^2)x^2y^2$
8. From the points on the circle $x^2 + y^2 = a^2$, tangents are drawn to the hyperbola $x^2 - y^2 = a^2$. Prove that the locus of the mid-points of the chord of contact is
$(x^2 - y^2)^2 = a^2(x^2 + y^2)$
9. Prove that the locus of the mid-points of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ which subtend right angle at the centre is
$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2 \left(\frac{1}{a^2} - \frac{1}{b^2}\right) = \left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right)$.
10. Tangents are drawn from a point P to the parabola $y^2 = 4ax$. If the chord of contact of the parabola be a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, find the locus of the point P.
11. Chords of the hyperbola $x^2 - y^2 = a^2$ touch the parabola $y^2 = 4ax$. Prove that the locus of their mid-points is the curve $y^2(x - a) = x^3$.
12. Prove that the locus of the mid-points of the rectangular hyperbola $xy = c^2$ of constant length $2d$ is
$(x^2 + y^2)(xy - c^2) = d^2xy$
13. A variable chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is a tangent to the circle $x^2 + y^2 = c^2$. Prove that the locus of its mid-points is
$$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2 = c^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right)$$
14. A variable chord of the circle $x^2 + y^2 = a^2$ touch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Prove that the locus of its mid-points is $(x^2 + y^2)^2 = a^2x^2 - b^2y^2$.
15. A tangent to the parabola $y^2 = 4ax$ meets the hyperbola $xy = c^2$ in two points P and Q. Prove that the locus of the mid-point of PQ lies on a parabola.
16. From a point P, tangents are drawn to the circle $x^2 + y^2 = a^2$. If the chord of contact of the circle is a normal chord of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, prove that the locus of the point P is $\left(\frac{a^2}{x^2} - \frac{b^2}{y^2}\right) = \left(\frac{a^2 + b^2}{a^2}\right)^2$.

17. The normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meets the axes in M and N, and lines MP and NP are drawn at right angles to the axes. Prove that the locus of P is the hyperbola $a^2x^2 - b^2y^2 = (a^2 + b^2)^2$.
18. Prove that the locus of the point of intersection of tangents to a hyperbola which meet at a constant angle β is the curve
$$(x^2 + y^2 + b^2 - a^2)^2 = 4\cot^2\beta(a^2b^2 - b^2x^2 + a^2b^2)$$
19. Prove that the chords of a hyperbola, which touch the conjugate hyperbola, are bisected at the point of contact.
20. A straight line is drawn parallel to the conjugate axis of a hyperbola to meet it and the conjugate hyperbola in the points P and Q. Show that the tangents at P and Q meet the curve $\frac{y^4}{b^4}\left(\frac{y^2}{b^2}+\frac{x^2}{a^2}\right)=\frac{4x^2}{a^2}$ and the normals meet on the axis of x.
21. Prove that the locus of the mid-points of the chords of the circle $x^2 + y^2 = 16$, which are tangent to the hyperbola $9x^2 - 16y^2 = 144$ is $(x^2 + y^2)^2 = 16x^2 - 9y^2$.
22. Find the asymptotes of the curve $2x^2 + 5xy + 2y^2 + 4x + 5y = 0$. Also, find the general equation of all hyperbolas having the same asymptotes.
23. Find the equation of the hyperbola whose asymptotes are the straight lines $x + 2y + 3 = 0$ and $3x + 4y + 5 = 0$ and pass through the point $(1, -1)$.
24. Let C be the centre of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and the tangent at any point P meets the asymptotes at the points Q and R. Prove that the equation to the locus of the centre of the circle circumscribing, the ΔCQR is $4(a^2x^2 - b^2y^2) = (a^2 + b^2)^2$.
25. If P, Q, R be three points on the rectangular hyperbola $xy = c^2$, whose abscissae are x_1, x_2, x_3, prove that the area of ΔPQR is
$$\frac{c^2}{2}\times\frac{(x_1-x_2)(x_2-x_3)(x_3-x_1)}{x_1\,x_2\,x_3}$$

Level IV

(Tougher Problems for JEE Advanced)

1. A tangent to the parabola $x^2 = 4ay$ meets the hyperbola $xy = k^2$ in two points P and Q. Prove that the locus of the mid-point of PQ lies on the parabola.
2. Find the equation of the chord of the hyperbola $25x^2 - 16y^2 = 400$ which is bisected at the point $(6, 2)$.
3. Find the locus of the mid-points of the chords of the circle $x^2 + y^2 = 16$ which are tangents to the hyperbola $9x^2 - 16y^2 = 144$.
4. A tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ cuts the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in P and Q. Prove that the locus of the mid-point of PQ is $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)$.
5. If a triangle is inscribed in a rectangular hyperbola, prove that its orthocentre lies on the curve.
6. Prove that the locus of the poles of the normal chords of the rectangular hyperbola $xy = c^2$ is the curve $(x^2 - y^2)^2 + 4c^2xy = 0$.
7. If a circle cuts a rectangular hyperbola $xy = c^2$ in A, B, C and D and the parameters of these four points be t_1, t_2, t_3 and t_4 respectively. Prove that the centre of the circle through A, B and C is
$$\left(\frac{c}{2}\left(t_1+t_2+t_3+\frac{1}{t_1t_2t_3}\right),\frac{c}{2}\left(\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}+t_1t_2t_3\right)\right)$$
8. If a triangle is inscribed in a rectangular hyperbola, prove that the orthocentre of the triangle lies on the curve.
9. If a circle cuts a rectangular hyperbola $xy = c^2$ in A, B, C, D and the parameters of the four points be t_1, t_2, t_3, and t_4 respectively, prove that the centre of the mean position of the four points bisects the distance between the centres of the two curves.
10. A circle of variable radius cuts the rectangular hyperbola $x^2 - y^2 = 9a^2$ in points P, Q, R and S. Find the equation of the locus of the centroid of ΔPQR.

Integer Type Questions

1. Find the eccentricity of the hyperbola conjugate to the hyperbola $\frac{x^2}{4}-\frac{y^2}{12}=1$.
2. If the foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{b^2}=1$ and the hyperbola $\frac{x^2}{144}-\frac{y^2}{81}=\frac{1}{25}$ coincide, find the value of $(b^2 + 1)$.
3. If e_1 and e_2 be the eccentricities of a hyperbola and its conjugate, find the value of $\left(\frac{1}{e_1^2}+\frac{1}{e_2^2}+3\right)$.
4. Find the number of tangents to the hyperbola $\frac{x^2}{4}-\frac{y^2}{3}=1$ from the point $(4, 3)$.
5. Find the number of points on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=3$ from which mutually perpendicular tangents can be drawn to the circle $x^2 + y^2 = a^2$.

6. If the latus rectum of the hyperbola $9x^2 - 16y^2 - 18x - 32y - 151 = 0$ is m, find the value of $(2m - 3)$.
7. If the number of possible tangents can be drawn to the curve $4x^2 - 9y^2 = 36$, which are perpendicular to the straight line $5x + 2y = 10$ is m, find the value of $(m + 4)$.
8. If values of m for which the line $y = mx + 2\sqrt{5}$ touches the hyperbola $16x^2 - 9y^2 = 144$ are the roots of $x^2 - (a + b)x - 4 = 0$, find the value of $(a + b + 3)$.
9. The curve $xy = c$, $(c > 0)$ and the circle $x_2 + y_2 = 1$ touch at two points, find the distance between the points of contact.
10. If the product of the perpendicular distances from any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ of eccentricity $e = \sqrt{3}$ from its asymptote is equal to 6, find the length of the transverse axis of the hyperbola.
11. The tangent to the hyperbola $xy = c^2$ at the point P intersects the x-axis at T and the y-axis at T'. The normal to the hyperbola at P intersects the x-axis at N and N', respectively. The area of Δs PNT and $PN'T'$ are Δ and Δ' respectively, find the value of $\left(\frac{c^2}{\Delta} + \frac{c^2}{\Delta'} + 4\right)$.
12. Find the area of the triangle formed by any tangent to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ and its asymptotes.

Comprehension Link Passage

Passage I

The locus of the foot of the perpendicular from any focus of a hyperbola upon any tangent to the hyperbola is an auxiliary circle of the hyperbola. Consider the foci of a hyperbola as (–3, –2) and (5, 6) and the foot of the perpendicular from the focus (5, 6) upon a tangent to the hyperbola as (2, 5).

1. The conjugate axis of the hyperbola is
   (a) $4\sqrt{11}$ (b) $2\sqrt{11}$ (c) $4\sqrt{22}$ (d) $2\sqrt{22}$
2. The directrix of the hyperbola corresponding to the focus (5, 6) is
   (a) $2x + 2y - 1 = 0$ (b) $2x + 2y - 11 = 0$
   (c) $2x + 2y - 7 = 0$ (d) $2x + 2y - 9 = 0$
3. The point of contact of the tangent with the hyperbola is
   (a) $\left(\frac{2}{9}, \frac{31}{3}\right)$ (b) $\left(\frac{7}{4}, \frac{23}{4}\right)$
   (c) $\left(\frac{2}{3}, 9\right)$ (d) $\left(\frac{7}{9}, 7\right)$

Passage II

The portion of the tangent intercepted between the asymptotes of the hyperbola is bisected at the point of contact.

Consider a hyperbola whose centre is at the origin. A line $x + y = 2$ touches this hyperbola at $P(1, 1)$ and intersects the asymptotes at A and B such that $AB = 6\sqrt{2}$.

1. The equation of the asymptotes are
   (a) $5xy + 2x^2 + 3y^2 = 0$ (b) $3x^2 + 4y^2 + 6xy = 0$
   (c) $2x^3 + 2y^2 - 5xy = 0$ (d) $2x^2 + y^2 - 5xy = 0$
2. The angle subtended by AB at the centre of the hyperbola is
   (a) $\sin^{-1}\left(\frac{4}{5}\right)$ (b) $\sin^{-1}\left(\frac{2}{5}\right)$
   (c) $\sin^{-1}\left(\frac{3}{5}\right)$ (d) $\sin^{-1}\left(\frac{1}{5}\right)$
3. The equation of the tangent to the hyperbola at $\left(-1, \frac{7}{2}\right)$ is
   (a) $5x + 2y = 2$ (b) $3x + 2y = 4$
   (c) $3x + 4y = 11$ (d) $3x - 4y = 10$

Passage III

A point P moves such that the sum of the slopes of the normals drawn from it to the hyperbola $xy = 16$ is equal to the sum of the ordinates of the feet of normals. The locus of the P is a curve C.

1. The equation of the curve C is
   (a) $x^2 = 4y$ (b) $x^2 = 16y$ (c) $x^2 = 12y$ (d) $y^2 = 8x$
2. If the tangent to the curve C cuts the co-ordinate axes at A and B, the locus of the mid-point of AB is
   (a) $x^2 = 4y$ (b) $x^2 = 2y$
   (c) $x^2 + 2y = 0$ (d) $x^2 + 4y = 0$
3. The area of the equilateral triangle, inscribed in the curve C, having one vertex as the vertex of the curve C is
   (a) $772\sqrt{3}$ s. u. (b) $776\sqrt{3}$ s. u.
   (c) $760\sqrt{3}$ s. u. (d) $768\sqrt{3}$ s. u.

Passage IV

The vertices of ΔABC lie on a rectangular hyperbola such that the orthocentre of the triangle is (3, 2) and the asymptotes of the rectangular hyperbola are parallel to the co-ordinate axes. The two perpendicular tangents of the hyperbola intersect at the point (1, 1).

1. The equation of the asymptotes is
   (a) $xy - x + y - 1 = 0$ (b) $xy - x - y + 1 = 0$
   (c) $2xy + x + y$ (d) $2xy = x + y + 1$
2. The equation of the rectangular hyperbola is
   (a) $xy = 2x + y - 2$ (b) $xy = 2x + y + 5$
   (c) $xy = x + y + 1$ (d) $xy = x + y + 10$
3. The number of real tangents that can be drawn from the point (1, 1) to the rectangular hyperbola is
   (a) 4 (b) 0 (c) 3 (d) 2

Passage V

A line is drawn through the point $P(-1, 2)$ meets the hyperbola $xy = c^2$ at the (points A and B lie on the same side of P) and Q is a point on the line segment AB.

1. If the point Q be chosen such that PA, PQ, and PB are in AP, the locus of the point Q is

(a) $x = y + 2xy$ (b) $x = y + xy$
(c) $2x = y + 2xy$ (d) $2x = y + xy$

2. If the point Q be chosen such that PA, PQ and PB are in GP, the locus of the point Q is
(a) $xy - y + 2x - c^2 = 0$ (b) $xy + y - 2x + c^2 = 0$
(c) $xy + y + 2x + c^2 = 0$ (d) $xy - y - 2x - c^2 = 0$

3. If the point Q be chosen such that PA, PQ, and PB are in HP, the locus of the point Q is
(a) $2x - y = 2c^2$ (b) $x - 2y = 2c^2$
(c) $2x + y + 2c^2 = 0$ (d) $x + 2y = 2c^2$

Passage VI

The graph of the conic $x^2 - (y - 1)^2 = 1$ has one tangent line with positive slope that passes through the origin, the point of tangency being (a, b).

1. The value of $\sin^{-1}\left(\frac{a}{b}\right)$ is
(a) $5\pi/12$ (b) $\pi/6$ (c) $\pi/3$ (d) $\pi/4$

2. The length of the latus rectum of the conic is
(a) 1 (b) $\sqrt{2}$ (c) 2 (d) none

3. The eccentricity of the conic is
(a) 4/3 (b) $\sqrt{3}$ (c) 2 (d) none

Matrix Match
(For JEE-Advanced Examination Only)

1. Match the following columns
Let z, z_1 and z_2 be three complex numbers and $b, c \in R^+$. Then the locus of z

Column I		Column II	
(A)	is an ellipse, if	(P)	$\|z - c\| = b$
(B)	is a hyperbola, if	(Q)	$\|z - z_1\| + \|z - z_2\| = 2b$, where $2b > \|z_1 - z_2\|$
(C)	is a straight line, if	(R)	$\|z - z_1\| - \|z - z_2\| = 2b$ Where $2b < \|z_1 - z_2\|$
(D)	is a circle, if	(S)	$\|z - z_1\| + \|z - z_2\| = \|z_1 - z_2\|$

2. Match the following columns:
The locus of a variable point P, whose co-ordinates are given by

Column I		Column II	
(A)	$x = 3\left(\frac{1-t^2}{1+t^2}\right)$ and $y = \left(\frac{8t}{1-t^2}\right)$ is	(P)	an ellipse
(B)	$x = \frac{1}{2}(e^t + e^{-t})$ and $y = \frac{1}{2}(e^t - e^{-t})$ is	(Q)	a hyperbola
(C)	$x = \frac{2}{(e^{i\theta} + e^{-i\theta})}$ and $y = i \times \left(\frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}}\right)$ is	(R)	a circle
(D)	$x = \frac{i}{2}(e^{-i\theta} - e^{i\theta})$ and $y = \left(\frac{e^{i\theta} + e^{-i\theta}}{\sin 2\theta}\right)$ is	(S)	a parabola

3. Match the following columns
The locus of the point of intersection of two perpendicular tangents to a conic is a director circle to the given conic.

Column I		Column II	
(A)	The director circle of $\frac{x^2}{16} - \frac{y^2}{9} = 1$ is	(P)	$x^2 + y^2 = 45$
(B)	The director circle of $\frac{x^2}{20} + \frac{y^2}{25} = 1$ is	(Q)	$x^2 + y^2 = 7$
(C)	The director circle of $x^2 - y^2 = 16$ is	(R)	$x^2 + y^2 = 0$
(D)	The director circle of $x^2 + y^2 = 25$ is	(S)	$x + 2 = 0$
(E)	The director circle of $y^2 = 8x$ is	(T)	$x^2 + y^2 = 50$
(F)	The director circle of $xy = 1$ is	(U)	$x + 4 = 0$

4. Match the following columns:
The equation of the common tangent between the given curves

Column I		Column II	
(A)	$x^2 + y^2 = 9$ and $\frac{x^2}{16} - \frac{y^2}{9} = 1$ is	(P)	$y = x + 7$
(B)	$\frac{x^2}{25} - \frac{y^2}{9} = 1$ and $y = 3\sqrt{\frac{2}{7}}\,x + \frac{16}{\sqrt{7}}$ is	(Q)	$\frac{x^2}{9} - \frac{y^2}{25} = 1$

(C)	$y^2 = 8x$ and $xy = -1$ is	(R)	$y = x + 2$
(D)	$x^2 - y^2 = 9$ and $x^2 + y^2 = 4$	(S)	$y = \sqrt{\frac{3}{5}}\, x + 6\sqrt{\frac{2}{3}}$

5. Match the following columns:

Column I		Column II	
(A)	The product of the perpendiculars from the foci of any tangent to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ is	(P)	16
(B)	The product of the perpendiculars from the foci of any tangent to the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ is	(Q)	9
(C)	The product of the perpendiculars from any point on the on the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$ to its asymptotes is	(R)	144/25
(D)	The product of the perpendiculars from any point on the hyperbola $\frac{x^2}{2} - y^2 = 1$ to its asymptotes is	(S)	2/3

6. Match the following columns:

Column I		Column II	
(A)	Two intersecting circles	(P)	have a common tangent
(B)	Two mutually external circles	(Q)	have a common normal
(C)	Two circles, one strictly inside the other	(R)	do not have a common tangent
(D)	Two branches of a hyperbola	(S)	do not have a common normal.

7. Match the following columns:

Column I		Column II	
(A)	Tangents are drawn from a point on the circle $x^2 + y^2 = 11$ to the hyperbola $\frac{x^2}{25} - \frac{y^2}{14} = 1$, the angle between the tangent is	(P)	$\sin^{-1}\left(\frac{3}{5}\right)$
(B)	Tangents are drawn from a point (4, 3) to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$, the angle between their tangents is	(Q)	$\frac{\pi}{2}$
(C)	The angle between the asymptotes to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ is	(R)	$\sin^{-1}\left(\frac{24}{25}\right)$
(D)	The angle between the asymptotes to the rectangular hyperbola $x^2 - y^2 = 2013$ is	(S)	$\frac{\pi}{3}$

Questions asked in Previous Years' JEE-Advanced Examinations

1. The equation $\frac{x^2}{1-r} - \frac{y^2}{1+r} = 1, r > 1$ represents a/an
   (a) ellipse (b) hyperbola
   (c) circle (d) None **[IIT-JEE, 1981]**
2. Each of the four inequalities given below defines a region in the xy-plane. One of these four regions does not have the following property. For any two points (x_1, y_1) and (x_2, y_2) in the region, the point $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ is also in the region. The inequality defining the region is
   (a) $x^2 + 2y^2 \le 1$ (b) $\max.\{|x|, |y|\} \le 1$
   (c) $x^2 - y^2 \le 1$ (d) $y^2 - x \le 0$
   [IIT-JEE, 1981]

No questions asked from 1982 to 1993.

3. The equation $2x^2 + 3y^2 - 8x - 18y + 35 = k$ represents
   (a) no locus if $k > 0$
   (b) an ellipse if $k < 0$
   (c) a point if $k = 0$
   (d) a hyperbola if $k > 0$ **[IIT-JEE, 1994]**

No questions asked in 1995.

4. An ellipse has eccentricity 1/2 and one focus at $S(1/2, 1)$. Its one directrix is the common tangent (nearer to S) to the circle $x^2 + y^2 = 1$ and $x_2 - y_2 = 1$. The equation of the ellipse in the standard form is … **[IIT-JEE, 1996]**
5. A variable straight line of slope 4 intersects the hyperbola $xy = 1$ at two points. Find the locus of the point which divides the line segment between these two points in the ratio 1 : 2. **[IIT-JEE, 1997]**
6. If the circle $x^2 + y^2 = a^2$ intersects the hyperbola $xy = c^2$ in four points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ and $S(x_4, y_4)$, then
   (a) $x_1 + x_2 + x_3 + x_4 = 0$ (b) $y_1 + y_2 + y_3 + y_4 = 0$
   (c) $x_1 \cdot x_2 \cdot x_3 \cdot x_4 = c^4$ (d) $y_1 \cdot y_2 \cdot y_3 \cdot y_4 = c^4$
   [IIT-JEE, 1998]

7. The angle between a pair of tangents drawn from a point P to the parabola $y^2 = 4ax$ is 45°. Show that the locus of the point P is a hyperbola.

8. If $x = 9$ is the chord of contact of the hyperbola $x^2 - y^2 = 9$, the equation of the corresponding pair of tangents is
(a) $9x^2 - 8y^2 + 18x - 9 = 0$
(b) $9x^2 - 8y^2 - 18x - 9 = 0$
(c) $9x^2 - 8y^2 - 18x + 9 = 0$
(d) $9x^2 - 8y^2 + 18x + 9 = 0$ **[IIT-JEE, 1999]**

9. Let $P(a \sec\theta, b\tan\theta)$ and $Q(a\sec\varphi, b\tan\varphi)$, where $\theta + \varphi = \frac{\pi}{2}$, be two points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
If (h, k) is the point of the intersection of the normals at P and Q, then k is
(a) $\left(\frac{a^2+b^2}{a}\right)$ (b) $-\left(\frac{a^2+b^2}{a}\right)$
(c) $\left(\frac{a^2+b^2}{b}\right)$ (d) $-\left(\frac{a^2+b^2}{b}\right)$
[IIT-JEE, 1999]

No questions asked in 2000 to 2002.

10. For a hyperbola $\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$, which of the following remains constant with the change of α?
(a) abscissae of vertices (b) abscissae of foci
(c) eccentricity (d) directrix
[IIT-JEE, 2003]

11. The point of contact of the line $2x + \sqrt{6}y = 2$ and the hyperbola $x^2 - 2y^2 = 4$ is
(a) $(4, -\sqrt{6})$ (b) $(\sqrt{6}, 1)$
(c) $(1/2, 1/\sqrt{6})$ (d) $(1/6, 3/2)$
[IIT-JEE, 2004]

12. Tangents are drawn to the circle $x^2 + y^2 = 9$ from a points on the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$. Find the locus of the mid-point of the chord of contact. **[IIT-JEE, 2005]**

13. Let a hyperbola passes through the focus of an ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$. The transverse and conjugate axis of this hyperbola coincide with the major and minor axes of the given ellipse, also the product of eccentricities of the given ellipse and hyperbola is 1, then the
(a) hyperbola is $\frac{x^2}{9} - \frac{y^2}{16} = 1$
(b) hyperbola is $\frac{x^2}{9} - \frac{y^2}{25} = 1$
(c) focus of the hyperbola is (5, 0)
(d) vertex of the hyperbola is $(5\sqrt{3}, 0)$
[IIT-JEE, 2006]

14. A hyperbola, having the transverse axis of length $2\sin\theta$ is confocal with the ellipse $3x^2 + 4y^2 = 12$, then its equation is
(a) $x^2 \operatorname{cosec}^2\theta - y^2\sec^2\theta = 1$
(b) $x^2\sec^2\theta - y^2\operatorname{cosec}^2\theta = 1$
(c) $x^2\sin^2\theta - y^2\cos^2\theta = 1$
(d) $x^2\cos^2\theta - y^2\sin^2\theta = 1$ **[IIT-JEE, 2007]**

15. Consider a branch of the hyperbola $x^2 - 2y^2 - 2\sqrt{2}x - 4\sqrt{2}y - 6 = 0$ with the vertex at the point A. Let B be one of the end points of its latus rectum. If C be the focus of the hyperbola nearest to the point A, the area of ΔABC is
(a) $\left(1 - \sqrt{\frac{2}{3}}\right)$ (b) $\left(\sqrt{\frac{3}{2}} - 1\right)$
(c) $\left(1 + \sqrt{\frac{2}{3}}\right)$ (d) $\left(\sqrt{\frac{3}{2}} + 1\right)$
[IIT-JEE, 2008]

16. An ellipse intersects the hyperbola $2x^2 - 2y^2 = 1$ orthogonally. The eccentricity of the ellipse is reciprocal of that of the hyperbola. If the axes of the ellipse are along the co-ordinate axes, then the
(a) ellipse is $x^2 + 2y^2 = 2$
(b) ellipse is $x^2 + 2y^2 = 4$
(c) foci of the ellipse are $(\pm 1, 0)$
(d) foci of the ellipse are $(\pm\sqrt{2}, 0)$.
[IIT-JEE, 2009]

17. Match the following columns:

Column I		Column II	
(A)	Circle	(P)	The locus of the point (h, k) for which the line $hx + ky = 1$ touches the circle $x_2 + y_2 = 4$.
(B)	Parabola	(Q)	A point z in the complex plane satisfying $\lvert z+2\rvert - \lvert z-2\rvert = \pm 3$
(C)	Ellipse	(R)	The points of the conic have parametric representations $x = \sqrt{3}\left(\frac{1-t^2}{1+t^2}\right)$ and $y = \left(\frac{2t}{1+t^2}\right)$

(D)	Hyperbola	(S)	The eccentricity of the conic lies in the interval $1 \le x < \infty$.
		(T)	The points z in the complex plane satisfying $Re(z+1)^2 = \|z\|^2 + 1$

[IIT-JEE, 2009]

Comprehension

The circle $x^2 + y^2 - 8x = 0$ and the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ intersect at the points A and B.

18. The equation of the common tangent with positive slope to the circle as well as to the hyperbola is
   (a) $2x - \sqrt{5}y - 20 = 0$ (b) $2x - \sqrt{5}y + 4 = 0$
   (c) $3x - 4y + 8 = 0$ (d) $4x - 3y + 4 = 0$
19. The equation of a circle with AB as its diameter is
   (a) $x^2 + y^2 - 12x + 24 = 0$
   (b) $x^2 + y^2 + 12x + 24 = 0$
   (c) $x^2 + y^2 + 24x - 12 = 0$
   (d) $x^2 + y^2 - 24x - 12 = 0$ **[IIT-JEE, 2010]**
20. The line $2x + y = 1$ is the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If this line passes through the point of intersection of the nearest directrix and the x-axis, the eccentricity of the hyperbola is …
   [IIT-JEE, 2010]
21. Let $P(6, 3)$ be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If the normal at the point P intersects the x-axis at $(9, 0)$, the eccentricity of the hyperbola is
   (a) $\sqrt{\frac{5}{2}}$ (b) $\sqrt{\frac{3}{2}}$ (c) $\sqrt{2}$ (d) $\sqrt{3}$
22. Let the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be reciprocal to that of the ellipse $x^2 + 4y^2 = 4$. If the hyperbola passes through a focus of the ellipse, then
   (a) the hyperbola is $\frac{x^2}{3} - \frac{y^2}{2} = 1$
   (b) a focus of hyperbola is $(2, 0)$
   (c) the eccentricity of hyperbola is $\frac{2}{\sqrt{3}}$
   (d) the hyperbola is $x^2 - 3y^2 = 3$ **[IIT-JEE, 2011]**
23. Tangents are drawn to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$, parallel to the straight line $2x - y = 1$. The points of contact of the tangents on the hyperbola are
   (a) $\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ (b) $\left(-\frac{9}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$
   (c) $(3\sqrt{3}, -2\sqrt{2})$ (d) $(-3\sqrt{3}, 2\sqrt{2})$
   [IIT-JEE, 2012]

No questions asked in between 2013-2014.

24. Consider the hyperbola $H : x^2 - y^2 = 1$ and a circle S with center $N(x_2, 0)$. Suppose that H and S touch each other at a point $P(x_1, y_1)$ with $x_1 > 1$ and $y_1 > 0$. The common tangent to H and S at P intersects the x-axis at point M. If (l, m) is the centroid of the triangle ΔPMN, then the correct expression(s) is (are)
   (a) $\frac{dI}{dx_1} = 1 - \frac{1}{3x_1^2}$ for $x_1 > 1$
   (b) $\frac{dm}{dx_1} = \frac{x_1}{3(\sqrt{x_1^2 - 1})}$ for $x_1 > 1$
   (c) $\frac{dI}{dx_1} = 1 + \frac{1}{3x_1^2}$ for $x_1 > 1$
   (d) $\frac{dm}{dy_1} = \frac{1}{3}$ for $y_1 > 0$ **[IIT-JEE-2015]**

No questions asked in 2016.

Answers

Level II

1. (b) 2. (a) 3. (a) 4. (b) 5. (d)
6. (b) 7. (b) 8. (d) 9. (b) 10. (a)
11. (d) 12. (b) 13. (d) 14. (a) 15. (c)
16. (b) 17. (c) 18. (c) 19. (b) 20. (b)
21. (d) 22. (b) 23. (d) 24. (c) 25. (a)
26. (a) 27. (b) 28. (c) 29. (a) 30. (b)
31. (a) 32. () 33. (c) 34. (b) 35. (a)
36. (c) 37. (d) 38. (b) 39. (c) 40. (d)
41. (d) 42. (b) 43. (a) 44. (a) 45. (a)
46. (a) 47. (d) 48. (c) 49. (d) 50. (b)
51. (a) 52. (a) 53. (a) 54. (c) 55. (c)

Level III

1. $\dfrac{x^2}{9}-\dfrac{y^2}{4}=1$
2. $y=\pm x\pm\sqrt{a^2-b^2}$
3. $y=\pm\sqrt{\dfrac{7}{10}}\,x+\sqrt{\dfrac{11}{5}}$
4. 72
10. $4a^2x^2+b^2y^2=4a^4$
22. $2x^2+5xy+2y^2+4x+5y+2=0$
23. $3x^2+10xy+8y^2+4x+6y+1=0$

Level IV

2. $75x-16y=418$
3. $(x^2+y^2)^2=16x^2-9y^2$
10. $\left(x-\dfrac{2h}{3}\right)^2+\left(y-\dfrac{2k}{3}\right)^2=a^2$

INTEGER TYPE QUESTIONS

1. 2	2. 8	3. 4	4. 1	5. 0
6. 6	7. 4	8. 3	9. 2	10. 6
11. 6	12. 6			

COMPREHENSIVE LINK PASSAGES

Passage I:	1. (d)	2. (b)	3. (c)
Passage II:	1. (a)	2. (c)	3. (b)
Passage III:	1. (b)	2. (c)	3. (d)
Passage IV:	1. (b)	2. (c)	3. (d)
Passage V:	1. (c)	2. (b)	3. (a)
Passage VI:	1. (d)	2. (c)	3. (d)

MATRIX MATCH

1. (A) $\to$ Q; (B) $\to$ R; (C) $\to$ S; (D) $\to$ P
2. (A) $\to$ P; (B) $\to$ Q; (C) $\to$ Q; (D) $\to$ Q
3. (A) $\to$ Q; (B) $\to$ P; (C) $\to$ R; (D) $\to$ T; (E) $\to$ S; (F) $\to$ R
4. (A) $\to$ Q; (B) $\to$ P; (C) $\to$ R; (D) $\to$ S
5. (A) $\to$ Q; (B) $\to$ P; (C) $\to$ R; (D) $\to$ S
6. (A) $\to$ P, Q; (B) $\to$ P, Q; (C) $\to$ Q, R; (D) $\to$ Q, R
7. (A) $\to$ Q; (B) $\to$ P; (C) $\to$ R; (D) $\to$ S

Hints and Solutions

Level I

1.

(i) The equation of the given hyperbola is $\dfrac{x^2}{9}-\dfrac{y^2}{4}=1$

(a) Centre: (0, 0)

(b) Vertices: $A(a, 0)=A(3, 0)$
and $A(-a, 0)=A(-3, 0)$

(c) Co-vertices: $B(0, b)=B(0, 2)$
and $B'(0, -b)=B'(0, -2)$

(d) The length of the transverse axis $=2a=6$

(e) The length of the conjugate axis $=2b=4$

(f) The length of the latus rectum $=\dfrac{2b^2}{a}=\dfrac{8}{3}$

(g) Eccentricity $=e=\sqrt{1+\dfrac{b^2}{a^2}}=\sqrt{1+\dfrac{4}{9}}=\dfrac{\sqrt{13}}{3}$

(h) Equation of the directrices:

$$x=\pm\frac{a}{e}=\pm\frac{2}{\sqrt{13}/3}=\pm\frac{6}{\sqrt{13}}$$

(ii) The equation of the given hyperbola is

$$\frac{x^2}{16}-\frac{y^2}{9}=-1$$

(a) Centre: (0, 0)

(b) Vertices: $A(0, b)=A(0, 3)$
and $A(0, -b)=A(0, -3)$

(c) Co-vertices: $B(a, 0)=B(4, 0)$
and $B'(-a, 0)=B'(-4, 0)$

(d) The length of the transverse axis $=2a=6$

(e) The length of the conjugate axis $=2b=8$

(f) The length of the latus rectum $=\dfrac{2a^2}{b}=\dfrac{16}{3}$

(g) Eccentricity $=e=\sqrt{1+\dfrac{a^2}{b^2}}=\sqrt{1+\dfrac{9}{16}}=\dfrac{5}{4}$

(h) Equation of the directrices :

$$y=\pm\frac{b}{e}=\pm\frac{3}{5/4}=\pm\frac{12}{5}.$$

(iii) The equation of the hyperbola is

$$9x^2-16y^2-36x+96y-252=0$$
$$\Rightarrow\quad 9(x^2-4x)-16(y^2-6y)=252$$
$$\Rightarrow\quad 9(x-2)^2-16(y-3)^2=252+36-144=144$$
$$\Rightarrow\quad \frac{9(x-2)^2}{144}-\frac{16(y-3)^2}{144}=1$$
$$\Rightarrow\quad \frac{(x-2)^2}{16}-\frac{(y-3)^2}{9}=1$$

(a) Centre : (0, 0)

$\Rightarrow\quad X=0, Y=0$

$\Rightarrow\quad x-2=0, y-3=0$

$\Rightarrow\quad x=2$ and $y=3$

Hence, the centre is (2, 3).

(b) Vertices : $(\pm a, 0)$

$\Rightarrow \quad X = \pm a,\ Y = 0$

$\Rightarrow \quad x - 2 = 4,\ y - 3 = 0$

$\Rightarrow \quad x = 2 \pm 4,\ y = 3$

Hence, the vertices are (6, 3) and (–2, 3).

(c) Co-vertices: $(0, \pm b)$

$\Rightarrow \quad X = 0,\ Y = \pm b$

$\Rightarrow \quad x - 2 = 0,\ y - 3 = \pm 3$

$\Rightarrow \quad x = 2,\ y = 3 \pm 3$

Hence, the co-vertices are (2, 6) and (2, 0)

(d) The length of the transverse axis = $2a = 8$

(e) The length of the conjugate axis = $2b = 6$

(f) Eccentricity = $e = \sqrt{1 + \frac{9}{16}} = \frac{5}{4}$

(g) Co-ordinates of Foci : $(\pm ae, 0)$

$X = \pm 5,\ Y = 0$

$\Rightarrow \quad X - 2 = \pm 5,\ y - 3 = 0$

$\Rightarrow \quad X = 2 \pm 5,\ y = 3$

Hence, the co-ordinates of the foci are (7, 3) and (–3, 3)

2. The equation of the hyperbola with centre (1, 0) is $\frac{(x-1)^2}{a^2} - \frac{y^2}{b^2} = 1$

Here, $2a = 6 \Rightarrow a = 3$

Also, one focus = (6, 0)

$\Rightarrow \quad 1 + ae = 6$

$\Rightarrow \quad ae = 5$

$\Rightarrow \quad 3e = 5$

$\Rightarrow \quad e = 5/3$

Therefore,

$$b^2 = a^2(e^2 - 1) = 9\left(\frac{25}{9} - 1\right) = 16$$

Hence, the equation of the hyperbola is

$$\frac{(x-1)^2}{9} - \frac{y^2}{16} = 1$$

3. Since the focus is (5, 2) and the vertex is (4, 2), so the axis of the hyperbola is parallel to x-axis and $a = 1$, $ae = 2$

Let the equation of the hyperbola be

$$\frac{(x-3)^2}{a^2} - \frac{(y-2)^2}{b^2} = 1 \qquad \ldots\text{(i)}$$

As we know that, the relation in a, b and e with respect to a hyperbola is

$$b^2 - a^2(e^2 - 1) - a^2e^2 - a^2 - 4 - 1 - 3$$

Now from Eq. (i), we get

$$\frac{(x-3)^2}{1} - \frac{(y-2)^2}{3} = 1$$

4. Since the focus is (–3, 2) and the vertex is (–3,4), so the axis of the hyperbola is parallel to y-axis and $be = 2$

$\Rightarrow \quad b \times \frac{5}{2} = 2$

$\Rightarrow \quad b = \frac{4}{5}$

Also, $a^2 = b^2(e^2 - 1) = \frac{16}{25}\left(\frac{25}{4} - 1\right) = \frac{84}{25}$

Hence, the equation of the hyperbola is

$$\frac{(x+3)^2}{a^2} - \frac{(y-2)^2}{b^2} = 1$$

$$\Rightarrow \quad \frac{(x+3)^2}{84} - \frac{(y-2)^2}{16} = \frac{1}{25}$$

5. From the definition of the hyperbola, we can write

$$\frac{SP}{PM} = e,$$

where S = focus, $P = (x, y)$

$\Rightarrow \quad SP^2 = e^2\, PM^2$

$$\Rightarrow \quad (x-2)^2 + (y-1)^2 = 4\left\{\left(\frac{x + 2y - 1}{\sqrt{1+4}}\right)^2\right\}$$

$\Rightarrow \quad 5\{(x - 2)^2 + (y - 1)^2\} = 4(x + 2y - 1)^2$

6. Given relation is

$2\,ae = 16$

$\Rightarrow \quad ae = 8$

$\Rightarrow \quad a = 8/e \quad \Rightarrow \quad a = 2$

Now, $b^2 - a^2(e^2 - 1) - a^2e^2 - a^2 = 64 - 4 = 60$

Hence, the equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\Rightarrow \quad \frac{x^2}{4} - \frac{y^2}{60} = 1$$

7. Since foci are (6, 4) and (–6, 4), so the axis of the hyperbola is parallel to x-axis.

It is given that the distance between two foci is 12, so

$2\,ae = 12$

$\Rightarrow \quad ae = 6$

$\Rightarrow \quad 2\,a = 6$

$\Rightarrow \quad a = 3$

Also,

$b^2 - a^2(e^2 - 1) - a^2e^2 - a^2 - 36 - 9 = 27$

Hence, the equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\Rightarrow \quad \frac{x^2}{9} - \frac{y^2}{27} = 1$$

8. Given relation is

$$\frac{2b^2}{a} = \frac{1}{2} \times (2a) = a$$

$$\Rightarrow \quad \frac{2b^2}{a} = a$$

$\Rightarrow \quad 2b^2 = a^2$

Also, relation in a, b and e is

$b^2 = a^2(c^2 - 1)$

$$\Rightarrow \quad \frac{a^2}{2} = a^2(e^2 - 1)$$

$$\Rightarrow \quad (e^2 - 1) = \frac{1}{2}$$

$$\Rightarrow \quad e = \sqrt{\frac{3}{2}}$$

9. Let the equation of the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and its conjugate is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$.
Therefore,

$$e_1 = \sqrt{1 + \frac{b^2}{a^2}} \text{ and } e_2 = \sqrt{1 + \frac{a^2}{b^2}}$$

$$\Rightarrow \quad e_1^2 = \left(\frac{a^2 + b^2}{a^2}\right) \text{ and } e_2^2 = \left(\frac{a^2 + b^2}{b^2}\right)$$

$$\Rightarrow \quad \frac{1}{e_1^2} + \frac{1}{e_2^2} = \left(\frac{a^2}{a^2 + b^2}\right) + \left(\frac{b^2}{a^2 + b^2}\right)$$

$$= \left(\frac{a^2 + b^2}{a^2 + b^2}\right) = 1$$

10. Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the equation of the hyperbola is is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
Also, it is given that,

$$2b = 2a$$

$$\Rightarrow \quad b = a$$

Let e_1 and e_2 be the eccentricities of the ellipse and the hyperbola.

Then $e_1 = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{a^2}{a^2}} = 0$

and $e_2 = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{a^2}{a^2}} = \sqrt{2}$

Thus, $e_1^2 + e_2^2 = 2$

11. The equation of the hyperbola is

$$4(2y - x - 3)^2 - 9(2x + y - 1)^2 = 80$$

The centre of the hyperbola is obtained from the equations

$$2y - x - 3 = 0 \text{ and } 2x + y - 1 = 0$$

Solving, we get,

$$x = -2/5$$

and $y = 13/10$

Hence, the centre is $\left(-\frac{2}{5}, \frac{13}{10}\right)$.

12. The equation of the given hyperbola is

$$3x^2 - 5y^2 - 6x + 20y - 32 = 0$$

$$\Rightarrow \quad 3(x^2 - 2x) - 5(y^2 - 4y) = 32$$

$$\Rightarrow \quad 3(x - 1)^2 - 5(y - 2)^2 = 32 + 3 - 20 = 15$$

$$\Rightarrow \quad \frac{3(x-1)^2}{15} - \frac{5(y-2)^2}{15} = 1$$

$$\Rightarrow \quad \frac{(x-1)^2}{5} - \frac{(y-2)^2}{3} = 1$$

13. The given straight lines are

$$\frac{x}{a} - \frac{y}{b} = 2013 \quad \ldots(i)$$

and

$$\frac{x}{a} + \frac{y}{b} = \frac{1}{2013} \quad \ldots(ii)$$

Multiplying Eqs (i) and (ii), we get

$$\left(\frac{x}{a} - \frac{y}{b}\right)\left(\frac{x}{a} + \frac{y}{b}\right) = 2013 \times \frac{1}{2013} = 1$$

$$\Rightarrow \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

which represents a hyperbola.

14. We have,

$$x = 3\left(\frac{1 + t^2}{1 - t^2}\right) \text{ and } y = \frac{4t}{t^2 - 1}$$

$$\Rightarrow \quad \frac{x^2}{9} - \frac{y^2}{4} = \left(\frac{1 + t^2}{1 - t^2}\right)^2 - \left(\frac{2t}{1 - t^2}\right)^2$$

$$\Rightarrow \quad \frac{x^2}{9} - \frac{y^2}{4} = \frac{(1 + t^2)^2 - 4t^2}{(1 - t^2)^2} = \left(\frac{1 - t^2}{1 - t^2}\right)^2 = 1$$

$$\Rightarrow \quad \frac{x^2}{9} - \frac{y^2}{4} = 1$$

which represents a hyperbola.

15. We have,

$$x = \frac{1}{2}(e^t + e^{-t}) \text{ and } y = \frac{1}{2}(e^t - e^{-t})$$

$$\Rightarrow \quad 2x = (e^t + e^{-t}) \text{ and } 2y = (e^t - e^{-t})$$

$$\Rightarrow \quad 4x^2 - 4y^2 = (e^t + e^{-t})^2 - (e^t - e^{-t})^2$$

$$\Rightarrow \quad 4x^2 - 4y^2 = 2 + 2$$

$$\Rightarrow \quad x^2 - y^2 = 1$$

Which represents a rectangular hyperbola.

16. The given equation of the hyperbola is

$$\frac{x^2}{2014 - \lambda} + \frac{y^2}{2013 - \lambda} = 1$$

$$\Rightarrow \quad (2013 - \lambda)x^2 + (2014 - \lambda)y^2 - (2013 - \lambda)(2014 - \lambda) = 0$$

$$\Rightarrow \quad (2013 - \lambda)x^2 + (\lambda - 2014)y^2 - (2013 - \lambda)(2014 - \lambda) = 0$$

The given equation represents a hyperbola, if

$$h^2 - ab > 0$$

$$\Rightarrow \quad 0 - (2013 - \lambda)(\lambda - 2014) > 0$$

$$\Rightarrow \quad (2013 - \lambda)(\lambda - 2014) < 0$$

$$\Rightarrow \quad 2013 < \lambda < 2014$$

$$\Rightarrow \quad \lambda \in (2013, 2014)$$

17. The given equation of the hyperbola is

$$\frac{x^2}{144}-\frac{y^2}{81}=\frac{1}{25}$$

We have,

$$e=\sqrt{1+\frac{b^2}{a^2}}=\sqrt{1+\frac{81}{144}}=\sqrt{\frac{225}{144}}=\frac{15}{12}=\frac{5}{4}$$

Also, $a^2=\frac{144}{25}$

Thus, the foci are

$$(\pm\, ae, 0)=\left(\pm\frac{12}{5}\times\frac{5}{4}, 0\right)=(\pm 3, 0).$$

Now, for the ellipse,

$ae = 3$

$\Rightarrow\ a^2e^2 = 9$

Thus, $b^2 = a^2(1 - e^2)$

$= a^2 - a^2e^2$

$= 16 - 9 = 7$

Hence, the value of b^2 is 7.

18. Let LL' be the latus rectum of the given hyperbola.

Therefore, $L\left(ae, \frac{b^2}{a}\right)$ and $L'\left(ae, -\frac{b^2}{a}\right)$ and the centre of the hyperbola is $C(0, 0)$

Now, slope of $CL = m_1 = \frac{(b^2/a)}{ae}=\frac{b^2}{a^2e}$

and the slope of $m_2=\frac{(-b^2/a)}{ae}=-\frac{b^2}{a^2e}$

Since, the latus rectum subtends right angle at the centre, so

$$m_1\times m_2=-1$$

$$\Rightarrow\ \left(\frac{b^2}{a^2e}\right)\times\left(-\frac{b^2}{a^2e}\right)=-1$$

$$\Rightarrow\ \left(\frac{b^4}{a^4e^2}\right)=1$$

$\Rightarrow\ b^4 = a^4e^2$

$\Rightarrow\ a^4(e^2-1)^2 = a^4e^2$

$\Rightarrow\ (e^2-1)^2 = e^2$

$\Rightarrow\ e^4 - 3e^2 + 1 = 0$

$$\Rightarrow\ e^2=\frac{3\pm\sqrt{5}}{2}$$

$$\Rightarrow\ e^2=\frac{3+\sqrt{5}}{2}$$

$$\Rightarrow\ e^2=\frac{3+\sqrt{5}}{2}=\frac{6+2\sqrt{5}}{4}=\left(\frac{\sqrt{5}+1}{2}\right)^2$$

$$\Rightarrow\ e=\left(\frac{\sqrt{5}+1}{2}\right)$$

19. We have,

$$2x_1^2-3y_1^2-1=2-48-1=-47<0$$

Thus, the point (1, 4) lies outside of the hyperbola.

20. Since the point $(\lambda, -1)$ is an exterior point of the curve

$4x^2 - 3y^2 = 1$, so

$4\lambda^2 - 3 - 1 < 0$

$\Rightarrow\ 4\lambda^2 - 4 < 0$

$\Rightarrow\ \lambda^2 - 1 < 0$

$\Rightarrow\ (\lambda + 1)(\lambda - 1) < 0$

$\Rightarrow\ -1 < \lambda < 1$

$\Rightarrow\ \lambda \in (-1, 1)$

Thus, the length of the interval, where λ lies is 2.

Therefore, $m = 2$

Hence, the value of $m + 10 = 2 + 10 = 12$.

21. As we know that if the line $y = mx + c$ be a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, the co-ordinates of the point of contact is $\left(\pm\frac{a^2m}{c}, \pm\frac{b^2}{c}\right)$.

The equation of the given hyperbola is

$25x^2 - 9y^2 = 225$

$$\Rightarrow\ \frac{x^2}{9}-\frac{y^2}{25}=1 \quad \ldots\text{(i)}$$

Also, the given line is

$25x + 12y = 45$

$\Rightarrow\ 12y = -25x + 45$

$$\Rightarrow\ y=-\frac{25}{12}x+\frac{45}{12} \quad \ldots\text{(ii)}$$

Here, $a = 3$, $b = 5$ and $m = -25/12$.

Thus, the common point is

$$\left(\pm\frac{a^2m}{c}, \pm\frac{b^2}{c}\right)=\left(\mp 5, \pm\frac{20}{3}\right).$$

22. As we know that, the line $y = mx + c$ will be a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, if

$$c^2 = a^2m^2 - b^2 \quad \ldots\text{(i)}$$

The equation of the hyperbola is

$9x^2 - 5y^2 = 45$

$$\Rightarrow\ \frac{x^2}{5}-\frac{y^2}{9}=1$$

Here, $a^2 = 5$, $b^2 = 9$, $m = 3$, $c = \lambda$

Therefore, from Eq. (i), we get

$\lambda^2 = a^2m^2 - b^2 = 15 - 9 = 6$

$\Rightarrow\ \lambda=\pm\sqrt{6}$

23. The equation of any tangent to the hyperbola

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$y=mx+\sqrt{a^2m^2-b^2} \quad \ldots\text{(i)}$$

The equation of the given tangent is

$$y = mx + \sqrt{9m^2 - 4} \quad \ldots\text{(ii)}$$

Since Eqs (i) and (ii) are identical, so

$a^2 = 9$, $b^2 = 4$

Hence, the equation of the hyperbola is

$$\frac{x^2}{9} - \frac{y^2}{4} = 1$$

$\Rightarrow \quad 4x^2 - 9y^2 = 36$

24. The equation of any tangent to the parallel to $5x - 4y + 7 = 0$ is

$5x - 4y + \lambda = 0$

$\Rightarrow \quad 4y = 5x + \lambda$

$$\Rightarrow \quad y = \frac{5}{4}x + \frac{\lambda}{4} \quad \ldots\text{(i)}$$

The equation of the given hyperbola is

$4x^2 - 9y^2 = 36$

$$\Rightarrow \quad \frac{x^2}{9} - \frac{y^2}{4} = 1 \quad \ldots\text{(ii)}$$

Since, the line (i) is a tangent to the hyperbola (ii), so

$$\frac{\lambda^2}{16} = 9\left(\frac{25}{16}\right) - 4$$

$$\Rightarrow \quad \frac{\lambda^2}{16} = \frac{225 - 64}{16}$$

$\Rightarrow \quad \lambda^2 = 161$

$\Rightarrow \quad \lambda = \pm\sqrt{161}$

Hence, the equations of the tangents are

$$5x - 4y \pm \sqrt{161} = 0$$

25. The equation of any line perpendicular to

$3x + 4y + 10 = 0$ is

$4x - 3y + \lambda = 0$

$$\Rightarrow \quad y = \left(\frac{4}{3}\right)x + \frac{\lambda}{3} \quad \ldots\text{(i)}$$

The equation of the given hyperbola is

$9x^2 - 16y^2 = 144$

$$\Rightarrow \quad \frac{x^2}{16} - \frac{y^2}{9} = 1 \quad \ldots\text{(ii)}$$

The line (i) will be a tangent to the hyperbola (ii), if

$$\left(\frac{\lambda}{3}\right)^2 = 16\left(\frac{4}{3}\right)^2 - 9$$

$\Rightarrow \quad \lambda^2 = 256 - 81 = 175$

$\Rightarrow \quad \lambda = \pm\sqrt{175} = \pm 5\sqrt{7}$

Hence, the equation of the tangents are $4x - 37 \pm 5\sqrt{7} = 0$.

26. The given hyperbola is

$x^2 - 9y^2 = 9$

$$\Rightarrow \quad \frac{x^2}{9} - \frac{y^2}{1} = 1 \quad \ldots\text{(i)}$$

The given line is $5x + 12y - 9 = 0$

$$\Rightarrow \quad y = \left(-\frac{5}{12}\right)x + \frac{3}{4} \quad \ldots\text{(ii)}$$

If the line (ii) be a tangent to the hyperbola (i), the co-ordinates of the point of contact can be

$$\left(\pm\frac{a^2m}{c}, \pm\frac{b^2}{c}\right) = \left(\pm(-5), \pm\frac{4}{3}\right)$$

27. The equation of the given hyperbola is

$4x^2 - 9y^2 = 36$

$$\Rightarrow \quad \frac{x^2}{9} - \frac{y^2}{4} = 1$$

Here, $a^2 = 9$, $b^2 = 4$.

The equation of any tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is}$$

$$y = mx + \sqrt{a^2m^2 - b^2}$$

$$\Rightarrow \quad y = mx + \sqrt{9m^2 - 4}$$

which is passing through (3, 2). So

$(2 - 3m)^2 = 9m^2 - 4$

$\Rightarrow \quad 4 - 12m + 9m^2 = 9m^2 - 4$

$\Rightarrow \quad 4 - 12m = -4$

$\Rightarrow \quad m = \frac{2}{3}, m = \infty$

Hence, the equations of the tangents are

$$x + 3 = 0 \text{ and } y = \frac{2}{3}x.$$

28. We have,

$$2x_1^2 - 3y_1^2 - 12 = 2.1 - 3.4 - 12$$

$$= 2 - 12 - 12 = 2 - 24 = -22 < 0$$

So, the point (1, –2) lies outside of the hyperbola.

Thus, the number of tangents is 2.

29. The equation of the given hyperbola is

$3x^2 - 4y^2 = 12$

$$\Rightarrow \quad \frac{x^2}{4} - \frac{y^2}{3} = 1 \quad \ldots\text{(i)}$$

Here, $a^2 = 4$, $b^2 = 3$ and $m = 4$

The equation of any tangent to the hyperbola (i) is

$$y = mx + \sqrt{a^2m^2 - b^2}$$

$$\Rightarrow \quad y = 4x \pm \sqrt{64 - 3}$$

$$\Rightarrow \quad y = 4x \pm \sqrt{61}$$

Hence, the equations of tangents are $y = 4x + \sqrt{61}$ and $y = 4x - \sqrt{61}$.

30. The equation of tangent to the curve

$x^2 - y^2 - 8x + 2y + 11 = 0$ is

$xx_1 - yy_1 - 4(x - x_1) + (y - y_1) + 1 = 0$

$\Rightarrow \quad 2x - y - 4(x + 2) + (y + 1) + 11 = 0$

$\Rightarrow \quad -2x + 4 = 0$

$\Rightarrow \quad x - 2 = 0$

31. When $y = 2$, then $4x^2 = 36$

$\Rightarrow \quad x = \pm 3$

Hence, the points are (3, 2) and (–3, 2).

The equation of the tangent at (3, 2) is

$$12x - 6y = 24$$

$\Rightarrow \quad 2x - y = 4$

Also, the equation of the tangent at (–3, 2) is

$$-12x - 6y = 24$$

$\Rightarrow \quad 2x + y + 4 = 0$

32. The equation of the given hyperbola is

$$9x^2 - 16y^2 = 144$$

$\Rightarrow \quad \dfrac{x^2}{16} - \dfrac{y^2}{9} = 1 \quad \ldots(i)$

The equation of any tangent to the hyperbola (i) can be considered as $y = mx + \sqrt{16m^2 - 9}$

which is passing through (4, 3). So

$$(3 - 4m)^2 = 16m^2 - 9.$$

$\Rightarrow \quad 9 - 24m + 16m^2 = 16m^2 - 9$

$\Rightarrow \quad 24m = 18$

$\Rightarrow \quad m = \dfrac{3}{4}$ and $m = \infty$

Let θ be the angle between them. Then

$$\tan(\theta) = \left|\frac{\frac{3}{4} - \infty}{1 + \frac{3}{4}.\infty}\right| = \left|\frac{\frac{3}{4\infty} - 1}{\frac{1}{\infty} + \frac{3}{4}}\right| = \frac{4}{3}$$

$\Rightarrow \quad \theta = \tan^{-1}\left(\dfrac{4}{3}\right)$

Hence, the angle between the tangents is

$$\tan^{-1}\left(\frac{4}{3}\right).$$

33. The equation of any tangent to the hyperbola

$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is

$y = m_1 x + \sqrt{a^2 m_1^2 - b^2} \quad \ldots(i)$

The equation of any tangent to the hyperbola

$\dfrac{x^2}{(-b^2)} - \dfrac{y^2}{(-a^2)} = 1$ is

$y = m_2 x + \sqrt{(-b^2)m_2^2 - (-a^2)} \quad \ldots(ii)$

Since the equations (i) and (ii) are identical, so

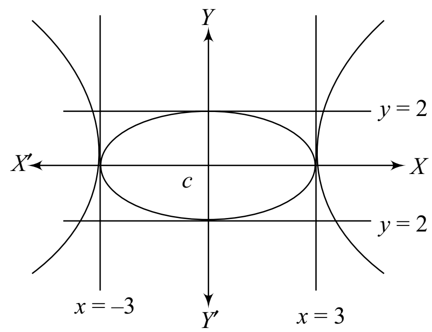
$$a^2 m_1^2 - b^2 = (-b^2)m_2^2 - (-a^2)$$

$\Rightarrow \quad m_1^2 = 1$ and $m_2^2 = 1$

Thus, $m_1 = \pm 1 = m_2$

Hence, the equations of the common tan gents are $y = \pm x + \sqrt{a^2 - b^2}$

34. The equation of the given curves are $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$ and $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$.

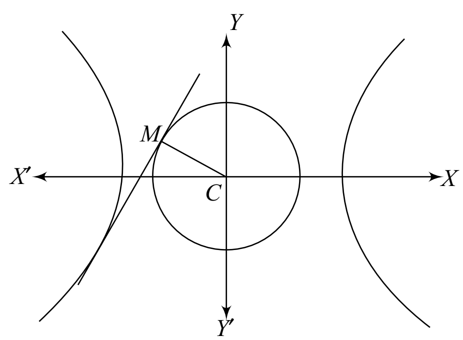


Here, the length of the major axis of the ellipse is equal to the length of the transverse axis and also the length of the minor axis is equal to the length of the conjugate axis. Thus, the equations of the common tangents are $x = \pm 3$ and $y = \pm 2$.

35. The equation of any tangent to the hyperbola $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$ is

$$y = mx + \sqrt{16m^2 - 9}$$

$\Rightarrow \quad m\,x - y + \sqrt{16m^2 - 9} = 0 \quad \ldots(i)$



If the tangent (i) is also the tangent to the circle $x^2 + y^2 = 9$, the length of the perpendicular from the centre of the circle is equal to the radius of the circle. So

$$\left|\frac{0 - \sqrt{16m^2 - 9}}{\sqrt{m^2 + 1}}\right| = 3$$

$\Rightarrow \quad 16m^2 - 9 = 9(m^2 + 1)$

$\Rightarrow \quad 16m^2 - 9m^2 = 9 + 1 = 10$

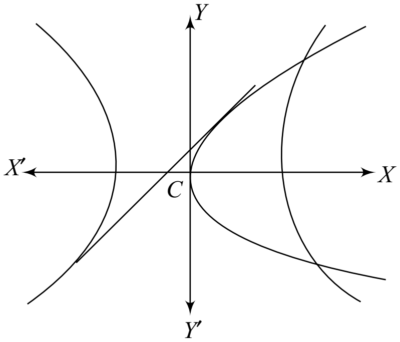
$\Rightarrow \quad 7m^2 = 10$

$\Rightarrow \quad m = \pm\sqrt{\dfrac{7}{10}}$

Hence, the equation of tangents are $y = \pm\sqrt{\dfrac{7}{10}}\,x + \sqrt{\dfrac{11}{5}}$.

36. The equation of any tangent to the parabola $y^2 = 8x$ is

$y = mx + \dfrac{2}{m} \quad \ldots(i)$



Since the tangent to the parabola is also a tangent to the hyperbola $\frac{x^2}{9}-\frac{y^2}{5}=1$, so

$$5x^2-9\left(mx+\frac{2}{m}\right)^2=45$$

$$\Rightarrow \quad 5x^2-9\left(m^2x^2+\frac{4}{m^2}+4m\right)=45$$

$$\Rightarrow \quad (5-9m^2)x^2-36x-9\left(\frac{4}{m^2}+5\right)=0$$

Now,

$$D=0$$

$$\Rightarrow \quad (36)^2+36(5-9m^2)\left(\frac{4}{m^2}+5\right)=0$$

$$\Rightarrow \quad 36+\frac{20}{m^2}+25-36-45m^2=0$$

$$\Rightarrow \quad \frac{4}{m^2}+5-9m^2=0$$

$$\Rightarrow \quad 9m^4-5m^2-4=0$$

$$\Rightarrow \quad (m^2-1)(9m^2+4)=0$$

$$\Rightarrow \quad (m^2-1)=0$$

$$\Rightarrow \quad m=\pm 1$$

Hence, the equations of the common tangents are

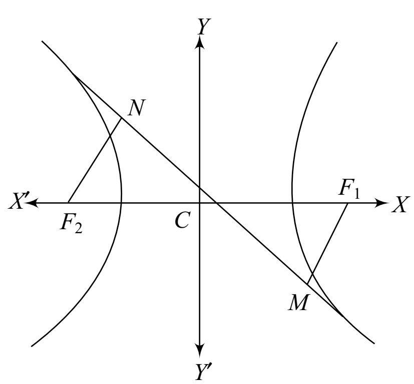
$$y=\pm x\pm 2.$$

37. As we know that the locus of the perpendicular tangents is the director circle.
Hence, the equation of the director circle is

$$x^2+y^2=a^2-b^2=16-9=7$$

$$\Rightarrow \quad x^2+y^2=7$$

38. Let F_1 and F_2 be two foci of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.



Then $F_1=(ae, 0)$ and $F_2=(-ae, 0)$.
The equation of any tangent to the hyperbola is

$$\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1 \qquad \ldots\text{(i)}$$

Let p_1 and p_2 be two perpendiculars from foci upon the tangent (i).

$$\Rightarrow \quad p_1p_2=\frac{a^2(e^2-1)(e^2\sec^2\theta-1)}{((e^2-1)\sec^2\theta+\tan^2\theta)}$$

$$\Rightarrow \quad p_1p_2=\frac{a^2(e^2-1)(e^2\sec^2\theta-1)}{((e^2-1)\sec^2\theta+(\sec^2\theta-1))}$$

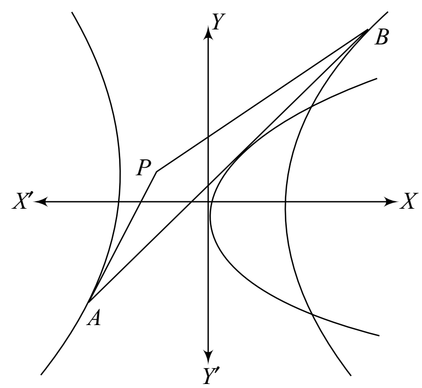
$$\Rightarrow \quad p_1p_2=\frac{a^2(e^2-1)(e^2\sec^2\theta-1)}{(e^2\sec^2\theta-1)}$$

$$\Rightarrow \quad p_1p_2=a^2(e^2-1)$$

$$\Rightarrow \quad p_1p_2=b^2$$

Hence, the product of the lengths of the perpendiculars is b^2.

39. Let $P(\alpha, \beta)$ be the point of intersection of tangents at A and B.



Clearly, the point of intersection of the tangents at A and B is the chord of contact.
Therefore, the chord of contact AB is

$$\frac{\alpha x}{a^2}-\frac{\beta y}{b^2}=1$$

$$\Rightarrow \quad y=\frac{b^2x\alpha}{a^2y\beta}-\frac{b^2}{\beta}$$

which is a tangent to the parabola $y^2=4ax$. So

$$-\frac{b^2}{\beta}=\frac{a}{\left(\frac{b^2\alpha}{a^2\beta}\right)}$$

$$\Rightarrow \quad \beta^2=\left(-\frac{b^4}{a^3}\alpha\right)$$

Hence, the locus of $P(\alpha, \beta)$ is

$$y^2=\left(-\frac{b^4}{a^3}x\right)$$

Then, $p_1 = \left| \dfrac{e\sec\theta - 1}{\sqrt{\dfrac{\sec^2\theta}{a^2} + \dfrac{\tan^2\theta}{b^2}}} \right|$

and $p_2 = \left| \dfrac{e\sec\theta + 1}{\sqrt{\dfrac{\sec^2\theta}{a^2} + \dfrac{\tan^2\theta}{b^2}}} \right|$

$$\Rightarrow \quad p_1 p_2 = \frac{(e\sec\theta - 1)(e\sec\theta + 1)}{\dfrac{\sec^2\theta}{a^2} + \dfrac{\tan^2\theta}{b^2}}$$

$$= \frac{a^2 b^2 (e^2 \sec^2\theta - 1)}{(b^2 \sec^2\theta + a^2 \tan^2\theta)}$$

$$= \frac{a^4 (e^2 - 1)(e^2 \sec^2\theta - 1)}{(a^2(e^2 - 1)\sec^2\theta + a^2 \tan^2\theta)}$$

40. The equation of the normal to the curve $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$ at $(8, 3\sqrt{3})$ is

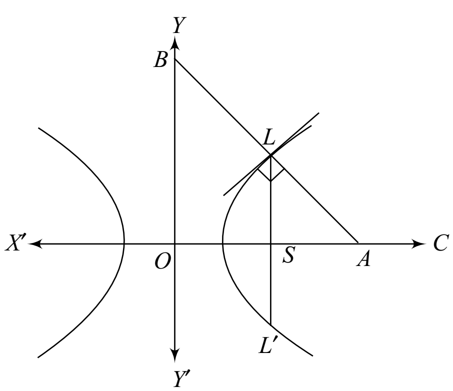
$$\frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 + b^2$$

$$\Rightarrow \quad \frac{16x}{8} + \frac{9y}{3\sqrt{3}} = 16 + 9$$

$$\Rightarrow \quad 2x + \sqrt{3}\, y = 25$$

Hence, the required equation of the normal is $2x + \sqrt{3}\, y = 25$.

41. Let, one end of the latus rectum of the given hyperbola is $L\left(ae, \dfrac{b^2}{a}\right)$.



The equation of the normal to the given hyperbola at L is

$$\frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 + b^2$$

$$\Rightarrow \quad \frac{ax}{e} + ay = a^2 + b^2$$

$$\Rightarrow \quad \frac{x}{\left(\dfrac{e(a^2 + b^2)}{a}\right)} + \frac{y}{\left(\dfrac{a^2 + b^2}{a}\right)} = 1$$

Hence, the area of ΔOAB

$$= \frac{1}{2} \times \frac{e(a^2 + b^2)}{a} \times \left(\frac{a^2 + b^2}{a}\right)$$

$$= \frac{1}{2} \times \frac{e(a^2 + b^2)^2}{a^2}$$

$$= \frac{1}{2} \times a^2 e^5$$

42. The equation of any normal at $(a \sec \varphi, b \tan \varphi)$ to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is

$$ax \cos + by \cot \varphi = a^2 + b^2 \qquad \ldots\text{(i)}$$

The equation of any line perpendicular to (i) and passing through the origin is

$$(b \cot \varphi)x - (a \cos \varphi)y = 0$$

$$\Rightarrow \quad bx - a \sin \varphi y = 0 \qquad \ldots\text{(ii)}$$

If we eliminate φ between Eqs (i) and (ii), we get the required locus of the foot of the perpendicular.
From (ii), we get,

$$\sin(\varphi) = \frac{bx}{ay}$$

$$\Rightarrow \quad \cos(\varphi) = \frac{\sqrt{(a^2 y^2 - b^2 x^2)}}{ay}$$

and $\cot(\varphi) = \dfrac{\sqrt{(a^2 y^2 - b^2 x^2)}}{bx}$

From Eq. (i), we get

$$ax \times \frac{\sqrt{(a^2 y^2 - b^2 x^2)}}{ay} + by \times \frac{\sqrt{(a^2 y^2 - b^2 x^2)}}{bx}$$
$$= a^2 + b^2$$

$$\Rightarrow \quad (x^2 + y^2)\left(\sqrt{(a^2 y^2 - b^2 x^2)}\right) = (a^2 + b^2)xy$$

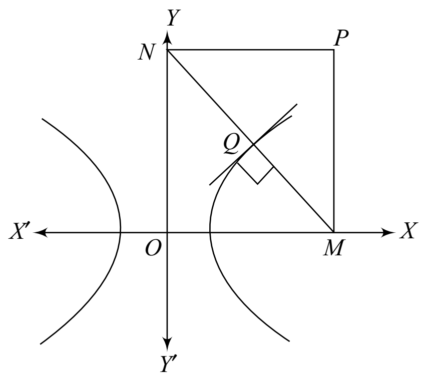
$$\Rightarrow \quad (x^2 + y^2)^2 (a^2 y^2 - b^2 x^2) = (a^2 + b^2)^2 x^2 y^2$$

which is the required locus of the foot of the perpendicular.

43. The equation of any normal to the hyperbola

$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is

$$ax \cos\varphi + by \cot\varphi = (a^2 + b^2) \qquad \ldots\text{(i)}$$



Since the normal (i) meets the x-axis at M and y-axis at N respectively. Then,

$$M=\left(\frac{a^2+b^2}{a\cos\varphi},0\right)\text{ and }N=\left(0,\left(\frac{a^2+b^2}{b}\right)\tan\varphi\right)$$

Let the co-ordinates of the point P be (α, β).
Since PM and PN are perpendiculars to the axes, so the co-ordinates of P are

$$\left(\left(\frac{a^2+b^2}{a}\right)\sec\varphi,\left(\frac{a^2+b^2}{b}\right)\tan\varphi\right)$$

Therefore,

$$\alpha=\left(\frac{a^2+b^2}{a}\right)\sec\varphi\text{ and }\beta=\left(\frac{a^2+b^2}{b}\right)\tan\varphi$$

$$\Rightarrow\quad \alpha\left(\frac{a}{a^2+b^2}\right)=\sec\varphi\text{ and }\beta\left(\frac{b}{a^2+b^2}\right)=\tan\varphi$$

As we know that,

$$\sec^2\varphi-\tan^2\varphi=1$$

$$\alpha^2\left(\frac{a}{a^2+b^2}\right)^2-\beta^2\left(\frac{b}{a^2+b^2}\right)^2=1$$

$$\Rightarrow\quad \alpha^2a^2-\beta^2b^2=(a^2+b^2)^2$$

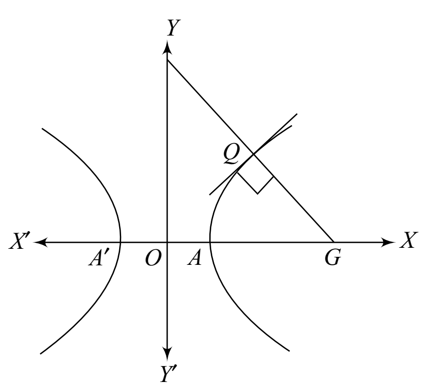
Hence, the locus of (α, β) is

$$a^2x^2-b^2y^2=(a^2+b^2)^2$$

44. The equation of any normal to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1\text{ at }(\phi)\text{ is}$$

$$ax\cos\varphi+by\cot\varphi=a^2+b^2$$



Thus, the co-ordinates of $G=\left(\dfrac{a^2+b^2}{a\cos\varphi},0\right)$

Clearly, the vertices, $A=(a, 0)$ and $A'=(-a, 0)$

Now, $AG=\left(\dfrac{a^2+b^2}{a\cos\varphi}-a\right)$

and $\quad A'G=\left(\dfrac{a^2+b^2}{a\cos\varphi}+a\right)$

Therefore,

$$AG.\,A'G=\left(\frac{a^2+b^2}{a\cos\varphi}-a\right)\left(\frac{a^2+b^2}{a\cos\varphi}+a\right)$$

$$=\left(\left(\frac{a^2+b^2}{a}\right)^2\sec^2\varphi-a^2\right)$$

$$=(a^2e^2\sec^2\varphi-a^2)$$

$$=a^2(e^2\sec^2\varphi-1)$$

$\Rightarrow\quad m=2, n=2, p=2$

Hence,

$$(m+n+p)^2+36=36+36=72.$$

45. The equation of the normal to the hyperbola $xy=c^2$ at $\left(ct,\dfrac{c}{t}\right)$ is

$$xt^3-yt-ct^4+c=0$$

$$\Rightarrow\quad ct^4-xt^3+yt-c=0$$

which is passing through (α, β), so

$$ct^4-\alpha t^3+\beta t-c=0.$$

Let its four roots are t_1, t_2, t_3, t_4.

Therefore, $t_1+t_2+t_3+t_4=\dfrac{\alpha}{c}$,

$$\Sigma(t_1t_2)=0,\ \Sigma(t_1t_2t_3)=-\frac{\beta}{c}$$

and $\quad\Sigma(t_1t_2t_3t_4)=1.$

(i) $x_1+x_2+x_3+x_4=c(t_1+t_2+t_3+t_4)$

$$=c\left(\frac{\alpha}{c}\right)=\alpha$$

(ii) $y_1+y_2+y_3+y_4=c\left(\dfrac{1}{t_1}+\dfrac{1}{t_2}+\dfrac{1}{t_3}+\dfrac{1}{t_4}\right)$

$$=c\left(\frac{\sum(t_1t_2t_3)}{\sum(t_1t_2t_3t_4)}\right)$$

$$=c\left(\frac{\left(-\dfrac{\beta}{c}\right)}{-1}\right)=\beta$$

(iii) $x_1^2 + x_2^2 + x_3^2 + x_4^2 = c^2(t_1^2 + t_2^2 + t_3^2 + t_4^2)$

$$= c^2\left\{\left(\sum t_1\right)^2 - 2\sum(t_1t_2)\right\}$$

$$= c^2\left\{\left(\frac{\alpha}{c}\right)^2 - 0\right\} = \alpha^2$$

(iv) $y_1^2 + y_2^2 + y_3^2 + y_4^2 = \left(\sum y_1\right)^2 - 2\sum(y_1y_2)$

$$= (\beta)^2 - 2c^2\sum\left(\frac{1}{t_1t_2}\right)$$

$$= (\beta)^2 - 2c^2\left(\frac{\sum t_1t_2}{t_1t_2t_3t_4}\right)$$

$$= (\beta)^2$$

(v) $x_1 \cdot x_2 \cdot x_3 \cdot x_4 = c^4(t_1t_2t_3t_4) = -c^4$

(vi) $y_1 \cdot y_2 \cdot y_3 \cdot y_4 = c^4\left(\frac{1}{t_1t_2t_3t_4}\right)$

$$= c^4\left(\frac{1}{-1}\right) = -c^4$$

46. the equation of any normal to the given hyperbola at (x, y) is

$$\frac{x}{a^2}(k - y) = \frac{y}{b^2}(x - b)$$

$$\Rightarrow \left(\frac{1}{a^2} + \frac{1}{b^2}\right)xy - \frac{y}{b} - \frac{kx}{a^2} = 0$$

$$\Rightarrow y = \frac{b^2kx}{a^2(e^2x - b)} \quad \ldots\text{(i)}$$

The equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \ldots\text{(ii)}$$

Solving Eqs (i) and (ii), we get,

$$\frac{x^2}{a^2} - \frac{b^4k^2x^2}{b^2a^4(e^2x - b)^2} = 1$$

$$\Rightarrow a^2e^4x^4 - 2ba^2e^2x^3 - (a^2b^2 + b^2k^2 + a^4e^4)x^2 + 2ba^4e^2x + a^4b^2 - 0 \quad \ldots\text{(iii)}$$

Let x_1, x_2, x_3, x_4 are the roots of Eq. (iii).

Then, $x_1 + x_2 + x_3 + x_4 = \frac{2b}{e^2}$,

$$\sum(x_1x_2) = -\frac{a^2b^2 + b^2k^2 + a^4e^2}{a^2e^4}$$

$$\sum(x_1x_2x_3) = -\frac{2ba^2}{e^2}$$

and $\sum(x_1x_2x_3x_4) = \frac{a^2b^2}{e^4}$

Therefore, $\sum\left(\frac{1}{x_1}\right) = \frac{\sum(x_1x_2x_3)}{x_1x_2x_3x_4} = \frac{2e^2}{b}$

(i) $(x_1 + x_2 + x_3 + x_4)\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}\right)$

$$= \left(\sum x_1\right)\left(\sum\left(\frac{1}{x_1}\right)\right) = \frac{2b}{e^2} \times \frac{2e^2}{b} = 4$$

(ii) Similarly we can have

$$(y_1 + y_2 + y_3 + y_4)\left(\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} + \frac{1}{y_4}\right)$$

$$= \left(\sum y_1\right)\left(\sum\left(\frac{1}{y_1}\right)\right)$$

$$= 4.$$

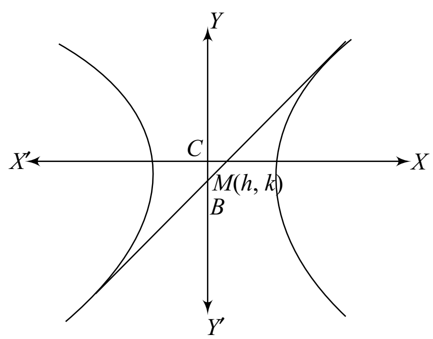
47. The equation of the chord of contact of tangents drawn from the point (2, 3) is

$$\frac{2x}{9} - \frac{y}{2} = 1$$

$$\Rightarrow 4x - 9y = 18$$

48. Any tangent to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ is

$$\frac{x}{3}\sec\theta - \frac{y}{2}\tan\theta = 1 \quad \ldots\text{(i)}$$



Let the tangent intersects the x-axis at A and y-axis at B, respectively.

Then $A = (3\cos\theta, 0)$ and $B = (0, -2\cot\theta)$

Let (h, k) be the mid-point of AB.

Therefore,

$$2h = 3\cos\theta \text{ and } 2k = -2\cot\theta$$

$$\Rightarrow \sec\theta = \frac{3}{2h} \text{ and } \tan\theta = -\frac{1}{k}$$

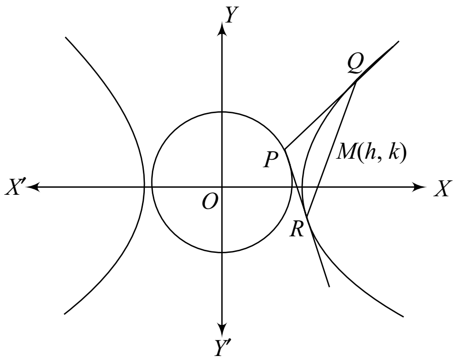
As we know that,

$$\sec^2\theta - \tan^2\theta = 1$$

$$\Rightarrow \frac{9}{4h^2} - \frac{1}{k^2} = 1$$

Hence, the locus of (h, k) is $\frac{9}{4x^2} - \frac{1}{y^2} = 1$.

49. Any point on the circle $x^2 + y^2 = a^2$ be $(a \cos \theta, a \sin \theta)$.



The equation of the chord of contact from the point $(a \cos \theta, a \sin \theta)$ to the hyperbola $x^2 - y^2 = a^2$ is

$(a \cos \theta)x - (a \sin \theta)y = a^2$

$\Rightarrow \quad x \cos \theta - y \sin \theta = a$ ...(i)

Let the mid-point be (h, k).

The equation of the chord bisected at (h, k) to the hyperbola $x^2 - y^2 = a^2$ is

$hx - ky = h^2 - k^2$ ...(ii)

Since the Eqs (i) and (ii) are identical, so

$$\frac{h}{\cos\theta} = \frac{-k}{\sin\theta} = \frac{h^2 - k^2}{a}$$

$$\Rightarrow \quad \cos\theta = \frac{ah}{h^2 - k^2} \text{ and } \sin\theta = \frac{-ak}{h^2 - k^2}$$

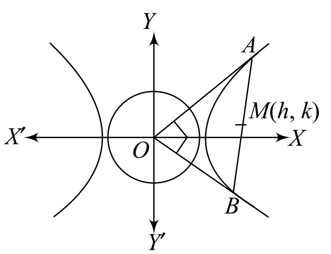
Squaring and adding, we get

$$\left(\frac{ah}{h^2 - k^2}\right)^2 + \left(\frac{-ak}{h^2 - k^2}\right)^2 = 1$$

$\Rightarrow \quad a^2(h^2 + k^2) = (h^2 - k^2)^2$

Hence, the locus of (h, k) is $a^2(x^2 + y^2) = (x^2 - y^2)^2$.

50. Let the mid-point be (h, k).



The equation of the chord bisected at (h, k) to the given hyperbola is

$T = S_1$

$$\Rightarrow \quad \frac{hx}{a^2} - \frac{ky}{b^2} = \frac{h^2}{a^2} - \frac{k^2}{b^2} \quad \text{...(i)}$$

The equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{...(ii)}$$

Since the chord (i) subtends right angle at the centre, so we can write

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{\left(\frac{hx}{a^2} - \frac{ky}{b^2}\right)^2}{\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2}$$

$$\Rightarrow \quad \frac{1}{a^2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2 x^2 - \frac{1}{b^2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2 y^2$$

$$= \frac{h^2}{a^4}x^2 + \frac{k^2}{b^4}y^2 - \frac{2hk}{a^2b^2}xy \quad \text{...(iii)}$$

Equation (iii) will be a right angle, if co-efficient of x^2 + co-efficient of $y^2 = 0$

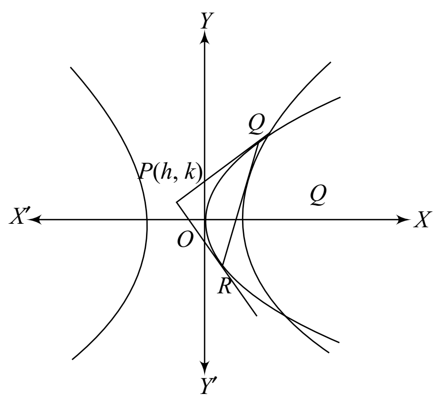
$$\Rightarrow \quad \frac{1}{a^2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2 - \frac{h^2}{a^4} - \frac{1}{b^2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2 - \frac{k^2}{b^4} = 0$$

$$\Rightarrow \quad \left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2 \left(\frac{1}{a^2} - \frac{1}{b^2}\right) = \left(\frac{h^2}{a^4} + \frac{k^2}{b^4}\right)$$

Hence, the locus of (h, k) is

$$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2 \left(\frac{1}{a^2} - \frac{1}{b^2}\right) = \left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right).$$

51. Let the point on the parabola be (h, k).



The equation of the chord of contact of the parabola is

$yk = 2a(x + h)$ ...(i)

$$\Rightarrow \quad y = \frac{2a}{k}x + \frac{2ah}{k} \quad \text{...(ii)}$$

Since, the line (ii) is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, so,

$c^2 = a^2m^2 - b^2$

$$\Rightarrow \quad \left(\frac{2ah}{k}\right)^2 = a^2\left(\frac{2a}{k}\right)^2 - b^2$$

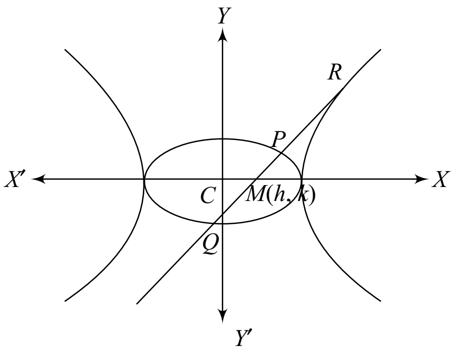
$\Rightarrow \quad 4a^2h^2 = 4a^4 - k^2b^2$

Hence, the locus of (h, k) is

$4a^2x^2 = 4a^4 - y^2b^2$

52. Any tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at $R(\theta)$ is

$$\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1 \quad \ldots(i)$$



Let the mid-point of PQ be (h, k).
Then the equation of the chord bisected at (h, k) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$\frac{hx}{a^2}+\frac{ky}{b^2}=\frac{h^2}{a^2}+\frac{k^2}{b^2} \quad \ldots(ii)$$

Therefore, the Eqs (i) and (ii) are identical. So

$$\frac{\sec\theta/a}{\frac{h}{a^2}}=\frac{-\tan\theta/b}{\frac{k}{b^2}}=\frac{1}{\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)}$$

$$\Rightarrow \quad \frac{\sec\theta}{\frac{h}{a}}=\frac{-\tan\theta}{\frac{k}{b}}=\frac{1}{\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)}$$

$$\Rightarrow \quad \sec\theta=\frac{\frac{h}{a}}{\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)} \text{ and } \tan\theta=\frac{\frac{k}{b}}{\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)}$$

We know that,

$$\sec^2\theta-\tan^2\theta=1$$

$$\Rightarrow \quad \frac{\left(\frac{h}{a}\right)^2}{\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)^2}-\frac{\left(\frac{k}{b}\right)^2}{\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)^2}=1$$

$$\Rightarrow \quad \left(\frac{h^2}{a^2}-\frac{k^2}{b^2}\right)=\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)^2$$

Hence, the locus of (h, k) is $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2$.

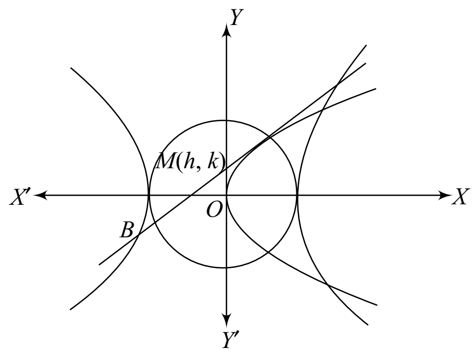
53. If (h, k) be the mid-point of the chord of the hyperbola $x^2-y^2=a^2$, then

$$T=S_1$$

$$\Rightarrow \quad hx-ky=h^2-k^2$$

$$\Rightarrow \quad ky=hx+(k^2-h^2)$$

$$\Rightarrow \quad y=\left(\frac{h}{k}\right)x+\left(\frac{k^2-h^2}{k}\right) \quad \ldots(i)$$



If the line (i) touches the parabola $y^2 = 4ax$, so

$$c=\frac{a}{m}$$

$$\Rightarrow \quad \left(\frac{h^2-k^2}{k}\right)=\frac{a}{(h/k)}=\frac{ak}{h}$$

$$\Rightarrow \quad h(h^2-k^2)=ak^2$$

Hence, the locus of (h, k) is

$$x(x^2-y^2)=ay^2$$

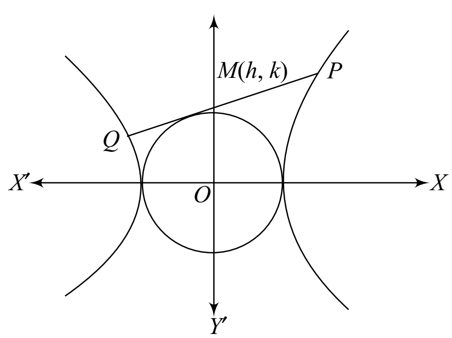
54. If (h, k) be the mid-point of the chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then

$$T=S_1$$

$$\Rightarrow \quad \frac{hx}{a^2}-\frac{ky}{b^2}=\frac{h^2}{a^2}-\frac{k^2}{b^2}$$

$$\Rightarrow \quad \frac{ky}{b^2}=\frac{hx}{a^2}-\left(\frac{h^2}{a^2}-\frac{k^2}{b^2}\right)$$

$$\Rightarrow \quad y=\left(\frac{b^2h}{a^2k}\right)x-\frac{b^2}{k}\left(\frac{h^2}{a^2}-\frac{k^2}{b^2}\right) \quad \ldots(i)$$



If the line (i) be a tangent to the circle $x^2+y^2=c^2$, then

$$C^2=A^2(1+m^2)$$

$$\Rightarrow \quad \frac{b^4}{k^2}\left(\frac{h^2}{a^2}-\frac{k^2}{b^2}\right)^2 = c^2\left(1+\frac{b^4h^2}{a^4k^2}\right)$$

$\Rightarrow \quad (b^2h^2 - a^2k^2)^2 = c^2(a^4k^2 + b^2h^2)$

Hence, the locus of (h, k) is

$(b^2x^2 - a^2y^2)^2 = c^2(a^4y^2 + b^4x^2)$

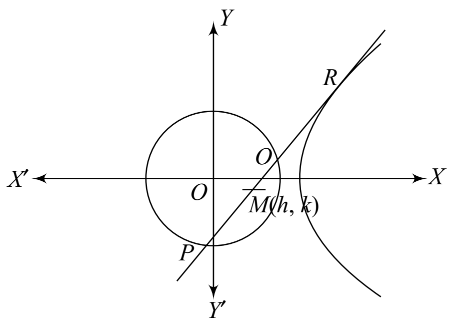
$$\Rightarrow \quad \left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^2 = c^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)$$

55. If (h, k) be the mid-point of the chord of the circle $x^2 + y^2 = a^2$, then

$T = S_1$

$\Rightarrow \quad hx + ky = h^3 + k^2$

$$\Rightarrow \quad y = \left(-\frac{h}{k}\right)x + \left(\frac{h^2+k^2}{k}\right) \qquad \text{(i)}$$



If the line (i) be a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$,

so, $\quad c^2 = a^2m^2 - b^2$

$$\Rightarrow \quad \left(\frac{h^2+k^2}{k}\right)^2 = a^2\left(\frac{h^2}{k^2}\right) - b^2$$

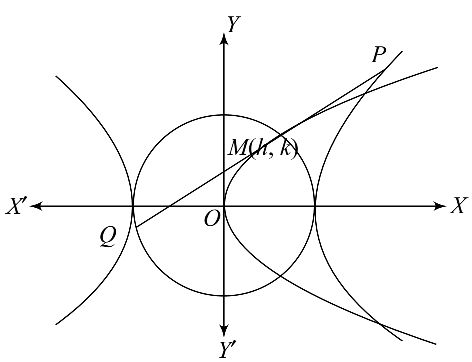
$\Rightarrow \quad (h^2 + k^2)^2 = (a^2h^2 - b^2k^2)$

Hence, the locus of (h, k) is

$(x^2 + y^2)^2 = (a^2x^2 - b^2y^2)$

56. Any tangent to the parabola $y^2 = 4ax$ at $(at^2, 2at)$ is

$$yt = x + at^2 \qquad \text{…(i)}$$



If (h, k) be the mid-point of the chord of the hyperbola $xy = c^2$, then

$$xk + yh = c^2 \qquad \text{…(ii)}$$

Therefore, the Eqs (i) and (ii) are identical.

Thus, $\frac{k}{1} = \frac{-h}{t} = \frac{-c^2}{at^2}$

$$\Rightarrow \quad t = \frac{-h}{k}, t^2 = \frac{-c^2}{ak}$$

Eliminating t, we get,

$h^2a = -kc^2$

Hence the locus of (h, k) is

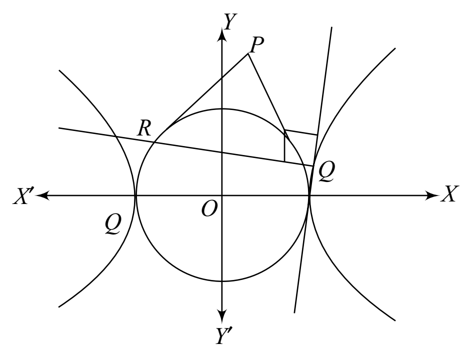
$yc^2 + x^2a - 0$

$$\Rightarrow \quad y = \left(-\frac{a}{c^2}\right)x^2.$$

57. Let the point P be (h, k).

Then the equation of the chord of contact of the circle $x^2 + y^2 = a^2$ is

$$hx + ky = a^2 \qquad \text{…(i)}$$



The equation of the normal chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at (ϕ) is

$$ax \cos \varphi - \text{by} \cot \varphi = a^2 + b^2 \qquad \text{…(ii)}$$

Equations (i) and (ii) are identical. Therefore,

$$\frac{a\cos\varphi}{h} = -\frac{b\cot\varphi}{k} = \frac{(a^2+b^2)}{a^2}$$

$$\sec\varphi = \frac{a^3}{h(a^2+b^2)} \text{ and } \tan\varphi = -\frac{a^2b}{k(a^2+b^2)}$$

We have,

$\sec^2\varphi - \tan^2\varphi = 1$

$$\Rightarrow \quad \left(\frac{a^3}{h(a^2+b^2)}\right)^2 - \left(-\frac{a^2b}{k(a^2+b^2)}\right)^2 = 1$$

$$\Rightarrow \quad \left(\frac{a^6}{h^2}-\frac{a^4b^2}{k^2}\right) = (a^2+b^2)^2$$

$$\Rightarrow \quad \left(\frac{a^2}{h^2}-\frac{b^2}{k^2}\right) = \left(\frac{a^2+b^2}{a^2}\right)^2$$

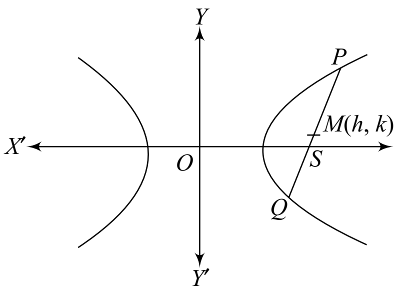
Hence, the locus of (h, k) is

$$\left(\frac{a^2}{x^2}-\frac{b^2}{y^2}\right) = \left(\frac{a^2+b^2}{a^2}\right)^2$$

58. If (h, k) be the mid-point of the chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then

$$T = S_1$$

$$\Rightarrow \quad \frac{hx}{x^2}-\frac{ky}{b^2}=\frac{h^2}{a^2}-\frac{k^2}{b^2} \qquad \ldots\text{(i)}$$



which is passing through the focus $(ae, 0)$ of the given hyperbola.

Therefore,

$$\frac{aeh}{a^2}=\frac{h^2}{a^2}-\frac{k^2}{b^2}$$

$$\Rightarrow \quad \frac{h^2}{a^2}-\frac{k^2}{b^2}=\frac{eh}{a}$$

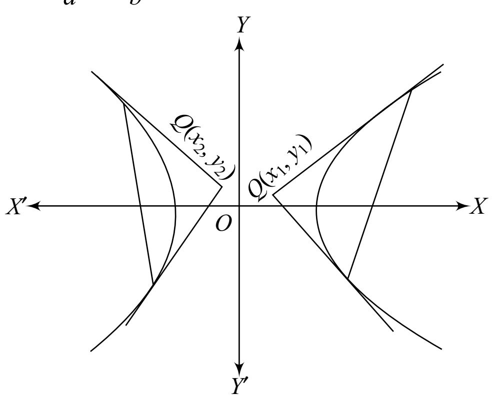
Hence, the locus of (h, k) is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\left(\frac{e}{a}\right)x$$

59. The equations of the chord of contact of the tangents to the given hyperbola at (x_1, y_1) and (x_2, y_2) are

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1 \qquad \ldots\text{(i)}$$

and $$\frac{xx_2}{a^2}-\frac{yy_2}{b^2}=1 \qquad \ldots\text{(ii)}$$



The slopes of the lines (i) and (ii) are

$$m_1=\left(\frac{b^2x_1}{a^2y_1}\right) \text{ and } m_2=\left(\frac{b^2x_1}{a^2y_1}\right)$$

Since, (i) and (ii) meet at right angles, so

$$m_1m_2=-1$$

$$\Rightarrow \quad \left(\frac{b^2x_1}{a^2y_1}\right)\times\left(\frac{b^2x_1}{a^2y_1}\right)=-1$$

$$\Rightarrow \quad \frac{x_1x_2}{y_1y_2}=-\frac{a^4}{b^4}$$

Thus, $m = 4$ and $n = 4$

Hence, the value of $\left(\frac{m+n}{4}\right)^{10}=\left(\frac{4+4}{4}\right)^{10}=2^{10}=1024.$

60. The equation of the polar with respect to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$$

$$\Rightarrow \quad \frac{x(-ae)}{a^2}-\frac{y\cdot 0}{b^2}=1$$

$$\Rightarrow \quad x=-\frac{a}{e}$$

61. The equation of the polars from points (x_1, y_1) and (x_2, y_2) to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{ are}$$

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1 \qquad \ldots\text{(i)}$$

and $$\frac{xx_2}{a^2}-\frac{yy_2}{b^2}=1 \qquad \ldots\text{(ii)}$$

Now, slopes of (i) and (ii) are

$$m_1=\frac{b^2x_1}{a^2y_1} \text{ and } m_2=\frac{b^2x_2}{a^2y_2}$$

Since the polars of the given points are perpendicular, so

$$m_1 \times m_2 = -1$$

$$\Rightarrow \quad \left(\frac{b^2x_1}{a^2y_1}\right)\times\left(\frac{b^2x_2}{a^2y_2}\right)=-1$$

$$\Rightarrow \quad \frac{x_1x_2}{y_1y_2}=-\frac{a^4}{b^4}$$

$$\Rightarrow \quad \frac{x_1x_2}{y_1y_2}+\frac{a^4}{b^4}=0$$

62. Let (x_1, y_1) be the pole of the hyperbola. Then the equation of the polar from a point (x_1, y_1) w.r.t. the hyperbola $x^2 - 3y^2 = 3$ is

$$xx_1 - 3yy_1 = 3 \qquad \ldots\text{(i)}$$

The equation of the given polar is

$$x - y = 3 \qquad \ldots\text{(ii)}$$

Therefore, the Eqs (i) and (ii) are identical. So

$$\frac{x_1}{1}=\frac{-3y_1}{-1}=\frac{3}{3}$$

$$\Rightarrow \quad x_1=1,\ y_1=\frac{1}{3}$$

Hence, the pole is $\left(1, \frac{1}{3}\right)$.

63. Let the pole be (h, k).
The equation of the polar from the point (h, k) w.r.t. the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$\frac{hx}{a^2} - \frac{ky}{b^2} = 1 \quad \ldots(i)$$

The equation of the normal chord of the given hyperbola at (ϕ) is

$$ax \cos \varphi - by \cot \varphi = (a^2 + b^2) \quad \ldots(ii)$$

Equations (i) and (ii) are identical. Therefore,

$$\frac{a\cos\varphi}{(h/a^2)} = \frac{b\cot\varphi}{(k/b^2)} = \frac{(a^2+b^2)}{1}$$

$$\Rightarrow \quad \cos\varphi = (a^2+b^2)\frac{h}{a^3}, \cot\varphi = (a^2+b^2)\frac{k}{b^3}$$

$$\Rightarrow \quad \sec\varphi = \frac{a^3}{(a^2+b^2)h}, \tan\varphi = \frac{b^3}{(a^2+b^2)k}$$

We know that,

$$\sec^2\varphi - \tan^2\varphi = 1$$

$$\Rightarrow \quad \left(\frac{a^3}{(a^2+b^2)h}\right)^2 - \left(\frac{b^3}{(a^2+b^2)k}\right)^2 = 1$$

$$\Rightarrow \quad \left(\frac{a^3}{h}\right)^2 - \left(\frac{b^3}{k}\right)^2 = (a^2+b^2)^2$$

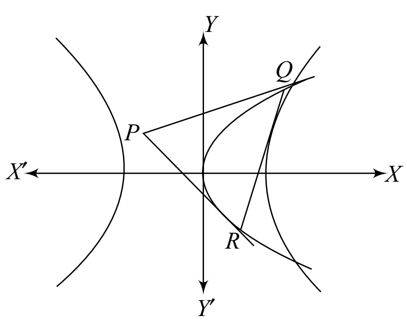
Hence, the locus of (h, k) is

$$(a^6y^2 - b^6x^2) = (a^2 + b^2)^2(x^2y^2)$$

64. Let (h, k) be the pole.
The equation of the polar from the point (h, k) w.r.t. the parabola $y^2 = 4ax$ is

$$yk = 2a(x + h) = 2ax + 2ah$$

$$\Rightarrow \quad y = \left(\frac{2a}{k}\right)x + \left(\frac{2ah}{k}\right) \quad \ldots(i)$$



If the line (i) be a tangent to the hyperbola $x^2 - y^2 = a^2$, then

$$c^2a^2m^2 - a^2$$

$$\Rightarrow \quad \left(\frac{2ah}{k}\right)^2 = a^2\left(\frac{2a}{k}\right)^2 - a^2$$

$$\Rightarrow \quad 4h^2 = 4a^2 - k^2$$

$$\Rightarrow \quad h^2 + \frac{k^2}{4} = a^2$$

Hence, the locus of (h, k) is

$$x^2 + \frac{y^2}{4} = a^2$$

$$\Rightarrow \quad 4x^2 + y^2 = 4a^2$$

65. Let (h, k) be the pole.
Then the equation of the polar w.r.t. the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$\frac{hx}{a^2} - \frac{ky}{b^2} = 1$$

$$\Rightarrow \quad y = \left(\frac{b^2h}{a^2k}\right)x + \left(-\frac{b^2}{k}\right) \quad \ldots(i)$$

The foci of the given hyperbola are $(ae, 0)$ and $(-ae, 0)$
The equation of the circle is

$$(x - ae)(x + ae) + y^2 = 0$$

$$\Rightarrow \quad x^2 + y^2 = (ae)^2 \quad \ldots(ii)$$

If the line (i) be a tangent to the circle (ii), then

$$c^2 = a^2(1 + m^2)$$

$$\Rightarrow \quad \frac{b^4}{k^2} = (ae)^2\left(1 + \frac{b^4k^2}{a^4h^2}\right)$$

$$\Rightarrow \quad \frac{b^4}{k^2} = e^2\left(\frac{a^4h^2 + b^4k^2}{a^2k^2}\right)$$

$$\Rightarrow \quad \frac{a^2b^4}{e^2} = (a^4h^2 + b^4k^2)$$

$$\Rightarrow \quad (a^4h^2 + b^4k^2) = \frac{a^4b^4}{(a^2+b^2)}$$

$$\Rightarrow \quad \left(\frac{h^2}{a^4} + \frac{k^2}{b^4}\right) = \frac{1}{(a^2+b^2)}$$

Hence, the locus of (h, k) is $\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right) = \frac{1}{(a^2+b^2)}$

66. The equation of the chord is

$$7x + y = 2y = -7x + 2 \quad \ldots(i)$$

Hence, the equation of the diameter is

$$y = \frac{b^2x}{a^2m} = \frac{7x}{3 \times (-7)} = -\frac{x}{3}$$

$$x + 3y = 0$$

67. The given line is $3x + 4y + 10 = 0$

$$y = \left(-\frac{3}{4}\right)x + \left(-\frac{5}{2}\right) \quad \ldots(i)$$

Hence, the equation of the diameter corresponds to the line (i) is

$$y = \frac{b^2x}{a^2m} = \frac{4x}{9\left(-\frac{3}{4}\right)} = -\frac{16}{27}x$$

$$\Rightarrow \quad 16x + 27y = 0$$

68. The slope of the given chord is $m = \left(-\frac{2}{3}\right)$.

Hence, the equation of the diameter parallel to the given chord is

$$y = -\frac{b^2 x}{a^2 m} = -\frac{4x}{9\left(-\frac{2}{3}\right)} = -\frac{2}{3}x$$

$\Rightarrow \quad 2x + 3y = 0$

69. Let the equation of the diameter, which is conjugate to $x = 2y$ is

$y = m_1 x$.

As we know that two diameters $y = \left(\frac{1}{2}\right)x$ and $y = m_1 x$ are conjugates, if

$$m_1 m_2 = \frac{b^2}{a^2}$$

$\Rightarrow \quad m_1 \times \frac{1}{2} = \frac{16}{9}$

$\Rightarrow \quad m_1 = \frac{32}{9}$

Hence, the equation of the conjugate diameters is $y = \frac{32}{9}x$

$\Rightarrow \quad 32x = 9y$

70. Equations of the asymptotes to the hyperbola

$xy - 2x - 3y = 0$ is

$xy - 2x - 3y + \lambda = 0$,

where λ is any constant such that it represents two straight lines.

Therefore,

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$\Rightarrow \quad 0 + 2 \times \left(\frac{-3}{2}\right) \times (-1) \times \left(\frac{1}{2}\right) - 0 - 0 - \lambda\left(\frac{1}{2}\right)^2 = 0$

$\Rightarrow \quad \lambda = 6$

Hence, the required asymptotes are

$xy - 2x - 3y + 6 = 0$

$\Rightarrow \quad (x - 2)(y - 3) = 0$

$\Rightarrow \quad x = 2$ and $y = 3$

71. The equations of the asymptotes of the given curve is

$3x^2 + 10xy + 8y^2 + 14x + 22y + \lambda = 0$,

where λ is any constant such that it represents two straight lines.

Therefore,

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$\Rightarrow \quad 3 \cdot 8 \cdot \lambda - +2 \cdot 7 \cdot 11 \cdot 5 - 3 \cdot 121 - 8 \cdot 49 - \lambda \cdot 25 = 0$

$\Rightarrow \quad 24\lambda - + 770 - 363 - 392 - 25\lambda = 0$

$\Rightarrow \quad \lambda = 15$

Hence, the combined equation of the given asymptotes is

$3x^2 + 10xy + 8y^2 + 14x + 22y + 15 = 0$

72. Since the asymptotes are perpendicular to each other, so the hyperbola is rectangular. Hence, its eccentricity is $\sqrt{2}$.

73. The combined equation of the asymptotes is

$(2x - y - 3)(3x + y - 7) = 0$

$\Rightarrow \quad 6x^2 - xy - y^2 - 23x + 4y + 21 = 0$

Let the equation of the hyperbola be

$6x^2 - xy - y^2 - 23x + 4y + \lambda = 0$,

where λ is any constant such that it represents two straight lines which passes through (1, 1), so $\lambda = 15$.

Hence, the equation of the hyperbola becomes

$6x^2 - xy - y^2 - 23x + 4y + 15 = 0$.

74. The combined equation of the asymptotes parallel to the lines $2x + 3y = 0$ and $3x + 2y = 0$ is

$(2x + 3y + \lambda)(3x + 2y + \mu) = 0$

which is passing through (1, 2).

Therefore,

$(2 \cdot 1 + 3 \cdot 2 + \lambda)(3 \cdot 1 + 2 \cdot 2 + \mu) = 0$

$\Rightarrow \quad (8 + \lambda)(7 + \mu) = 0$

$\Rightarrow \quad \lambda = -8, \mu = -7$

Thus, the combined equation of the assymptotes is

$(2x + 3y - 8)(3x + 2y - 7) = 0$

Let the equation of the hyperbola be

$2x + 3y = 0$ and $2x - 3y = 0$

which is passing through (5, 3). So

$(2 \cdot 5 + 3 \cdot 3 - 8)(3 \cdot 5 + 2 \cdot 3 - 7) + \lambda = 0$

$\Rightarrow \quad 11 \times 14 + \lambda = 0$

$\Rightarrow \quad \lambda = -154$

Hence, the equation of the hyperbola is

$(2x + 3y - 8)(3x + 2y - 7) - 154 = 0$.

75. The equation of the given hyperbola is

$x^2 - 2y^2 = 2$.

So, the equations of its asymptotes are $x - \sqrt{2}\,y = 0$ and $x + \sqrt{2}\,y = 0$.

Let any point on the hyperbola be $P\ (\sqrt{2}\sec\varphi, \tan\varphi)$.

Let PM and PN are two perpendiculars from the point P to the asymptotes.

Then, $PM \cdot PN$

$$= \left|\frac{\sqrt{2}\sec\varphi - \sqrt{2}\tan\varphi}{\sqrt{1+2}}\right| \times \left|\frac{\sqrt{2}\sec\varphi + \sqrt{2}\tan\varphi}{\sqrt{1+2}}\right|$$

$$= \frac{2}{3} \times (\sec\varphi - \tan\varphi) \times (\sec\varphi + \tan\varphi)$$

$$= \frac{2}{3} \times (\sec^2\varphi - \tan^2\varphi)$$

$$= \frac{2}{3}.$$

76. The equation of the given hyperbola is $\frac{x^2}{9} - \frac{y^2}{4} = 1$.

Thus, the equations of the asymptotes are

$$\left(\frac{x}{3} + \frac{y}{2}\right)\left(\frac{x}{3} - \frac{y}{2}\right) = 0$$

$\Rightarrow \quad \left(\frac{x}{3} + \frac{y}{2}\right) = 0$ and $\left(\frac{x}{3} - \frac{y}{2}\right) = 0$

$\Rightarrow \quad 2x + 3y = 0$ and $2x - 3y = 0$

The equation of any tangent to the hyperbola

$$\frac{x^2}{9}-\frac{y^2}{4}=1 \text{ is}$$

$$\frac{x}{3}\sec\varphi-\frac{y}{2}\tan\varphi=1$$

Let the points of intersection of $2x + 3y = 0$, $2x - 3y = 0$ and $\frac{x}{3}\sec\varphi-\frac{y}{2}\tan\varphi=1$ are O, P and Q respectively.

Therefore, $O = (0, 0)$,

$$P=\left(\frac{3}{\sec\varphi+\tan\varphi},-\frac{2}{\sec\varphi+\tan\varphi}\right)$$

and $$Q=\left(\frac{3}{\sec\varphi-\tan\varphi},\frac{2}{\sec\varphi-\tan\varphi}\right)$$

Hence, the area of ΔOPQ

$$=\frac{1}{2}\begin{vmatrix} 0 & 0 \\ \dfrac{3}{\sec\varphi+\tan\varphi} & -\dfrac{2}{\sec\varphi+\tan\varphi} \\ \dfrac{3}{\sec\varphi-\tan\varphi} & \dfrac{2}{\sec\varphi-\tan\varphi} \\ 0 & 0 \end{vmatrix}$$

$$=\frac{1}{2}(6+6)=6 \text{ s. u.}$$

77. Let the point P be $(a \sec\varphi, b\tan\varphi)$.

The equations of the asymptotes of the given hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are

$bx - ay = 0$ and $bx + ay = 0$

Let PM and PN be two perpendiculars from the point P to the transverse axis and the asymptote

$bx - ay = 0$

Thus, $PM = b\tan\varphi$

and $$PN=\left|\frac{ab\sec\varphi-ab\tan\varphi}{\sqrt{b^2+a^2}}\right|$$

It is given that, $PM = PN$

$$\Rightarrow \quad b\tan\varphi=\left|\frac{ab\sec\varphi-ab\tan\varphi}{\sqrt{b^2+a^2}}\right|$$

$$\Rightarrow \quad \tan\varphi=a\left|\frac{\sec\varphi-\tan\varphi}{\sqrt{b^2+a^2}}\right|$$

78. Let the equation of the hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad \ldots\text{(i)}$$

and any point on the hyperbola (i) be

$P(a\sec\varphi, b\tan\varphi)$

The equation of the tangent to the hyperbola (i) at P is

$$\frac{x}{a}\sec\varphi-\frac{y}{b}\tan\varphi=1 \qquad \ldots\text{(ii)}$$

The equations of the asymptotes of the hyperbola (i) are

$bx - ay = 0$ and $bx + ay = 0$

Let the points of the intersection of the asymptotes and the tangent are O, Q, R respectively.

Then, $O = (0, 0)$,

$Q = [a(\sec\varphi + \tan\varphi), b(\sec\varphi + \tan\varphi)]$ and

$R = [a(\sec\varphi - \tan\varphi), b(\sec\varphi - \tan\varphi)]$

Clearly, mid-point of QR is $(a\sec\varphi, b\tan\varphi)$, which is co-ordinates of P.

Thus, the area of ΔOQR

$$=\frac{1}{2}\left\|\begin{matrix} 0 & 0 \\ a(\sec\varphi+\tan\varphi) & b(\sec\varphi+\tan\varphi) \\ a(\sec\varphi-\tan\varphi) & -b(\sec\varphi-\tan\varphi) \\ 0 & 0 \end{matrix}\right\|$$

$$=\frac{1}{2}|-ab-ab|$$

$= ab$.

79. The equations of the asymptotes of the given hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are

$bx - ay = 0$ and $bx + ay = 0$

Let P be any point on the given hyperbola be

$(a\sec\varphi, b\tan\varphi)$.

It is given that, p_1 and p_2 be the lengths of perpendiculars from the point P to the asymptotes

$bx - ay = 0$ and $bx + ay = 0$

Thus, $$p_1=\left|\frac{ab\sec\varphi-ab\tan\varphi}{\sqrt{b^2+a^2}}\right|$$

and $$p_2=\left|\frac{ab\sec\varphi+ab\tan\varphi}{\sqrt{b^2+a^2}}\right|$$

Therefore, $\frac{1}{p_1p_2}$

$$=\left(\frac{a^2+b^2}{(ab\sec\varphi-ab\tan\varphi)(ab\sec\varphi+ab\tan\varphi)}\right)$$

$$=\left(\frac{a^2+b^2}{a^2b^2}\right)=\frac{1}{a^2}+\frac{1}{b^2}$$

Hence, the result.

80. The equation of the normal at $\left(ct_1,\frac{c}{t_1}\right)$ is

$$t_1^3x-t_1y-ct_1^4+c=0 \qquad \ldots\text{(i)}$$

Also it meets the hyperbola again at

$$\left(ct_2,\frac{c}{t_2}\right)$$

Therefore,

$$ct_2t_1^3 - c\frac{t_1}{t_2} - ct_1^4 + c = 0$$

$$\Rightarrow \quad t_2t_1^3 - \frac{t_1}{t_2} - t_1^4 + 1 = 0$$

$$\Rightarrow \quad t_2^2t_1^3 - t_1 - t_1^4t_2 + t_2 = 0$$

$$\Rightarrow \quad t_2t_1^3(t_2 - t_1) + (t_2 - t_1) = 0$$

$$\Rightarrow \quad (t_2t_1^3 + 1)(t_2 - t_1) = 0$$

$$\Rightarrow \quad (t_2t_1^3 + 1) = 0 \qquad (\because t_1 \neq t_2)$$

$$\Rightarrow \quad t_2t_1^3 = -1$$

Hence, the result.

81. Let P, Q and R are the vertices of a triangle such that

$$P = \left(ct_1, \frac{c}{t_1}\right), Q = \left(ct_2, \frac{c}{t_2}\right), R = \left(ct_3, \frac{c}{t_3}\right)$$

Now, slope of $QR = \dfrac{\frac{c}{t_3} - \frac{c}{t_2}}{ct_3 - ct_2} = -\dfrac{1}{t_2t_3}$

Therefore, slope of PM is t_2t_3.

The equation of the perpendicular PM on QR is

$$y - \frac{c}{t_1} = t_2t_3(x - ct_1) \qquad \ldots\text{(i)}$$

Similarly, the equation of the perpendicular BN on PR is

$$y - \frac{c}{t_2} = t_1t_3(x - ct_2) \qquad \ldots\text{(ii)}$$

Solving Eq. (i) and (ii), we get,

$$x = -\frac{c}{t_1t_2t_3} \text{ and } y = -ct_1t_2t_3.$$

Thus, the point $\left(-\dfrac{c}{t_1t_2t_3}, -ct_1t_2t_3\right)$ lies on the rectangular hyperbola $xy = c^2$.

Hence, the result.

82. The equation of the normal to the rectangular hyperbola $xy = c^2$ at t is

$$xt^2 - y = ct^3 - \frac{c}{t} \qquad \ldots\text{(i)}$$

Let the pole be (h, k).

Then the equation of the polar from the point (h, k) to the rectangular hyperbola

$xy = c^2$ is

$$xk + yh = 2c^2 \qquad \text{(ii)}$$

Therefore, the Eqs (i) and (ii) are identical. So

$$\frac{k}{t^2} = \frac{h}{-1} = \frac{2c^2}{ct^3 - \frac{c}{t}}$$

$$\Rightarrow \quad t^2 = -\frac{k}{h} \text{ and } h = -\frac{2ct}{t^4 - 1}$$

Eliminating t, we get

$$(h^2 - k^2)^2 + 4c^2hk = 0$$

Hence, the locus of (h, k) is

$$(x^2 - y^2)^2 + 4c^2xy = 0$$

83. The equations of the asymptotes of the rectangular hyperbola $xy = c^2$ are $x = 0$ and $y = 0$.

Clearly, the angle between the asymptotes

$$= \frac{\pi}{2}$$

$$\Rightarrow \quad 2\alpha = \frac{\pi}{2}$$

$$\Rightarrow \quad \alpha = \frac{\pi}{4}$$

Thus, the eccentricity,

$$e = \sqrt{2} = \sec\left(\frac{\pi}{4}\right) = \sec\alpha$$

84. Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c + 0 \qquad \ldots\text{(i)}$$

and the equation of the given hyperbola be

$$xy = 1 \qquad \ldots\text{(ii)}$$

Solving, we get

$$x^2 + \frac{1}{x^2} + 2gx + \frac{2f}{x} + c = 0$$

$$\Rightarrow \quad x^4 + 2gx^3 + cx^2 + 2fx + 1 = 0$$

Let its roots are x_1, x_2, x_3 and x_4.

Then, $x_1x_2x_3x_4 = 1$

Similarly, we can easily prove that

$$y_1y_2y_3y_4 = 1$$

85. Any point on the rectangular hyperbola

$$xy = c^2 \text{ is } P\left(ct, \frac{c}{t}\right)$$

The equation of any tangent to the rectangular hyperbola $xy = c^2$ at t is

$$\frac{x}{t} + yt = 2c \qquad \ldots\text{(i)}$$

The equation of any normal to the rectangular hyperbola $xy = c^2$ at t is

$$xt^3 - yt - xt^4 + c = 0 \qquad \ldots\text{(ii)}$$

Therefore, $a_1 = 2ct, a_2 = \dfrac{2c}{t}$

and $b_1 = c\left(t - \dfrac{1}{t^3}\right), b_2 = c\left(\dfrac{1}{t} - t^3\right)$

Now,

$$a_1a_2 + b_1b_2 = 2c^2t\left(t - \frac{1}{t^3}\right) + \frac{2c^2}{t}\left(\frac{1}{t} - t^3\right)$$

$$= 2c^2\left(t^2 - \frac{1}{t^2} + \frac{1}{t^2} - t^2\right) = 0$$

86. We have, $e_1 = \sqrt{2}$ and $e_2 = \sqrt{2}$

Now, $(e_1 + e_2)^2 = (\sqrt{2} + \sqrt{2})^2 = (2\sqrt{2})^2 = 8$

87. Any point on the given hyperbola

$\frac{x^2}{2} - y^2 = 1$ be $P(\sqrt{2}\sec\varphi, \tan\varphi)$.

The equations of the asymptotes of the given hyperbola are

$$x - \sqrt{2}\, y = 0 \text{ and } x + \sqrt{2}\, y = 0.$$

Let PM and PN be the lengths of perpendiculars from the point P on the asymptotes.

Thus,

$$PM.PN$$

$$= \left|\frac{\sqrt{2}\sec\varphi - \sqrt{2}\tan\varphi}{\sqrt{1+2}}\right| \times \left|\frac{\sqrt{2}\sec\varphi + \sqrt{2}\tan\varphi}{\sqrt{1+2}}\right|$$

$$= \frac{2}{3}$$

$$= (1 \pm 2\sqrt{2}, 2)$$

88. Let the points A, B, C be

$$\left(ct_1, \frac{c}{t_1}\right), \left(ct_2, \frac{c}{t_2}\right), \left(ct_3, \frac{c}{t_3}\right) \text{ respectively.}$$

(i) Then the area of the ΔABC

$$= \frac{1}{2}\begin{vmatrix} ct_1 & \frac{c}{t_1} & 1 \\ ct_2 & \frac{c}{t_2} & 1 \\ ct_3 & \frac{c}{t_3} & 1 \end{vmatrix}$$

$$= \frac{1}{2}\frac{c^2}{t_1 t_2 t_3}\begin{vmatrix} t_1^2 & 1 & t_1 \\ t_2^2 & 1 & t_2 \\ t_3^2 & 1 & t_3 \end{vmatrix}$$

$$= \frac{c^2}{2} \times \frac{(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)}{t_1 t_2 t_3}$$

(ii) The equations of the tangents at A, B and C are

$$\frac{x}{t_1} + yt_1 = 2c \quad \ldots\text{(i)}$$

$$\frac{x}{t_2} + yt_2 = 2c \quad \ldots\text{(ii)}$$

and $$\frac{x}{t_3} + yt_3 = 2c \quad \ldots\text{(iii)}$$

Thus, the points of intersections of (i) and (ii), (i) and (iii), (ii) and (iii) meet at P, Q, R respectively.

Thus, $P = \left(\frac{2ct_1t_2}{t_1 + t_2}, \frac{2c}{t_1 + t_2}\right)$,

$$Q = \left(\frac{2ct_1t_3}{t_1 + t_3}, \frac{2c}{t_1 + t_3}\right)$$

and $R = \left(\frac{2ct_2t_3}{t_2 + t_3}, \frac{2c}{t_2 + t_3}\right)$ respectively

Hence, the area of the ΔPQR

$$= \frac{1}{2}\begin{vmatrix} \frac{2ct_1t_2}{t_1 + t_2} & \frac{2c}{t_1 + t_2} & 1 \\ \frac{2ct_1t_3}{t_1 + t_3} & \frac{2c}{t_1 + t_3} & 1 \\ \frac{2ct_2t_3}{t_2 + t_3} & \frac{2c}{t_2 + t_3} & 1 \end{vmatrix}$$

$$= \frac{2c^2(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)}{(t_1 + t_2)(t_1 + t_2)(t_1 + t_2)}$$

89. The given rectangular hyperbola is

$xy = 18$ …(i)

Replacing x by $x \cos(45°) + y \sin(45°)$ and y by $-x \sin(45°) + y \cos(45°)$ in (i), we get

$$\left(\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}\right)\left(-\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}\right) = 18$$

$\Rightarrow \quad x^2 - y^2 = -18$

Hence, the length of the transverse axis

$= 2a = 2.6 = 12$ units.

90. Let (h, k) be any point.

The equation of the chord of contact of the tangents from (h, k) to the circle $x^2 + y^2 = 4$ is

$hx + ky = 4$.

Also, the given hyperbola is

$xy = 1$

$$\Rightarrow \quad x\left(\frac{4 - hx}{k}\right) = 1$$

$\Rightarrow \quad 4x - hx^2 = k$

$\Rightarrow \quad hx^2 - 4x + k = 0$

Thus, its roots are equal. So

$D = 0$

$\Rightarrow \quad 16 - 4hk = 0$

$\Rightarrow \quad hk = 4$

Hence, the locus of (h, k) is $xy = 4$.

91. The combined equation of the asymptotes of the given hyperbola is

$xy - hx - ky + \lambda = 0$

where λ is any constant such that it represents two straight lines.

Therefore,

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$

$$\Rightarrow \quad 0 + 2\left(-\frac{k}{2}\right)\left(-\frac{h}{2}\right)\left(\frac{1}{2}\right) - \frac{\lambda}{4} = 0$$

$\Rightarrow \quad \lambda = hk$

Hence, the asymptotes are

$xy - hx - ky + hk = 0$

$\Rightarrow (x - h)(y - k) = 0$

$\Rightarrow (x - h) = 0$ and $(y - k) = 0$

92. We have, $\theta = 2\tan^{-1}\left(\frac{b}{a}\right)$

$\Rightarrow \tan\left(\frac{\theta}{2}\right) = \frac{b}{a}$

Also,

$$e = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{1+\tan^2\left(\frac{\theta}{2}\right)} = \sec\left(\frac{\theta}{2}\right)$$

$\Rightarrow \cos\left(\frac{\theta}{2}\right) = \frac{1}{e}$

93. The equation of the given hyperbola is

$9x^2 - 16y^2 = 144$

$\Rightarrow \frac{x^2}{16} - \frac{y^2}{9} = 1$ ...(i)

Let any point on the given hyperbola be $P(8, k)$.
Since the point P lies on (i), so

$\frac{64}{16} - \frac{k^2}{9} = 1$

$\Rightarrow k^2 = 27$

$\Rightarrow k = 3\sqrt{3}$

Hence, the co-ordinates of P be $(8, 3\sqrt{3})$.

Thus, the equation of the reflected ray is

$$y - 3\sqrt{3} = \frac{0-3\sqrt{3}}{-5-8}(x-8)$$

$\Rightarrow 3\sqrt{3}\,x - 13y + 15\sqrt{3} = 0$

Level III

1. As we know that $y = mx + c$ will be the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if

$c^2 = a^2m^2 - b^2 = 9m^2 - 4$

Hence, the equation of the hyperbola is

$\frac{x^2}{9} - \frac{y^2}{4} = 1$

2. The equation of any tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $y = mx + \sqrt{a^2m^2 - b^2}$ ...(i)

and the equation of any tangent to the hyperbola $\frac{x^2}{b^2} - \frac{y^2}{a^2} = -1$, i.e. $\frac{x^2}{(-b^2)} - \frac{y^2}{(-a^2)} = 1$ is

$y = mx + \sqrt{(-b^2)m^2 + a^2}$ ...(ii)

If (i) and (ii) are the same, then

$a^2m^2 - b^2 = -b^2m^2 + a^2$

$\Rightarrow (a^2 + b^2)\,m^2 = (a^2 + b^2)$

$\Rightarrow m^2 = 1$

$\Rightarrow m = \pm 1$

Hence, the equation of the common tangent be $y = \pm x \pm \sqrt{a^2 - b^2}$.

3. The equation of any tangent to the hyperbola

$\frac{x^2}{16} - \frac{y^2}{9} = 1$ is

$y = mx + \sqrt{16m^2 - 9}$

$\Rightarrow mx - y + \sqrt{16m^2 - 9} = 0$

which is also a tangent of $x^2 + y^2 = 9$. So

$$\frac{\sqrt{16m^2-9}}{\sqrt{m^2+1}} = 3$$

$\Rightarrow \sqrt{16m^2 - 9} = 3\sqrt{m^2+1}$

$\Rightarrow 16m^2 - 9 = 9m^2 + 9$

$\Rightarrow 16m^2 - 9m^2 = 9 + 9$

$\Rightarrow 7m^2 = 18$

$\Rightarrow m = \pm\frac{3\sqrt{2}}{\sqrt{7}}$

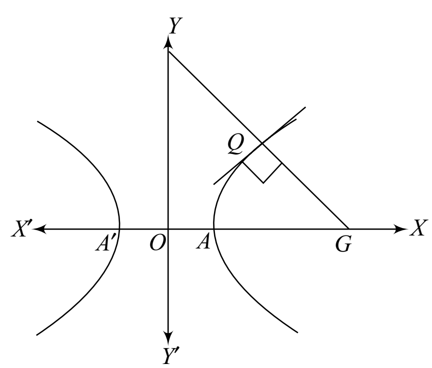
Hence, the equation of the common tangent be

$$y = \pm\left(\frac{3\sqrt{2}}{\sqrt{7}}\right)x + \sqrt{\frac{288}{7} - 9}$$

$\Rightarrow \sqrt{7}y = \pm(3\sqrt{2})x + 15$

4. The equation of any normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (ϕ) is

$ax\cos\varphi + by\cot\varphi = a^2 + b^2$



Thus, the co-ordinates of $G = \left(\frac{a^2+b^2}{a\cos\varphi}, 0\right)$

Clearly, the vertices, $A = (a, 0)$ and $A' = (-a, 0)$

Now, $AG = \left(\dfrac{a^2+b^2}{a\cos\varphi} - a\right)$

and $\quad A'G = \left(\dfrac{a^2+b^2}{a\cos\varphi} + a\right)$

Therefore,

$$AG \cdot A'G = \left(\frac{a^2+b^2}{a\cos\varphi} - a\right)\left(\frac{a^2+b^2}{a\cos\varphi} + a\right)$$

$$= \left(\left(\frac{a^2+b^2}{a}\right)^2 \sec^2\varphi - a^2\right)$$

$= (a^2c^2 \sec^2 \varphi - a^2)$

$= a^2 (e^2 \sec^2 \varphi - 1)$

$\Rightarrow \quad m = 2, n = 2, p = 2$

Hence, $(m+ n + p)^2 + 36 = 36 + 36 = 72$.

5. The equation of the normal to the hyperbola $xy = c^2$ at $\left(ct, \dfrac{c}{t}\right)$ is

$xt^3 - yt - ct^4 + c = 0$

$\Rightarrow \quad ct^4 - xt^3 + yt - c = 0$

which is passing through (α, β), so $ct^4 - \alpha t^3 + \beta t - c = 0$

Let its four roots are $t_1 t_2, t_3, t_3$.

Therefore,

$$t_1 + t_2 + t_3 + t_4 = \frac{\alpha}{c}, \Sigma(t_1t_2) = 0,$$

$$\Sigma(t_1t_2t_3) = -\frac{\beta}{c} \text{ and } \Sigma(t_1t_2t_3t_4) = 0.$$

(i) $x_1 + x_2 + x_3 + x_4 = c(t_1 + t_2 + t_3 + t_4)$

$$= c\left(\frac{\alpha}{c}\right) = \alpha$$

(ii) $y_1 + y_2 + y_3 + y_4 = c\left(\dfrac{1}{t_1} + \dfrac{1}{t_2} + \dfrac{1}{t_3} + \dfrac{1}{t_4}\right)$

$$= c\left(\frac{\sum(t_1t_2t_3)}{\sum(t_1t_2t_3t_4)}\right) = c\left(\frac{\left(-\frac{\beta}{c}\right)}{-1}\right) = \beta$$

(iii) $x_1^2 + x_2^2 + x_3^2 + x_4^2 = c^2(t_1^2 + t_2^2 + t_3^2 + t_4^2)$

$$= c^2\left\{\left(\sum t_1\right)^2 - 2\sum(t_1t_2)\right\}$$

$$= c^2\left\{\left(\frac{\alpha}{c}\right)^2 - 0\right\} = \alpha^2$$

(iv) $y_1^2 + y_2^2 + y_3^2 + y_4^2 = \left(\sum y_1\right)^2 - 2\sum(y_1y_2)$

$$= (\beta)^2 - 2c^2 \sum\left(\frac{1}{t_1t_2}\right)$$

$$= (\beta)^2 - 2c^2\left(\frac{\sum t_1 t_2}{t_1t_2t_3t_4}\right)$$

$$= (\beta)^2$$

(v) $x_1x_2x_3x_4 = c^4(t_1t_2t_3t_4) = -c^4$

(vi) $y_1y_2y_3y_4 = c^4\left(\dfrac{1}{t_1t_2t_3t_4}\right) = c^4\left(\dfrac{1}{-1}\right) = -c^4$

6. The equation of any normal to the given hyperbola at (x, y) is

$$\frac{x}{a^2}(k - y) = \frac{y}{b^2}(x - b)$$

$$\Rightarrow \quad \left(\frac{1}{a^2} + \frac{1}{b^2}\right)xy - \frac{y}{b} - \frac{kx}{a^2} = 0$$

$$\Rightarrow \quad y = \frac{b^2kx}{a^2(e^2x - b)} \qquad \ldots\text{(i)}$$

The equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \ldots\text{(ii)}$$

Solving Eqs (i) and (ii), we get

$$\frac{x^2}{a^2} - \frac{b^4k^2x^2}{b^2a^4(e^2x - b)^2} = 1$$

$$\Rightarrow \quad a^2e^4x^4 - 2ba^2e^2x^3 - (a^2b^2 + b^2k^2 + a^4e^4)\, x^2 + 2ba^4e^2x + a^4b^2 = 0 \qquad \ldots\text{(iii)}$$

Let x_1, x_2, x_3, x_4 are the roots of Eq. (iii).

Then, $x_1 + x_2 + x_3 + x_4 = \dfrac{2b}{e^2}$,

$$\sum(x_1x_2) = -\frac{a^2b^2 + b^2k^2 + a^4e^2}{a^2e^4}$$

$$\sum(x_1x_2x_3) = -\frac{2ba^2}{e^2}$$

and $\quad \sum(x_1x_2x_3x_4) = \dfrac{a^2b^2}{e^4}$

Therefore, $\sum\left(\dfrac{1}{x_1}\right) = \dfrac{\sum(x_1x_2x_3)}{x_1x_2x_3x_4} = \dfrac{2e^2}{b}$

(i) $\quad (x_1 + x_2 + x_3 + x_4)\left(\dfrac{1}{x_1} + \dfrac{1}{x_2} + \dfrac{1}{x_3} + \dfrac{1}{x_4}\right)$

$$= \left(\sum x_1\right)\left(\sum\left(\frac{1}{x_1}\right)\right) = \frac{2b}{e^2} \times \frac{2e^2}{b} = 4$$

(ii) $\quad$ Similarly, we can have

$$(y_1 + y_2 + y_3 + y_4)\left(\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} + \frac{1}{y_4}\right)$$

$$= \left(\sum y_1\right)\left(\sum\left(\frac{1}{y_1}\right)\right)$$

$= 4$.

7. The equation of the normal to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ at $(a \sec \theta, b \tan \theta)$ is

$$ax\cos\theta + by\cot\theta = a^2 + b^2 \quad \ldots\text{(i)}$$

Let the point Q be (α, β).

The equation of any line perpendicular to (i) is

$(b\cot\theta)x - (a\cos\theta)y + k = 0$

which is passing through the centre. So

$k = 0$

Thus, $(b\cot\theta)x - (a\cos\theta)y = 0$

which meets at Q.

Thus,

$(b\cot\theta)\alpha - (a\cos\theta)\beta = 0$

$$\Rightarrow \quad \sin\theta = \frac{b\alpha}{a\beta}$$

$$\Rightarrow \quad \cos\theta = \frac{\sqrt{a^2\beta^2 - b^2\alpha^2}}{a\beta}$$

and $\quad \cot\theta = \dfrac{\sqrt{a^2\beta^2 - b^2\alpha^2}}{b\alpha}$

Putting the values of $\cos\theta$ and $\cot\theta$, in Eq. (i) we get

$$ax\left(\frac{\sqrt{a^2\beta^2 - b^2\alpha^2}}{a\beta}\right) + by\left(\frac{\sqrt{a^2\beta^2 - b^2\alpha^2}}{b\alpha}\right) = a^2 + b^2$$

Hence, the locus of Q is

$$\frac{x}{y}\left(\sqrt{a^2y^2 - b^2x^2}\right) + \frac{y}{x}\left(\sqrt{a^2y^2 - b^2x^2}\right) = a^2 + b^2$$

$$\Rightarrow \quad \left(\frac{x}{y} + \frac{y}{x}\right)\left(\sqrt{a^2y^2 - b^2x^2}\right) = a^2 + b^2$$

$$\Rightarrow \quad \left(\frac{x}{y} + \frac{y}{x}\right)^2 (a^2y^2 - b^2x^2) = (a^2 + b^2)^2$$

$$\Rightarrow \quad (x^2 + y^2)^2(a^2y^2 - b^2x^2) = (a^2 + b^2)^2$$

8. Any point on the circle $x^2 + y^2 = a^2$ is $(a\cos\theta, a\sin\theta)$. The chord of contact of this point with respect to the hyperbola $x^2 - y^2 = a^2$ is

$$x\cos\theta - y\sin\theta = a \quad \ldots\text{(i)}$$

If its mid-point be (h, k), then it is same as

$T = S_1$

i.e. $\quad hx - ky = h^2 - k^2$

Comparing Eqs (i) and (ii), we get

$$\frac{\cos\theta}{h} = \frac{\sin\theta}{k} = \frac{a}{h^2 - k^2}$$

We know that

$\sin^2\theta + \cos^2\theta = 1$

$$\Rightarrow \quad \left(\frac{ah}{h^2 - k^2}\right)^2 + \left(\frac{ak}{h^2 - k^2}\right)^2 = 1$$

$$\Rightarrow \quad a^2(h^2 + k^2) = (h^2 - k^2)^2$$

Hence, the locus of (h, k) is

$a^2(x^2 + y^2) = (x^2 - y^2)^2$

9. Let the mid-point be $M(h, k)$

The equation of the chord of the hyperbola

$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is

$T = S_1$

$$\Rightarrow \quad \frac{hx}{a^2} - \frac{ky}{b^2} = \frac{h^2}{a^2} - \frac{k^2}{b^2}$$

First we make it a homogeneous equation of 2nd degree.

$$\left(\frac{\frac{hx}{a^2} - \frac{ky}{b^2}}{\frac{h^2}{a^2} - \frac{k^2}{b^2}}\right)^2 = \left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$$

$$\Rightarrow \quad \left(\frac{h^2x^2}{a^4} + \frac{k^2y^2}{b^4} - 2(\ldots)xy\right) = \left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)$$

which subtends right angle at the centre, i.e.

co-efficient of x^2 + co-efficient of $y^2 = 0$

$$\left(\frac{h^2}{a^4} - \frac{1}{a^2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right) + \frac{k^2}{b^4} + \frac{1}{b^2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)\right) = 0$$

$$\Rightarrow \quad \left(\frac{h^2}{a^4} + \frac{k^2}{b^4}\right) = \left(\frac{1}{a^2} + \frac{1}{b^2}\right)\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)$$

Hence, the locus of $M(h, k)$ is

$$\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right) = \left(\frac{1}{a^2} + \frac{1}{b^2}\right)\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$$

10. Let the point P be (h, k).

The equation of the tangent to the parabola $y^2 = 4ax$ at P is

$yk = 2a(a + h)$

$$\Rightarrow \quad y = \frac{2ax}{k} + \frac{2ah}{k}$$

which is a tangent to the hyperbola. So

$c^2 = a^2m^2 - b^2$

$$\Rightarrow \quad \left(\frac{2ah}{k}\right)^2 = a^2\left(\frac{2a}{k}\right)^2 - b^2$$

$$\Rightarrow \quad \frac{4a^2h^2}{k^2} = \frac{4a^4}{k^2} - b^2$$

$$\Rightarrow \quad 4a^2h^2 + k^2b^2 = 4a^4$$

Hence, the locus of (h, k) is

$4a^2x^2 + b^2y^2 = 4a^2$

11. Let $M(h, k)$ be the mid-point of the hyperbola

$x^2 - y^2 = a^2$

The equation of the chord at M is

$T = S_1$

$\Rightarrow \quad hx - ky = h^2 - k^2$

$\Rightarrow \quad ky = hx - (h^2 - k^2)$

$$\Rightarrow \quad y = \left(\frac{h}{k}\right)x - \left(\frac{h^2 - k^2}{k}\right)$$

which is a tangent to the parabola $y^2 = 4ax$. So

$$c = \frac{a}{m}$$

$$\Rightarrow \quad \left(\frac{k^2 - h^2}{k}\right) = \frac{ak}{h}$$

$\Rightarrow \quad ak^2 = k^2h - h^3$

$\Rightarrow \quad k^2(h - a) = h^3$

Hence, the locus of $M(h, k)$ is

$y^2(x - a) = x^3$

12. Let $M(h, k)$ be the mid-point of the chord of length $2d$ inclined at an angle θ with the x-axis. Then its extremities are

$(h + d\cos\theta\ ck + d\sin\theta)$

and $\quad (h - d\cos\theta\ ck - d\sin\theta)$

These extremities lie on the hyperbola $xy = c^2$

So, $\quad (h + d\cos\theta)(k + d\sin\theta) = c^2$ …(i)

and $\quad (h - d\cos\theta)(k - d\sin\theta) = c^2$ …(ii)

Adding and subtracting Eqs (i) and (ii), we get

$hk + d^2\sin\theta\cos\theta = c^2$ …(iii)

and $\quad h\sin\theta + k\cos\theta = 0$

i.e. $\quad \tan\theta = -\dfrac{k}{h}$ …(iv)

Eliminating θ between Eqs (iii) and (iv), we get

$$(hk - c^2)\left(\frac{h^2 + k^2}{h^2}\right) = \frac{d^2k}{h}$$

$\Rightarrow \quad (hk - c^2)(h^2 + k^2) = d^2hk$

Hence, the locus of $M(h, k)$ is

$(xy - c^2)(x^2 + y^2) = d^2xy$

13. Let the mid-point be $M(h, k)$.

The equation of the chord of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is

$T = S_1$

$$\Rightarrow \quad \frac{hx}{a^2} - \frac{ky}{b^2} = \frac{h^2}{a^2} - \frac{k^2}{b^2}$$

$$\Rightarrow \quad \frac{ky}{b^2} = \frac{hx}{a^2} - \left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)$$

$$\Rightarrow \quad y = \left(\frac{b^2h}{a^2k}\right)x - \frac{b^2}{k}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)$$

which is a tangent to the circle $x^2 + y^2 = c^2$. So

$c^2 = a^2(1 + m^2)$

$$\Rightarrow \quad \left(\frac{b^2}{k}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)\right)^2 = c^2\left(1 + \frac{b^4h^2}{a^4k^2}\right)$$

$$\Rightarrow \quad \frac{b^4}{k^2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2 = c^2\left(1 + \frac{b^4h^2}{a^4k^2}\right)$$

Hence, the locus of the mid-point $M(h, k)$ is

$$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2 = c^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right)$$

14. Let the mid-point be $M(h, k)$.

The equation of the chord of the circle

$x^2 + y^2 = a^2$ is

$T = S_1$

$\Rightarrow \quad hx + ky = h^2 + k^2$

$\Rightarrow \quad ky = -hx + (h^2 + k^2)$

$$\Rightarrow \quad y = -\left(\frac{h}{k}\right)x + \left(\frac{h^2 + k^2}{k}\right)$$

which is a tangent to the hyperbola. So

$c^2 = a^2m^2 - b^2$

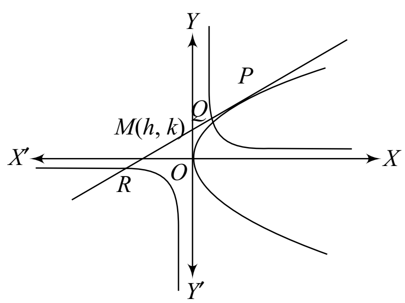
$$\Rightarrow \quad \left(\frac{h^2 + k^2}{k}\right)^2 = a^2\left(\frac{h}{k}\right)^2 - b^2$$

$\Rightarrow \quad (h^2 + k^2)^2 = (a^2h^2 - k^2b^2)$

Hence, the locus of $M(h, k)$ is

$(x^2 + y^2)^2 = (a^2x^2 - b^2y^2)$

15. Let the point P be $(at^2, 2at)$.



The equation of the tangent to the parabola at P is

$yt = x + at^2$ …(i)

Given hyperbola is

$xy = c^2$ …(ii)

The equation of the chord of the hyperbola $xy = c^2$ at $M(h, k)$ is

$xk + yh = 2hk$ …(ii)

So, Eqs (i) and (ii) are the same line. So

$$\frac{1}{k} = -\frac{t}{h} = -\frac{at^2}{2hk}$$

$$\Rightarrow \quad t = -\frac{h}{k} \text{ and } t^2 = -\frac{2h}{a}$$

$$\Rightarrow \quad \frac{h^2}{k^2} = -\frac{2h}{a}$$

$$\Rightarrow \quad \frac{h}{k^2} = -\frac{2}{a}$$

$$\Rightarrow \quad k^2 = -\frac{ah}{2}$$

Hence, the locus of $M(h, k)$ is

$$y^2 = -\frac{ax}{2}$$

which is a parabola.

16. The equation of the chord of contact of the circle

$x^2 + y^2 = a^2$ is $hx + ky = a^2$ …(i)

and the equation of the normal chord of the hyperbola is

$ax \cos \varphi + by \cot \varphi = a^2 + b^2$ …(ii)

Solving, we get

$$\frac{a\cos\varphi}{h} = \frac{b\cot\varphi}{k} = \frac{a^2+b^2}{a^2}$$

$$\Rightarrow \quad \cos\varphi = \frac{(a^2+b^2)h}{a^3}, \cot\varphi = \frac{(a^2+b^2)k}{a^2 b}$$

$$\Rightarrow \quad \cos\varphi = \frac{(a^2+b^2)h}{a^3}, \sin\varphi = \frac{bh}{ak}$$

We know that $\sin^2 \varphi + \cos^2 \varphi = 1$

$$\Rightarrow \quad \left(\frac{bh}{ak}\right)^2 + \left(\frac{(a^2+b^2)h}{a^3}\right)^2 = 1$$

$$\Rightarrow \quad \frac{h^2}{a^2}\left(\frac{b^2}{k^2} + \left(\frac{a^2+b^2}{a^2}\right)^2\right) = 1$$

$$\Rightarrow \quad \left(\frac{b^2}{k^2} + \left(\frac{a^2+b^2}{a^2}\right)^2\right) = \frac{a^2}{h^2}$$

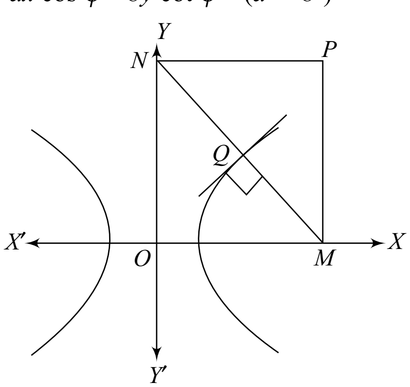
$$\Rightarrow \quad \left(\frac{a^2}{h^2} - \frac{b^2}{k^2}\right) = \left(\frac{a^2+b^2}{a^2}\right)^2$$

Hence, the locus of $M(h, k)$ is

$$\left(\frac{a^2}{x^2} - \frac{b^2}{y^2}\right) = \left(\frac{a^2+b^2}{a^2}\right)^2$$

17. The equation of any normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$ax \cos \varphi + by \cot \varphi = (a^2 + b^2)$ …(i)



Since the normal (i) meets the x-axis at M and y-axis at N respectively. Then,

$$M = \left(\frac{a^2+b^2}{a\cos\varphi}, 0\right) \text{ and } N = \left(0, \left(\frac{a^2+b^2}{b}\right)\tan\varphi\right)$$

Let the co-ordinates of the point P be (α, β).

Since PM and PN are perpendicular to the axes, so the co-ordinates of P are

$$\left(\left(\frac{a^2+b^2}{a}\right)\sec\varphi, \left(\frac{a^2+b^2}{b}\right)\tan\varphi\right)$$

Therefore,

$$\alpha = \left(\frac{a^2+b^2}{a}\right)\sec\varphi \text{ and } \beta = \left(\frac{a^2+b^2}{b}\right)\tan\varphi$$

$$\Rightarrow \quad \alpha\left(\frac{a}{a^2+b^2}\right) = \sec\varphi \text{ and } \beta\left(\frac{b}{a^2+b^2}\right) = \tan\varphi$$

As we know that, $\sec^2 \varphi - \tan^2 \varphi = 1$

$$\alpha^2\left(\frac{a}{a^2+b^2}\right)^2 - \beta^2\left(\frac{b}{a^2+b^2}\right)^2 = 1$$

$\Rightarrow \quad \alpha^2 a^2 - \beta^2 b^2 = (a^2 + b^2)^2$

Hence, the locus of (α, β) is

$a^2x^2 - b^2y^2 = (a^2 + b^2)^2$

18. Let the equation of the hyperbola be

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and the point of intersection be $P(h, k)$.

The equation of any tangent to the hyperbola is

$$y = mx + \sqrt{a^2m^2 - b^2}$$

which is passing through $P(h, k)$.

$$k = mh + \sqrt{a^2m^2 - b^2}$$

$\Rightarrow \quad (k - mh)^2 = (a^2m^2 - b^2)$

$\Rightarrow \quad (h^2 - a^2)m^2 - 2(kh)m + (k^2 + b^2) = 0$

It has two roots, say m_1 and m_2

$$\text{Thus, } m_1 + m_2 = \frac{2hk}{(h^2 - a^2)}$$

$$\text{and} \quad m_1 m_2 = \frac{k^2+b^2}{h^2-a^2}$$

Clearly, $\beta = \theta_1 - \theta_2$

$\Rightarrow \quad \tan \beta = \tan (\theta_1 - \theta_2)$

$$= \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$$

$$= \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$= \frac{\sqrt{(m_1+m_2)^2 - 4m_1m_2}}{1 + m_1m_2}$$

$$= \frac{\sqrt{\frac{4h^2k^2}{(h^2-a^2)^2} - 4\left(\frac{k^2+b^2}{h^2-a^2}\right)}}{1 + \left(\frac{k^2+b^2}{h^2-a^2}\right)}$$

$$\Rightarrow \quad \tan^2\beta\left(\frac{h^2+k^2+b^2-a^2}{h^2-a^2}\right)^2 = \frac{4h^2k^2-4(k^2+b^2)(h^2-a^2)}{(h^2-a^2)^2}$$

$\Rightarrow \quad (h^2 + k^2 + b^2 - a^2)\tan^2\beta$
$= 4(h^2k^2 - (k^2 + b^2)(h^2 - a^2))$

$\Rightarrow \quad (h^2 + k^2 + b^2 - a^2)^2 \tan^2\beta = 4(a^2k^2 + h^2b^2 + a^2b^2)$

$\Rightarrow \quad (h^2 + k^2 + b^2 - a^2)^2 = 4\cot\beta(a^2k^2 - h^2b^2 + a^2b^2)$

Hence, the locus of $P(h, k)$ is

$$(x^2 + y^2 + b^2 - a^2)^2 = 4\cot^2\beta(a^2y^2 - b^2x^2 + a^2b^2)$$

19. Let the hyperbola be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ …(i)

and its conjugate be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$ …(ii)

The equation of any tangent, say AB at (p, q) is

$$\frac{px}{a^2}-\frac{qy}{b^2}=-1 \quad \text{…(iii)}$$

where $\frac{p^2}{a^2}-\frac{q^2}{b^2}+1=0$

i.e. $b^2p^2 - a^2q^2 + a^2b^2 = 0$

Eliminating y between Eqs (i) and (iii), we get

$$\frac{x^2}{a^2}-\frac{1}{b^2}\left(1+\frac{xp}{a^2}\right)^2\left(\frac{b^2}{q}\right)^2=1$$

$$\Rightarrow \quad \left(\frac{1}{a^2}-\frac{p^2b^2}{a^4q^2}\right)x^2-\left(\frac{2pb^2}{a^2q^2}\right)x-\left(\frac{b^2}{p^2}+1\right)=0$$

$$\Rightarrow \quad \left(\frac{a^2b^2}{a^4q^2}\right)x^2-\left(\frac{2pb^2}{a^2q^2}\right)x-\left(\frac{b^2}{p^2}+1\right)=0$$

Let its roots are x_1 and x_2. Then

$$x_1+x_2=\frac{2p}{q^2}\cdot\frac{b^2}{a^2}\div\frac{a^2b^2}{a^4q^2}=2p$$

$$\Rightarrow \quad \frac{x_1+x_2}{2}=p$$

Similarly $\frac{y_1+y_2}{2}=q$

Hence, the point of contact is the mid-point of the chord AB.

20. Let the equation of the hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

and its conjugate be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$.

The equation of any line parallel to the conjugate axis be $x = k$.

Let P be $\left(k, \frac{b}{a}\sqrt{k^2-a^2}\right)$

and Q be $\left(k, \frac{b}{a}\sqrt{k^2+a^2}\right)$

The equation of the tangent at P to the hyperbola is

$$\frac{xk}{a^2}-\frac{y}{ab}\sqrt{k^2-a^2}=1 \quad \text{…(i)}$$

and the equation of the tangent at Q to the conjugate hyperbola is

$$\frac{xk}{a^2}-\frac{y}{ab}\sqrt{k^2+a^2}=-1 \quad \text{…(ii)}$$

Squaring and adding Eqs (i) and (ii), we get

$$k^2=\frac{1}{\frac{x^2}{a^4}+\frac{y^2}{a^2b^2}}$$

$$\Rightarrow \quad k=\frac{1}{\sqrt{\frac{x^2}{a^4}+\frac{y^2}{a^2b^2}}}$$

Solving, we get

$$x=\frac{1}{2k}\left(\frac{ay}{b}\right)^2$$

$$\Rightarrow \quad x^2=\frac{1}{4k^2}\left(\frac{ay}{b}\right)^4$$

$$\Rightarrow \quad x^2=\frac{1}{4}\left(\frac{ay}{b}\right)^4\times\left(\frac{x^2}{a^4}+\frac{y^2}{a^2b^2}\right)$$

$$\Rightarrow \quad \frac{4x^2}{a^2}=\frac{y^4}{b^4}\times\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)$$

21. Let the mid-point be $M(h, k)$.

The equation of the chord is

$T = S_1$

$\Rightarrow \quad hx + ky = h^2 + k^2$

$\Rightarrow \quad ky = -hx + (h^2 + k^2)$

$$\Rightarrow \quad y=-\left(\frac{h}{k}\right)x+\left(\frac{h^2+k^2}{k}\right) \quad \text{…(i)}$$

which is a tangent to the hyperbola

$9x^2 - 16y^2 = 144$

$$\Rightarrow \quad \frac{x^2}{16}-\frac{y^2}{9}=1 \quad \text{…(ii)}$$

So, $c^2 = a^2m^2 - b^2$

$$\Rightarrow \quad \left(\frac{h^2+k^2}{k}\right)^2=16\left(-\frac{h}{k}\right)^2-9$$

$\Rightarrow \quad (h^2 + k^2)^2 = 16h^2 - 9k^2$

Hence, the locus of $M(h, k)$ is

$$(x^2 + y^2)^2 = 16x^2 - 9y^2$$

22. Given hyperbola is
$$2x^2 + 5xy + 2y^2 + 4x + 5y = 0 \quad \ldots\text{(i)}$$
The equation of the asymptote of the above hyperbola is
$$2x^2 + 5xy + 2y^2 + 4x + 5y + k = 0 \quad \ldots\text{(ii)}$$
If (ii) is an asymptote of (i), then
$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$
$$\Rightarrow \quad 2\cdot 2\cdot k + 2\cdot\frac{5}{2}\cdot\frac{4}{2}\cdot\frac{5}{2} - 2\cdot\frac{25}{4} - 2\cdot 4 - k\cdot\frac{25}{4} = 0$$
$$\Rightarrow \quad 16\cdot k + 100 - 50 - 32 - 25k = 0$$
$$\Rightarrow \quad 9k = 18$$
$$\Rightarrow \quad k = 2$$
Putting $k = 2$ in Eq. (ii), we get
$$2x^2 + 5xy + 2y^2 + 4x + 5y + 2 = 0$$
$$\Rightarrow \quad (2x + y + 2)(x + 2y + 1) = 0$$
$$\Rightarrow \quad (2x + y + 2) = 0, (x + 2y + 1) = 0$$
Hence, the equation of the hyperbola is
$$(2x + y + 2)(x + 2y + 1) = c$$

23. Let the equation of the hyperbola is
$$(x + 2y + 3)(3x + 4y + 5) = c$$
which is passing through $(1, -1)$. So
$$(1 - 2 + 3)(3 - 4 + 5) = c$$
$$\Rightarrow \quad c = 2.4 = 8$$
Hence, the equation of the hyperbola is
$$(x + 2y + 3)(3x + 4y + 5) = 8$$
$$\Rightarrow \quad 3x^2 + 10xy + 8y^2 + 14x + 22y + 7 = 0$$

24. Let the point P be $(a \sec \varphi, b \tan \varphi)$.
The equation of the tangent to the hyperbola at P is
$$\frac{x}{a}\sec\varphi - \frac{y}{b}\tan\varphi = 1 \quad \ldots\text{(i)}$$
and the equation of the asymptotes to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is
$$\frac{x}{a} = \frac{y}{b} \quad \ldots\text{(ii)}$$
and $$\frac{x}{a} = -\frac{y}{b} \quad \ldots\text{(iii)}$$
Solving Eqs (i) and (ii), we get
$$Q = \left(\frac{a\cos\varphi}{1-\sin\varphi}, \frac{b\cos\varphi}{1-\sin\varphi}\right)$$
and $$R = \left(\frac{a\cos\varphi}{1+\sin\varphi}, \frac{-b\cos\varphi}{1+\sin\varphi}\right)$$
Let O be the centre of the circle passing through C, Q and R having its co-ordinate as (h, k).
Thus, $OC = OQ = OR$
Now, $OC = OQ$
$$\Rightarrow \quad h^2 + k^2 = \left(h - \frac{a\cos\varphi}{1-\sin\varphi}\right)^2 + \left(k - \frac{b\cos\varphi}{1-\sin\varphi}\right)^2$$
$$\Rightarrow \quad 2(ah + bk) = (a^2 + b^2)\left(\frac{\cos\varphi}{1-\sin\varphi}\right) \quad \ldots\text{(iv)}$$
Also, $OC = OR$
$$\Rightarrow \quad h^2 + k^2 = \left(h - \frac{a\cos\varphi}{1+\sin\varphi}\right)^2 + \left(k + \frac{b\cos\varphi}{1+\sin\varphi}\right)^2$$
$$\Rightarrow \quad 2(ah - bk) = (a^2 + b^2)\left(\frac{\cos\varphi}{1+\sin\varphi}\right) \quad \ldots\text{(v)}$$
Multiplying Eqs (iv) and (v), we get
$$4(a^2h^2 - b^2k^2) = (a^2 + b^2)^2$$
Hence, the locus of (h, k) is
$$4(a^2x^2 - b^2y^2) = (a^2 + b^2)^2$$

25. The area of the ΔPQR
$$= \frac{1}{2}\begin{vmatrix} x_1 & \frac{c^2}{x_1} & 1 \\ x_2 & \frac{c^2}{x_2} & 1 \\ x_3 & \frac{c^2}{x_3} & 1 \end{vmatrix}$$
$$= \frac{c^2}{2} \times \frac{1}{x_1x_2x_3}\begin{vmatrix} x_1^2 & 1 & x_1 \\ x_2^2 & 1 & x_2 \\ x_3^2 & 1 & x_3 \end{vmatrix}$$
$$= \frac{c^2}{2} \times \frac{(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)}{x_1x_2x_3}$$

Level IV

1. The equation of any tangent to the parabola
$x^2 = 4ay$ is $x = my + \frac{a}{m}$
$$\Rightarrow \quad mx - m^2y - a = 0 \quad \ldots\text{(i)}$$
Let the mid-point be $M(h, k)$.
The equation of the chord of the hyperbola
$xy = c^2$ is
$$xk + yh - 2c^2 = 0 \quad \ldots\text{(ii)}$$
Since the lines (i) and (ii) are the same line, so
$$\frac{m}{k} = -\frac{m^2}{h} = \frac{a}{2c^2}$$
$$\Rightarrow \quad m = \frac{ak}{2c^2} \text{ and } m^2 = -\frac{ah}{2c^2}$$
$$\Rightarrow \quad \left(\frac{ak}{2c^2}\right)^2 = -\frac{ah}{2c^2}$$
$$\Rightarrow \quad \frac{a^2k^2}{c^4} = -\frac{2ah}{c^2}$$
$$\Rightarrow \quad k^2 = -\left(\frac{2c^2}{a}\right)h$$
Hence, the locus of M is
$$y^2 = -\left(\frac{2c^2}{a}\right)x$$
which is a parabola

2. Given hyperbola is $25x^2 - 16y^2 = 400$
$$\frac{x^2}{16} - \frac{y^2}{25} = 1$$
The equation of the chord of the hyperbola bisected at (6, 2) is
$$T = S_1$$
$$\Rightarrow \quad \frac{6x}{16} - \frac{2y}{25} = \frac{36}{16} - \frac{4}{25}$$
$$\Rightarrow \quad 150x - 32y = 900 - 64$$
$$\Rightarrow \quad 150x - 32y = 836$$
$$\Rightarrow \quad 75x - 16y = 418$$

3. Let the mid-point be $M(h, k)$.
The equation of the chord bisected at M to the given circle is
$$hx + ky = h^2 + k^2$$
$$\Rightarrow \quad ky = -hx + (h^2 + k^2)$$
$$\Rightarrow \quad y = -\left(\frac{h}{k}\right)x + \left(\frac{h^2 + k^2}{k}\right)$$
which is a tangent to the given hyperbola. So
$$c^2 = a^2m^2 - b^2$$
$$\Rightarrow \quad \left(\frac{h^2 + k^2}{k}\right)^2 = 16\left(\frac{h}{k}\right)^2 - 9$$
$$\Rightarrow \quad (h^2 + k^2)^2 = 16h^2 - 9k^2$$
Hence, the locus of (h, k) is
$$(x^2 + y^2)^2 = 16x^2 - 9y^2$$

4. The equation of any tangent to the given hyperbola is
$$y = mx + \sqrt{a^2m^2 - b^2}$$
$$\Rightarrow \quad mx - y + \sqrt{a^2m^2 - b^2} = 0 \qquad \ldots\text{(i)}$$
Let the mid-point be $M(h, k)$.
The equation of the chord of the ellipse is
$$T = S_1$$
$$\Rightarrow \quad \frac{hx}{a^2} + \frac{ky}{b^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2} \qquad \ldots\text{(ii)}$$
Since the lines (i) and (ii) are the same line. So
$$\frac{m}{(h/a^2)} = \frac{-1}{(k/b^2)} = \frac{\sqrt{a^2m^2 - b^2}}{\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)}$$
$$\Rightarrow \quad m = -\frac{(h/a^2)}{(k/b^2)}$$
and $$\sqrt{a^2m^2 - b^2} = -\frac{\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)}{(k/b^2)}$$
Solving, we get
$$(a^2m^2 - b^2) = \frac{\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)^2}{(k/b^2)^2}$$
$$\Rightarrow \quad \left(\frac{k}{b^2}\right)^2\left(a^2\left(\frac{hb^2}{ka^2}\right)^2 - b^2\right) = \left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)^2$$
$$\Rightarrow \quad \left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right) = \left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)^2$$
Hence, the locus of M is
$$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2$$

5. Let the parameters of the vertices A, B and C of the points on the hyperbola $xy = c^2$ be t_1, t_2 and t_3 respectively.
Now the equation of the side BC is
$$x + yt_2t_3 - c(t_2 + t_3) = 0$$
Any line through A perpendicular to BC is
$$y - \frac{c}{t_1} = t_2t_3(x - ct_1)$$
$$\Rightarrow \quad y - xt_2t_3 = \frac{c}{t_1} - ct_1t_2t_3 \qquad \ldots\text{(i)}$$
Similarly, any line through B perpendicular to AC is
$$y - xt_1t_3 = \frac{c}{t_2} - ct_1t_2t_3 \qquad \ldots\text{(ii)}$$
Solving Eqs (i) and (ii), we get, the orthocentre as $\left(-\frac{c}{t_1t_2t_3}, -ct_1t_2t_3\right)$.
Clearly, it satisfies the hyperbola
$$xy = c^2$$
Hence, the result.

6. The equation of the normal to the rectangular hyperbola is
$$xt^3 - yt = c(t^4 - 1) \qquad \ldots\text{(i)}$$
Let the pole be $M(h, k)$.
The equation of the polar at M is
$$xk + yh = 2c^2 \qquad \ldots\text{(ii)}$$
Since the lines (i) and (ii) are the same line, so
$$\frac{t^3}{k} = \frac{-t}{h} = \frac{c(t^4 - 1)}{2c^2}$$
$$\Rightarrow \quad \frac{t^3}{k} = \frac{-t}{h} = \frac{(t^4 - 1)}{2c}$$
Solving, we get
$$t^2 = -\frac{k}{h}, \frac{t^2}{h^2} = \frac{(t^4 - 1)^2}{4c^2}$$
$$\Rightarrow \quad -\frac{k}{h^3} = \frac{(t^4 - 1)^2}{4c^2}$$

$$\Rightarrow \quad -\frac{k}{h^3} = \frac{\left(\frac{k^2}{h^2} - 1\right)^2}{4c^2}$$

$\Rightarrow \quad -4c^2kh = (k^2 - h^2)^2$

$\Rightarrow \quad (h^2 - k^2)^2 + 4c^2hk = 0$

Hence, the locus of (h, k) is

$(x^2 - y^2)^2 + 4c^2xy = 0$

7. Let the equation of the circle be

$x^2 + y^2 + 2gx + 2fy + k = 0$ …(i)

and the equation of the rectangular hyperbola is $xy = c^2$ …(ii)

Putting $x = ct$ and yc/t, then

$$c^2t^2 + \frac{c^2}{t^2} + 2g(ct) + 2f\left(\frac{c}{t}\right) + k = 0$$

$\Rightarrow \quad c^2t^4 + c^2 + 2g(ct^3) + 2f(ct) + kt^2 = 0$

$\Rightarrow \quad c^2t^4 + 2gct^3 + kt^2 + 2fct + c^2 = 0$

which is a bi-quadratic equation of t. So, it has four roots t_1, t_2, t_3 and t_4. Then

$$\Sigma t_1 = -\frac{2g}{c}$$

$$\Sigma t_1t_2 = \frac{k}{c^2}$$

$$\Sigma t_1t_2t_3 = -\frac{2f}{c}$$

and $\quad \Sigma t_1t_2t_3t_4 = 1$

Also, $\sum \frac{1}{t_1} = \frac{\sum t_1t_2t_3}{\sum t_1t_2t_3t_4} = -\frac{2f}{c}$

Now, $(-g, -f)$

$$= \left(\frac{c}{2}\left(-\frac{2g}{c}\right), \frac{c}{2}\left(-\frac{2f}{c}\right)\right)$$

$$= \left(\frac{c}{2}(t_1 + t_2 + t_3 + t_4), \frac{c}{2}\left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4}\right)\right)$$

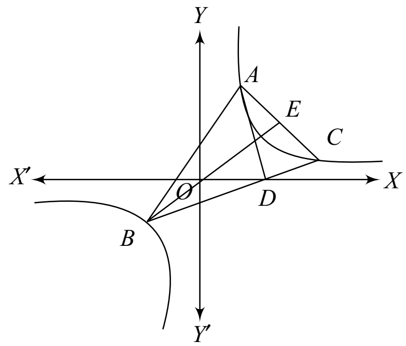
$$= \left(\frac{c}{2}\left(t_1 + t_2 + t_3 + \frac{1}{t_1t_2t_3}\right), \frac{c}{2}\left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + t_1t_2t_3\right)\right)$$

8. Let t_1, t_2 and t_3 are the vertices of ΔABC described on the rectangular hyperbola $xy = c^2$.

So the co-ordinates of A, B and C are

$\left(ct_1, \frac{c}{t_1}\right), \left(ct_2, \frac{c}{t_2}\right)$ and $\left(ct_3, \frac{c}{t_3}\right)$ respectively

Now, slope of $BC = -\frac{1}{t_2t_3}$



Slope of $AD = t_2t_3$

Now the equation of the altitude AD is

$$y - \frac{c}{t_1} = t_2t_3(x - ct_1) \quad \text{…(i)}$$

Similarly, equation of the altitude BE is

$$y - \frac{c}{t_2} = t_1t_3(x - ct_2) \quad \text{…(ii)}$$

Solving, Eqs (i) and (ii), we get the co-ordinates of the orthocentre as

$$\left(-\frac{c}{t_1t_2t_3}, -ct_1t_2t_3\right)$$

which lies on $xy = c^2$.

9. Let the equation of the circle be

$x^2 + y^2 + 2gx + 2fy + k = 0$ …(i)

and the equation of the rectangular hyperbola is

$xy = c^2$ …(ii)

Putting $x = ct$ and yc/t, then

$$c^2t^2 + \frac{c^2}{t^2} + 2g(ct) + 2f\left(\frac{c}{t}\right) + k = 0$$

$\Rightarrow \quad c^2t^4 + c^2 + 2g(ct^3) + 2f(ct) + kt^2 = 0$

$\Rightarrow \quad c^2t^4 + 2gct^3 + kt^2 + 2fct + c^2 = 0$

which is a bi-quadratic equation of t. So, it has four roots, say t_1, t_2, t_3 and t_4.

Then $\sum t_1 = -\frac{2g}{c}$

$$\sum t_1t_2 = \frac{k}{c^2}$$

$$\sum t_1t_2t_3 = -\frac{2f}{c}$$

and $\quad \sum t_1t_2t_3t_4 = 1$

Also, $\sum \frac{1}{t_1} = \frac{\sum t_1t_2t_3}{\sum t_1t_2t_3t_4} = -\frac{2f}{c}$

The centre of the mean position of the four points is

$$\left(\frac{c}{4}(t_1 + t_2 + t_3 + t_4), \frac{c}{4}\left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4}\right)\right)$$

$$= \left(\frac{c}{4}\sum t_1, \frac{c}{4}\sum \frac{1}{t_1}\right)$$

$$= \left(-\frac{g}{2}, -\frac{f}{2}\right)$$

Thus, the centres of the circle and the rectangular hyperbola are $(-g, -f)$ and $(0, 0)$.
and the mid-points of the centres of the circle and the hyperbola is $\left(-\frac{g}{2}, -\frac{f}{2}\right)$

Hence, the result.

10. Let the equation of the circle be
$$x^2 + y^2 - 6\alpha x - 6\beta y + k = 0 \quad \ldots(i)$$
and the equation of the rectangular hyperbola is
$$x^2 - y^2 = 9a^2 \quad \ldots(ii)$$
Eliminating y between Eqs (i) and (ii), we get
$$(x^2 + x^2 - 9a^2 - 6ax + k)^2 = 36\beta^2(x^2 - 9a)^2$$
$\Rightarrow \quad 4x^4 - 24\alpha x^3 + (\ldots)x^2 + (\ldots)x + .. = 0$
which is a bi-quadratic equation of x.
Let the abscissae of four points P, Q, R, S be x_1, x_2, x_3 and x_4, respectively.
Thus, $x_1 + x_2 + x_3 + x_4 = 6\alpha$
Similarly, $y_1 + y_2 + y_3 + y_4 = 6\beta$
Let (h, k) be the centroid of ΔPQR. So
$$h = \frac{x_1 + x_2 + x_3}{3}, k = \frac{y_1 + y_2 + y_3}{3}$$
$\Rightarrow \quad h = \frac{6\alpha - x_4}{3}, k = \frac{6\beta - y_4}{3}$
$\Rightarrow \quad x_4 = 6\alpha - 3h, y_4 = 6\beta - 3k$
Since (x_4, y_4) lies on the curve, so
$$x_4^2 - y_4^2 = 9a^2$$
$\Rightarrow \quad (6\alpha - 3h)^2 - (6\beta - 3k)^2 = 9a^2$
$\Rightarrow \quad (2\alpha - h)^2 - (2\beta - k)^2 = a^2$
Hence, the locus of (h, k) is
$$(2\alpha - x)^2 - (2\beta - y)^2 = a^2$$
$$(x - 2\alpha)^2 - (y - 2\beta)^2 = a^2$$

Integer Type Questions

1. $e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{12}{4}} = \sqrt{4} = 2$

2. We have,
$$e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{81}{144}} = \sqrt{\frac{225}{144}} = \frac{15}{12} = \frac{5}{4}$$
Also, $a^2 = \frac{144}{25}$
Thus, the foci are
$$(\pm ae, 0) = \left(\pm \frac{12}{5} \times \frac{5}{4}, 0\right) = (\pm 3, 0)$$
Now, for the ellipse,
$$ae = 3$$
$\Rightarrow \quad a^2e^2 = 9$
Thus, $\quad b^2 = a^2(1 - e^2) = a^2 - a^2e^2$
$\quad = 16 - 9 = 7$
Hence, the value of $(b^2 + 1)$ is 8.

3. Clearly, $\frac{1}{e_1^2} + \frac{1}{e_2^2} = \frac{1}{\left(1 + \frac{b^2}{a^2}\right)} + \frac{1}{\left(1 + \frac{a^2}{b^2}\right)}$
$$= \frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2}$$
$$= \frac{a^2 + b^2}{a^2 + b^2} = 1$$
Hence, the value of $\left(\frac{1}{e_1^2} + \frac{1}{e_2^2} + 3\right)$ is 4.

4. Clearly, the point (4, 3) lies on the hyperbola. So, the number of tangents is 1.

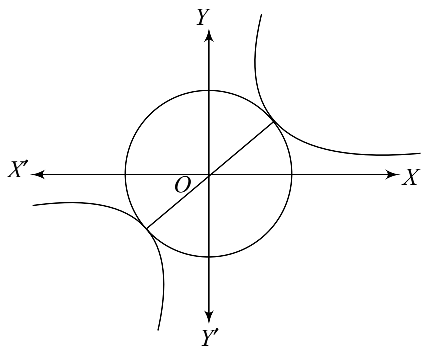
5. The director circle of the given circle $x^2 + y^2 = a^2$ is
$$x^2 + y^2 = 2a^2$$
So the radius of the circle is $a\sqrt{2}$, whereas the length of the transverse axis is $a\sqrt{3}$.
So, the director circle and the hyperbola will never intersect.
So, the number of points is zero.

6. Given hyperbola is
$$9x^2 - 16y^2 - 18x - 32y - 151 = 0$$
$\Rightarrow \quad 9(x^2 - 2x) - (y^2 + 2y) = 151$
$\Rightarrow \quad 9(x - 1)^2 - 16(y + 1)^2 = 144$
$\Rightarrow \quad \frac{(x-1)^2}{16} - \frac{(y+1)^2}{9} = 1$
Latus rectum $= m = \frac{2b^2}{a} = \frac{2.9}{4} = \frac{9}{2}$.
Hence, the value of $2m - 3 = 9 - 3 = 6$.

7. No real tangent can be drawn.
So, the value of m is zero.
Hence, the value of $(m + 4)$ is 4.

8. Given hyperbola is
$$16x^2 - 9y^2 = 144$$
$\Rightarrow \quad \frac{x^2}{9} - \frac{y^2}{16} = 1$
The equation of any tangent to the given hyperbola is
$$y = mx + \sqrt{a^2m^2 - b^2}$$
$\Rightarrow \quad y = mx + \sqrt{9m^2 - 16}$
It is given that,
$$\sqrt{(9m^2 - 16)} = 2\sqrt{5}$$
$\Rightarrow \quad (9m^2 - 16) = 20$
$\Rightarrow \quad 9m^2 = 36$
$\Rightarrow \quad m^2 = 4$
$\Rightarrow \quad m = \pm 2$
So, $\quad a + b = 2 - 2 = 0$
Hence, the value of $(a + b + 3)$ is 3.

9. Given curves are $xy = c, (c > 0)$
and $x^2 + y^2 = 1$



Hence, the distance between the points of contact = diameter of a circle = 2

10. The equation of the asymptotes of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are

$$\frac{x}{a}+\frac{y}{b}=0, \frac{x}{a}-\frac{y}{b}=0$$

i.e. $bx + ay = 0$, $bx - ay = 0$

Let any point on the hyperbola be

$P(a \sec \varphi, b \tan \varphi)$.

It is given that, $p_1 p_2 = 6$

$$\Rightarrow \left|\frac{ab\sec\varphi + ab\tan\varphi}{\sqrt{a^2+b^2}}\right|\left|\frac{ab\sec\varphi - ab\tan\varphi}{\sqrt{a^2+b^2}}\right| = 6$$

$$\Rightarrow \frac{a^2b^2(\sec^2\varphi - \tan^2\varphi)}{(a^2+b^2)} = 6$$

$$\Rightarrow \frac{a^2b^2}{(a^2+b^2)} = 6$$

$$\Rightarrow \frac{a^2\cdot 2a^2}{(a^2+2a^2)} = 6$$

$\Rightarrow$ $2a^2 = 18$

$\Rightarrow$ $a^2 = 9$

$\Rightarrow$ $2a = 6$

$\therefore$ Length of the transverse axis = 6

11. The equation of the tangent to the hyperbola $xy = c^2$ at $\left(ct, \frac{c}{t}\right)$ is

$$x + yt^2 = 2ct \quad \ldots\text{(i)}$$

Thus, $T = (2ct, 0)$ and $T' = \left(0, \frac{2c}{t}\right)$.

The equation of the normal to the hyperbola $xy = c^2$ at $\left(ct, \frac{c}{t}\right)$ is

$$xt^3 - yt - ct^4 + c = 0 \quad \ldots\text{(ii)}$$

Thus, $N = \left(ct - \frac{c}{t^3}, 0\right)$ and $N' = \left(0, \frac{c}{t} - ct^3\right)$.

Now, $\Delta = ar(\Delta PNT)$

$$= \frac{1}{2}\begin{vmatrix} ct & \frac{c}{t} & 1 \\ ct - \frac{c}{t^3} & 0 & 1 \\ 2ct & 0 & 1 \end{vmatrix}$$

$$= \frac{c^2}{2t}\left(t + \frac{1}{t^3}\right)$$

$$= \frac{c^2}{2}\left(1 + \frac{1}{t^4}\right)$$

and $\Delta = ar(\Delta PN'T')$

$$= \frac{1}{2}\begin{vmatrix} ct & \frac{c}{t} & 1 \\ 0 & \frac{c}{t} - ct^3 & 1 \\ 0 & \frac{2c}{t} & 1 \end{vmatrix}$$

$$= \left|\frac{ct}{2}\left(\frac{c}{t} - ct^3 - \frac{2c}{t}\right)\right|$$

$$= \frac{ct}{2}\left(ct^3 + \frac{c}{t}\right)$$

$$= \frac{c^2}{2}(t^4+1)$$

Now, $\frac{1}{\Delta}+\frac{1}{\Delta'} = \frac{2}{c^2}\left(\frac{t^4}{t^4+1}\right) + \frac{2}{c^2}\left(\frac{1}{t^4+1}\right)$

$$= \frac{2}{c^2}\left(\frac{t^4+1}{t^4+1}\right)$$

$$= \frac{2}{c^2}$$

Hence, the value of $\left(\frac{c^2}{\Delta}+\frac{c^2}{\Delta'}+4\right)$ is 6.

12. The equation of the given hyperbola is $\frac{x^2}{9}-\frac{y^2}{4}=1$.

Thus, the equations of the asymptotes are

$$\left(\frac{x}{3}+\frac{y}{2}\right)\left(\frac{x}{3}-\frac{y}{2}\right) = 0$$

$$\Rightarrow \left(\frac{x}{3}+\frac{y}{2}\right) = 0 \text{ and } \left(\frac{x}{3}-\frac{y}{2}\right) = 0$$

$\Rightarrow \quad 2x + 3y = 0$ and $2x - 3y = 0$

The equation of any tangent to the hyperbola

$\frac{x^2}{9} - \frac{y^2}{4} = 1$ is

$$\frac{x}{3}\sec\varphi - \frac{y}{2}\tan\varphi = 1$$

Let the points of intersection of $2x + 3y = 0$, $2x - 3y = 0$ and

$\frac{x}{3}\sec\varphi - \frac{y}{2}\tan\varphi = 1$ are

O, P and Q respectively.

Therefore, $O = (0, 0)$,

$$P = \left(\frac{3}{\sec\varphi + \tan\varphi}, -\frac{2}{\sec\varphi + \tan\varphi}\right)$$

and $\quad Q = \left(\frac{3}{\sec\varphi - \tan\varphi}, \frac{2}{\sec\varphi - \tan\varphi}\right)$

Hence, the area of ΔOPQ

$$= \frac{1}{2}\begin{vmatrix} 0 & 0 \\ \frac{3}{\sec\varphi + \tan\varphi} & -\frac{2}{\sec\varphi + \tan\varphi} \\ \frac{3}{\sec\varphi - \tan\varphi} & \frac{2}{\sec\varphi - \tan\varphi} \\ 0 & 0 \end{vmatrix}$$

$$= \frac{1}{2}(6 + 6) = 6 \text{ sq. u.}$$

Previous Years' JEE-Advanced Examinations

1. Given curve is

$$\frac{x^2}{1-r} - \frac{y^2}{1+r} = 1, r > 1$$

$\Rightarrow \quad (1 + r)x^2 - (1 - r)y^2 = (1 - r)(1 + r)$

$\Rightarrow \quad (1 + r)x^2 - (1 - r)y^2 + (r^2 - 1) = 0$

Now, $h^2 - ab = 0 - (1 + r)(r - 1)$

$= (1 - r^2) < 0$, as $r > 1$

So, it represents an ellipse.

2. Hints

3. Given curve is

$2x^2 + 3y^2 - 8x - 18y + 35 = k$

$\Rightarrow \quad 2(x^2 - 4x) + 3(y^2 - 6y) = k = 35$

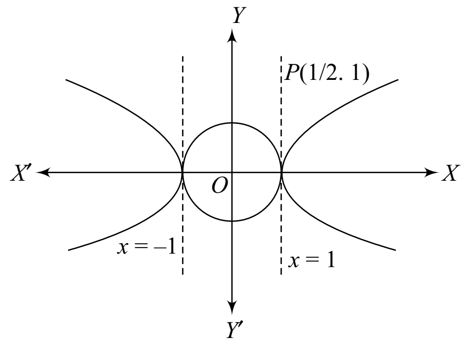
$\Rightarrow \quad 2\{(x - 2)^2 - 4\} + 3\{(y - 3)^2 - 9\} = k - 35$

$\Rightarrow \quad 2(x - 2)^2 + 3(y - 3)^2 = k - 35 + 8 + 27$

$\Rightarrow \quad 2(x - 2)^2 + 3(y - 3)^2 = k$

It represents a point if $k = 0$.

4. Clearly common tangents of the given curves are $x = 1$ and $x = -1$, respectively.



Thus, $x = 1$ is nearer to $P(1/2, 1)$.

Therefore, the directrix of the ellipse is $x = 1$.

Let $Q(x, y)$ be any point on the ellipse.

Now, the length of the perpendicular from Q to the directrix $x - 1 = 0$ is

$|x - 1|$.

By the definition of the ellipse, we have

$QP = e|x - 1|$

$$\Rightarrow \quad \sqrt{\left(x - \frac{1}{2}\right)^2 + (y-1)^2} = e|x - 1|$$

$$\Rightarrow \quad \left(x - \frac{1}{2}\right)^2 + (y-1)^2 = e^2|x - 1|^2$$

$$\Rightarrow \quad \left(x - \frac{1}{2}\right)^2 + (y-1)^2 = \frac{1}{4}(x^2 - 2x + 1)$$

$$\Rightarrow \quad x^2 - x + \frac{1}{4} + y^2 - 2y + 1 = \frac{x^2}{4} - \frac{1}{2}x + \frac{1}{4}$$

$$\Rightarrow \quad x^2 - x + y^2 - 2y + 1 = \frac{x^2}{4} - \frac{x}{2}$$

$\Rightarrow \quad 4x^2 - 4x + 4y^2 - 8y + 4 = x^2 - 2x$

$\Rightarrow \quad 3x^2 + 4y^2 - 2x - 8y + 4 = 0$

$\Rightarrow \quad (3x^2 - 2x) + (4y^2 - 8y) + 4 = 0$

$$\Rightarrow \quad 3\left(x^2 - \frac{2}{3}x\right) + 4(y^2 - 2y) + 4 = 0$$

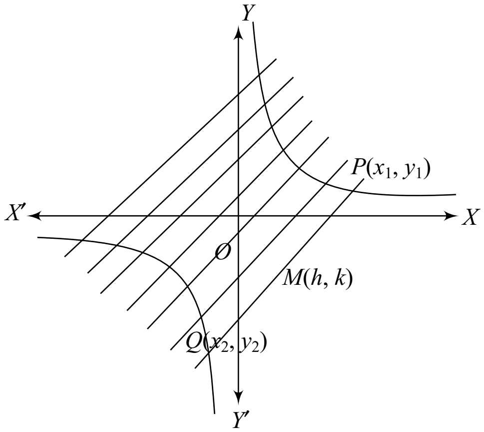
$$\Rightarrow \quad 3\left\{\left(x - \frac{1}{3}\right)^2 - \frac{1}{9}\right\} + 4\left\{(y-1)^2 - 1\right\} + 4 = 0$$

$$\Rightarrow \quad 3\left(x - \frac{1}{3}\right)^2 + 4(y-1)^2 = -4 + \frac{1}{3} + 4$$

$$\Rightarrow \quad 3\left(x - \frac{1}{3}\right)^2 + 4(y-1)^2 = \frac{1}{3}$$

$$\Rightarrow \quad \frac{\left(x - \frac{1}{3}\right)^2}{1/9} + \frac{(y-1)^2}{1/12} = 1$$

5. Let $M(h, k)$ be the point.
The equation of any line through $M(h, k)$ having slope 4 is
$$y - k = 4(x - h)$$
Suppose the line meets the curve $xy = 1$ at $P(x_1, y_1)$ and $Q(x_2, y_2)$.



Now,
$$y - k = 4(x - h)$$
$$\Rightarrow \quad \frac{1}{x} - k = 4(x - h)$$
$$\Rightarrow \quad 4x^2 - (4h - k)x - 1 = 0$$
Let its roots be x_1, x_2.
$$\therefore \quad x_1 + x_2 = \frac{4h - k}{4} \quad \ldots\text{(i)}$$
$$\text{and} \quad x_1 x_2 = -\frac{1}{4} \quad \ldots\text{(ii)}$$
$$\text{Also, } h = \frac{2x_1 + x_2}{3}$$
$$2x_1 + x_2 = 3h \quad \ldots\text{(iii)}$$
From Eqs (i) and (iii), we get
$$x_1 = 3h - \frac{4h - k}{4} = \frac{8h + k}{4}$$
$$\Rightarrow \quad x_2 = 3h - \frac{8h + k}{2} = -\frac{(2h + k)}{2}$$
Putting the values of x_1 and x_2 in Eq. (ii), we get
$$\frac{(8h + k)}{4} \times -\frac{(2h + k)}{2} = -\frac{1}{4}$$
$$\Rightarrow \quad (8h + k)(2h + k) = 2$$
$$\Rightarrow \quad 16h^2 + 10hk + k^2 - 2 = 0$$
Hence, the locus of $M(h, k)$ is
$$16x^2 + 10xy + y^2 - 2 = 0.$$

6. Given $x^2 + y^2 = a^2$ and $xy = c^2$
We have,
$$x^2 + \left(\frac{c^2}{x}\right)^2 = a^2$$
$$\Rightarrow \quad x^4 + c^4 = a^2x^2$$
$$\Rightarrow \quad x^4 - a^2x^2 + c^4 = 0$$
Let its roots be x_1, x_2, x_3 and x_4.
Thus, $x_1 + x_2 + x_3 + x_4 = 0$
and $x_1x_2x_3x_4 = c^4$
Similarly, $y_1 + y_2 + y_3 + y_4 = 0$
and $y_1y_2y_3y_4 = c^4$

7. Let the point P be (h, k)
The equation of any tangent to the parabola is
$$y = mx + \frac{a}{m} \text{ which is passing through } P(h, k).$$
$$\therefore \quad k = mh + \frac{a}{m}$$
$$\Rightarrow \quad m^2h - km + a = 0$$
Since it has two roots say m_1 and m_2. Thus,
$$m_1 + m_2 = \frac{k}{m} \text{ and } m_1 m_2 = \frac{a}{h}$$
$$\text{Now, } \tan(45°) = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$
$$\Rightarrow \quad \left(\frac{m_2 - m_1}{1 + m_1 m_2}\right)^2 = 1$$
$$\Rightarrow \quad (m_2 - m_1)^2 = (1 + m_1m_2)^2$$
$$\Rightarrow \quad (m_2 + m_1)^2 - 4m_1m_2 = (1 + m_1m_2)^2$$
$$\Rightarrow \quad \left(\frac{k}{h}\right)^2 - 4\left(\frac{a}{h}\right) = \left(1 + \frac{a}{h}\right)^2$$
$$\Rightarrow \quad \frac{k^2 - 4ah}{h^2} = \frac{(a + h)^2}{h^2}$$
$$\Rightarrow \quad (a + h)^2 = k^2 - 4ah$$
$$\Rightarrow \quad h^2 + 6ah + a^2 = k^2$$
$$\Rightarrow \quad (h + 3a)^2 = k^2 - 8a^2$$
Hence, the locus of $P(h, k)$ is
$$(x + 3a)^2 = y^2 - 8a^2$$

8. Let $P(h, k)$ be the point whose chord of contact w.r.t. the hyperbola $x^2 - y^2 = 9$ is
$$x = 9 \quad \ldots\text{(i)}$$
Also, the equation of the chord contact of the tangents from $P(h, k)$ is
$$hx - ky = 9 \quad \ldots\text{(ii)}$$
Since the Eqs (i) and (ii) are identical, so
$$h = 1 \text{ and } k = 0$$
Thus, the point P is $(1, 0)$.
The equations of the pair of tangents is
$$SS_1 = T^2$$
$$\Rightarrow \quad (x^2 - y^2 - 9)(1 - 0 - 9) = (x - 9)^2$$
$$\Rightarrow \quad -8(x^2 - y^2 - 9) = x^2 - 18x + 81$$
$$\Rightarrow \quad 9x^2 - 8y^2 - 18x + 9 = 0$$

9. The equation of the normal at $P(a \sec \theta, b \tan \theta)$ is
$$ax \cos \theta + by \cot \theta = a^2 + b^2.$$
$$\Rightarrow \quad ax + by \operatorname{cosec} \theta = (a^2 + b^2) \sec \theta$$
and the equation of the normal at $Q(a \sec \varphi, b \tan \varphi)$ is

$ax \cos\varphi + by \cot\varphi = a^2 + b^2$

$\Rightarrow \quad ax + by \operatorname{cosec}\varphi = (a^2 + b^2)\sec\varphi$

Solving, we get

$b(\operatorname{cosec}\theta - \operatorname{cosec}\varphi)\, y = (a^2 + b^2)(\sec\theta - \sec\varphi)$

$$\Rightarrow \quad y = \left(\frac{a^2+b^2}{b}\right)\left(\frac{\sec\theta - \sec\varphi}{\operatorname{cosec}\theta - \operatorname{cosec}\varphi}\right)$$

$$\Rightarrow \quad y = \left(\frac{a^2+b^2}{b}\right)\left(\frac{\operatorname{cosec}\varphi - \operatorname{cosec}\theta}{\operatorname{cosec}\theta - \operatorname{cosec}\varphi}\right)$$

$$\Rightarrow \quad y = -\left(\frac{a^2+b^2}{b}\right)$$

Thus, $k = -\left(\frac{a^2+b^2}{b}\right)$

10. Given curve is $\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$

We have

$b^2 = a^2(e^2 - 1)$

$\Rightarrow \quad a^2e^2 = a^2 + b^2$

$\Rightarrow \quad a^2e^2 = \cos^2\alpha + \sin^2\alpha = 1$

$\Rightarrow \quad (ae)^2 = 1$

$\Rightarrow \quad (ae) = \pm 1$

Abscissae of foci are ± 1 irrespective of the value of α.

11. Given hyperbola is $x^2 - 2y^2 = 4$

$$\Rightarrow \quad \frac{x^2}{4} - \frac{y^2}{2} = 1 \qquad \ldots\text{(i)}$$

Given line is

$$2x + \sqrt{6}y = 2$$

$$\Rightarrow \quad \sqrt{6}y = -2x + 2$$

$$\Rightarrow \quad y = -\frac{2}{\sqrt{6}}x + \frac{2}{\sqrt{6}}$$

$$\Rightarrow \quad y = -\sqrt{\frac{2}{3}}x + \sqrt{\frac{2}{3}}$$

As we know that if the line $y = mx + c$ be a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the co-ordinates of the point of contact is $\left(\pm\frac{a^2m}{c}, \pm\frac{b^2}{c}\right)$.

$$= \left(\pm\frac{4\left(-\sqrt{\frac{2}{3}}\right)}{\sqrt{\frac{2}{3}}}, \pm\frac{2}{\sqrt{\frac{2}{3}}}\right)$$

$= (\pm(-4), \pm\sqrt{6})$

$= (4, -\sqrt{6})$ satisfies the given curve.

12. Given hyperbola is $\frac{x^2}{9} - \frac{y^2}{4} = 1$.

The equation of any point on the above hyperbola is $P(3\sec\theta, 2\tan\theta)$.

and the equation of the chord of contact of the circle $x^2 + y^2 = 9$ relative to the point P is

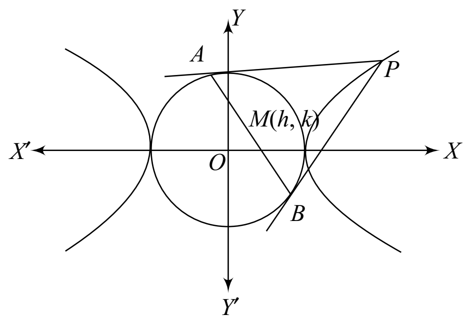
$$3x\sec\theta + 2y\sin\theta = 9 \qquad \ldots\text{(i)}$$

Let $M(h, k)$ be the mid-point of (i).

The equation of the chord of the circle bisected at M is

$T = S_1$

$$\Rightarrow \quad hx + ky = h^2 + k^2 \qquad \ldots\text{(ii)}$$



Clearly, Eqs (i) and (ii) are identical. So

$$\frac{3\sec\theta}{h} = \frac{2\tan\theta}{k} = \frac{9}{h^2+k^2}$$

$$\Rightarrow \quad \sec\theta = \frac{3h}{h^2+k^2}, \tan\theta = \frac{9k}{2(h^2+k^2)}$$

As we know that,

$\sec^2\theta - \tan^2\theta = 1$

$$\Rightarrow \quad \left(\frac{3h}{h^2+k^2}\right)^2 - \left(\frac{9k}{2(h^2+k^2)}\right)^2 = 1$$

$\Rightarrow \quad 36h^2 - 81k^2 = (h^2 + k^2)^2$

Hence, the locus of $M(h, k)$ is

$36x^2 - 81y^2 = (x^2 + y^2)^2$

13. Let e be the eccentricity of the given ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$

Thus, $e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$

Therefore, the eccentricity of the hyperbola is $\frac{5}{3}$.

Also, $ae = 5 \cdot \frac{3}{5} = 3$

Thus, the hyperbola passes through the focus, i.e. (3, 0) of the given ellipse.

So, the semi-transverse axis is 3, i.e. $a = 3$

So, the semi conjugate axis is 4, i.e. $b = 4$

Hence, the equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\Rightarrow \quad \frac{x^2}{9} - \frac{y^2}{16} = 1$$

14. Given, the transverse axis = 2 sin θ

$\Rightarrow$ $2a = 2\sin\theta$

$\Rightarrow$ $a = \sin\theta$

Given ellipse is

$3x^2 + 4y^2 = 12$

$\Rightarrow$ $\frac{x^2}{4} + \frac{y^2}{3} = 1$

The eccentricity of the ellipse,

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{3}{4}} = \frac{1}{2}$$

and the foci of the ellipse = $(\pm ae, 0)$

$$= \left(\pm 2 \cdot \frac{1}{2}, 0\right) = (\pm 1, 0)$$

Let e_1 be the eccentricity of the hyperbola.

Now, $b^2 = a^2(e_1^2 - 1)$

$= a^2 e_1^2 - a^2$

$= 1 - \sin^2\theta = \cos^2\theta$

Hence, the equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$\Rightarrow$ $\frac{x^2}{\sin^2\theta} - \frac{y^2}{\cos^2\theta} = 1$

$\Rightarrow$ $x^2 \operatorname{cosec}^2\theta - y^2 \sec^2\theta = 1$

15. Given curve is

$$x^2 - 2y^2 - 2\sqrt{2}x - 4\sqrt{2}y - 6 = 0$$

$\Rightarrow$ $(x^2 - 2\sqrt{2}\,x) - 2(y^2 + 2\sqrt{2}\,y) = 6$

$\Rightarrow$ $\left\{(x - \sqrt{2})^2 - 2\right\} - 2\left\{(y + \sqrt{2})^2 - 2\right\} = 6$

$\Rightarrow$ $(x - \sqrt{2})^2 - 2(y + \sqrt{2})^2 = 6 + 2 - 4$

$\Rightarrow$ $(x - \sqrt{2})^2 - 2(y + \sqrt{2})^2 = 4$

$\Rightarrow$ $\frac{(x - \sqrt{2})^2}{4} - \frac{(y + \sqrt{2})^2}{2} = 1$

Vertex: $(a, 0)$

$\Rightarrow$ $x - \sqrt{2} = 2,\ y + \sqrt{2} = 0$

$\Rightarrow$ $x = \sqrt{2} + 2,\ y = -\sqrt{2}$

Thus, $A(\sqrt{2} + 2, -\sqrt{2})$

Focus: $x - \sqrt{2} = ae,\ y + \sqrt{2} = 0$

$\Rightarrow$ $x = \sqrt{2} + 2 \cdot \frac{\sqrt{3}}{\sqrt{2}},\ y = -\sqrt{2}$

$\Rightarrow$ $x = \sqrt{2} + \sqrt{6},\ y = -\sqrt{2}$

Therefore, the focus is C: $(\sqrt{2} + \sqrt{6}, -\sqrt{2})$

End-point of L.R. = $\left(ae, \frac{b^2}{a}\right)$

$\Rightarrow$ $x - \sqrt{2} = \sqrt{6},\ y + \sqrt{2} = 1$

$\Rightarrow$ $x = \sqrt{2} + \sqrt{6},\ y = 1 - \sqrt{2}$

So, $B = (\sqrt{2} + \sqrt{6}, 1 - \sqrt{2})$.

Now, $ar(\Delta ABC) = \frac{1}{2} \times (\sqrt{6} - 2) \times 1$

$$= \left(\sqrt{\frac{3}{2}} - 1\right)$$

16. Given hyperbola is

$2x^2 - 2y^2 = 1$

$\Rightarrow$ $\frac{x^2}{1/2} - \frac{y^2}{1} = 1$

Thus, $e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{2}{2}} = \sqrt{2}$

The eccentricity of the ellipse is $\frac{1}{\sqrt{2}}$.

Also, $b^2 = a^2(1 - e^2) = a^2\left(1 - \frac{1}{2}\right) = \frac{a^2}{2}$

Thus, the equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$\Rightarrow$ $\frac{x^2}{a^2} + \frac{y^2}{a^2/2} = 1$

$\Rightarrow$ $x^2 + 2y^2 = a^2$

Let $P(h, k)$ be the point of intersection of the ellipse and the hyperbola.

Thus, $h^2 + 2k^2 = a^2$ and $2h^2 - 2k^2 = 1$ …(i)

The Equations of tangents at $P(h, k)$ to the ellipse and hyperbola are

$hx + 2ky = a^2$ and $2hx - 2ky = 1$

Now, $m(E_T) = -\frac{h}{2k}$ and $m(H_T) = \frac{h}{k}$

Since both the curves cut at right angles, so

$m(E_T) \times m(H_T) = -1$

$\Rightarrow$ $-\frac{h}{2k} \times \frac{h}{k} = -1$

$\Rightarrow$ $h^2 = 2k^2$ …(ii)

From Eqs (i) and (ii), we get,

$2h^2 - h^2 = 1$

$\Rightarrow$ $h^2 = 1$

and $h^2 + 2k^2 = a^2$

$\Rightarrow$ $h^2 + h^2 = a^2$

$\Rightarrow$ $a^2 = 1 + 1 = 2$

Thus, the equation of the ellipse is

$\Rightarrow$ $x^2 + 2y^2 = 2$

and its foci are $(\pm ae, 0) = (\pm 1, 0)$.

17. (P) As $hx + ky - 1 = 0$ touches the circle $x^2 + y^2 = 4$, so,

$$\left|\frac{0+0-1}{\sqrt{h^2+k^2}}\right| = 2$$

$$\Rightarrow \quad (h^2+k^2) = \frac{1}{4}$$

Thus, (h, k) lies on $(x^2+y^2) = \frac{1}{4}$

(Q) z lies on the hyperbola, since
$|SP - S'P| = 2a$.

(R) $$\left(\frac{x}{\sqrt{3}}\right)^2 + y^2 = \left(\frac{1-t^2}{1+t^2}\right)^2 + \left(\frac{2t}{1+t^2}\right)^2$$

$$= \frac{(1-t^2)^2 + 4t^2}{(1+t^2)^2}$$

$$= \frac{(1+t^2)^2}{(1+t^2)^2}$$

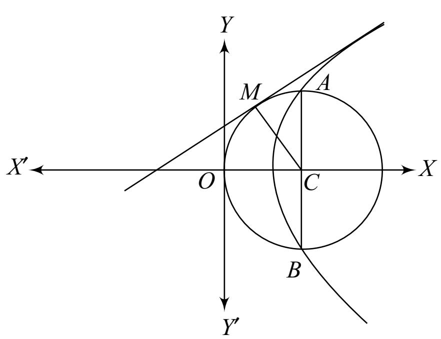
$$= 1$$

which represents an ellipse.

(S) For $x > 1$, the given conic is a hyperbola.
For $x = 1$, the conic is a parabola.

(T) Let $z = x + iy$
Given $Re(z+1)^2 = |z|^2 + 1$
$\Rightarrow (x+1)^2 = x^2 + y^2 + 1$
$\Rightarrow x^2 + 2x + 1 = x^2 + y^2 + 1$
$\Rightarrow y^2 = 2x$
which represents a parabola.

21. Given circle is
$x^2 + y^2 - 8x = 0$
$\Rightarrow (x-4)^2 + y^2 = 16$



The centre is (4, 0) and the radius = 4.

Given hyperbola is $\frac{x^2}{9} - \frac{y^2}{4} = 1$

The equation of any tangent to the given hyperbola is

$$y = mx + \sqrt{a^2m^2 - b^2}$$

$$\Rightarrow \quad y = mx + \sqrt{9m^2 - 4}$$

Now, $CM = 4$

$$\Rightarrow \quad \left|\frac{4m - 0 + \sqrt{9m^2-4}}{\sqrt{m^2+1}}\right| = 4$$

$$\Rightarrow \quad (4m + \sqrt{9m^2-4})^2 = 16(m^2+1)$$

$$\Rightarrow \quad 16m^2 + 9m^2 - 4 + 8m\sqrt{9m^2-4} = 16m^2 + 16$$

$$\Rightarrow \quad (9m^2 - 20)^2 = (-8m\sqrt{9m^2-4})^2$$

$\Rightarrow \quad 81m^4 + 400 - 360m = 64m(9m^2 - 4)$

$\Rightarrow \quad 495m^4 + 104m^2 - 400 = 0$

$\Rightarrow \quad (99m^2 + 100)(5m^2 - 4) = 0$

$\Rightarrow \quad (5m^2 - 4) = 0$

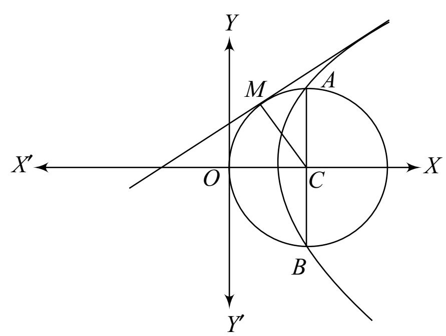
$$\Rightarrow \quad m = \frac{2}{\sqrt{5}}, \text{ since } m > 0$$

Thus, the equation of the tangent is

$$y = \frac{2}{\sqrt{5}}x + \frac{4}{\sqrt{5}}$$

$$\Rightarrow \quad 2x - \sqrt{5}y + 4 = 0$$

19.



Let the co-ordinates of A be
$(3 \sec\theta, 2\tan\theta)$
As A lies on the circle, so
$9\sec^2\theta = 4\tan^2\theta - 24\sec\theta = 0$
$\Rightarrow \quad 9\sec^2\theta + 4(\sec^2\theta - 1) - 24\sec\theta = 0$
$\Rightarrow \quad 13\sec^2\theta - 24\sec\theta - 4 = 0$
$\Rightarrow \quad (13\sec\theta + 2)(\sec\theta - 2) = 0$
$\Rightarrow \quad \sec\theta = 2$, since $|\sec\theta| \geq 1$
Thus, the points A and B are

$$(6, 2\sqrt{3}) \text{ and } (6, -2\sqrt{3})$$

The equation of the circle with AB as diameter is
$(x-6)^2 + y^2 = 12$
$\Rightarrow \quad x^2 + y^2 - 12x + 24 = 0.$

20. Given line is $2x + y = 1$ …(i)

which is passing through $\left(\frac{a}{e}, 0\right)$

$$\Rightarrow \quad 2 \cdot \frac{a}{e} = 1$$

$$\Rightarrow \quad a = \frac{e}{2}.$$

Since the equation (i) is a tangent to the given hyperbola, so

$c^2 = a^2m^2 - b^2$

$\Rightarrow \quad 1 = a^2, 4 - b^2$

$\Rightarrow \quad 4a^2 - b^2 = 1$

$\Rightarrow \quad 4a^2 - a^2(e^2 - 1) = 1$

$\Rightarrow \quad 5a^2 - a^2e^2 = 1$

$\Rightarrow \quad a^2(5 - e^2) = 1$

$\Rightarrow \quad \frac{e^2}{4}(5-e^2)=1$

$\Rightarrow \quad 5e^2 - e^4 = 4$

$\Rightarrow \quad e^4 - 5e^2 + 4 = 0$

$\Rightarrow \quad (e^2 - 1)(e^2 - 4) = 0$

$\Rightarrow \quad (e^2 - 4) = 0$, as $e \neq 1$

$\Rightarrow \quad e = 2$

21. The equation of the normal at P is

$$\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$$

$$\Rightarrow \quad \frac{a^2x}{6}+\frac{b^2y}{3}=a^2+b^2$$

which is passing through (9, 0). So

$$\frac{a^2\cdot 9}{6}+\frac{\mathrm{b}^2\cdot 0}{3}=a^2+b^2$$

$$\Rightarrow \quad \frac{3a^2}{2}=a^2+b^2$$

$$\Rightarrow \quad \frac{3a^2}{2}=a^2+a^2(e^2-1)$$

$$\Rightarrow \quad \frac{3}{2}=1+(e^2-1)$$

$$\Rightarrow \quad (e^2-1)=\frac{3}{2}-1=\frac{1}{2}$$

$$\Rightarrow \quad e^2=1+\frac{1}{2}=\frac{3}{2}$$

$$\Rightarrow \quad e=\sqrt{\frac{3}{2}}.$$

Hence, the eccentricity is $\sqrt{\frac{3}{2}}$.

22. Given ellipse is

$x^2 + 4y^2 = 4$

$$\Rightarrow \quad \frac{x^2}{4}+\frac{y^2}{1}=1$$

The eccentricity,

$$e=\sqrt{1-\frac{b^2}{a^2}}=\sqrt{1-\frac{1}{4}}=\frac{\sqrt{3}}{2}$$

Thus, the eccentricity of the hyperbola is $\frac{2}{\sqrt{3}}$.

Foci of the ellipse = $(\pm ae, 0) = (\pm\sqrt{3}, 0)$.

The hyperbola passing through the focus of the ellipse, so

$$\frac{3}{a^2}-0=1$$

$\Rightarrow \quad a^2 = 3$

Now, $b^2 = a^2\,(e^2 - 1)$

$$\Rightarrow \quad b^2=a^2\left(\frac{4}{3}-1\right)=\frac{a^2}{3}=1$$

Hence, the equation of the hyperbola is

$$\frac{x^2}{3}-\frac{y^2}{1}=1$$

$\Rightarrow \quad x^2 - 3y^2 = 3$

$\therefore \quad$ Focus of a hyperbola = (2, 0)

23. Given line is $2x - y = 1$. So,

$m = 2$

The equation of any tangent to the given hyperbola is

$$y=mx\pm\sqrt{a^2m^2-b^2}$$

$$\Rightarrow \quad y=2x\pm\sqrt{9.4-4}$$

$$\Rightarrow \quad y=2x\pm\sqrt{32}$$

$$\Rightarrow \quad y=2x\pm 4\sqrt{2}$$

$$\Rightarrow \quad 2x-y+4\sqrt{2}=0,\ 2x-y-4\sqrt{2}=0$$

$$\Rightarrow \quad \frac{x}{2\sqrt{2}}-\frac{y}{4\sqrt{2}}=1 \text{ and } -\frac{x}{2\sqrt{2}}+\frac{y}{4\sqrt{2}}=1$$

Comparing it with $\frac{xx_1}{9}-\frac{yy_1}{4}=1$. we get the points of contact as

$$\left(\frac{9}{2\sqrt{2}},\frac{1}{\sqrt{2}}\right) \text{ and } \left(-\frac{9}{2\sqrt{2}},-\frac{1}{\sqrt{2}}\right).$$

24. Tangent at P is $xx_1 - yy_1 = 1$ intersects the x-axis at $M\left(\frac{1}{x_1},0\right)$

Slope of normal $=-\frac{y_1}{x_1}=\frac{y_1-0}{x_1-x_2}$

$\Rightarrow \quad x_2 = 2x_1$

Thus, $N = (2x_1, 0)$

For centroid, $l=\frac{3x_1+\frac{1}{x_1}}{3},\ m=\frac{y_1}{3}$

$$\Rightarrow \quad \frac{dl}{dx_1}=1-\frac{1}{3x_1^2}$$

and $\quad \frac{dm}{dy_1}=\frac{1}{3},\ \frac{dm}{dx_1}=\frac{1}{3}\frac{dy_1}{dx_1}=\frac{x_1}{3\sqrt{x_1^2-1}}$